# Detection threshold in very sparse matrix completion

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*Work in collaboration with Raj Rao Nadakuditi (Univ. Michigan) and Simon Coste (Inria Paris)*

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# Matrix estimation

Let $P \in M_{n,m}(\mathbb{R})$ be a large rectangular matrix $n = \Theta(m)$.

We observe each entry independently with probability $d/n$. The other entries remain hidden.

$d =$ *average number of observed entries per row.*

We assume that the matrix $P$ is simple: notably small rank and some spectral incoherence.

# Completion and estimation

Matrix completion: can we reconstruct exactly $P$ thanks to this observation?

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Matrix estimation: we look for a matrix with a small mean square error

$$\mathrm{MSE}(\widehat{P}) = \sum_{i,j} |\widehat{P}_{ij} - P_{ij}|^2 = \|\widehat{P} - P\|_F^2.$$

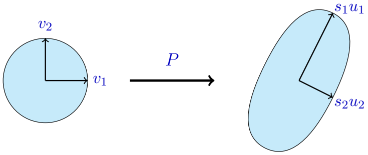
Best bounds for $d \ll \log n$: $\mathrm{MSE}(\widehat{P}) = O(mn/d)$.

*Candès-Tao 09, Candès-Recht 10, Keshavan-Montanari-Oh 09 ...*

Singular value decomposition of $P \in M_{n,m}(\mathbb{C})$ :

$$P = UDV^* = \sum_{k=1}^{n} s_k u_k v_k^*,$$

where $D = \mathrm{diag}(s_1, \ldots, s_n) \in M_{n,m}(\mathbb{C})$ and $s_1 \geqslant \ldots \geqslant s_n \geqslant 0$ are the singular values de $P$.




$$\|P\|_F^2 = \sum_k s_k^2.$$

# Matrix detection

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⋆ Fix $0 < \gamma < 1$. Find the smallest $d$ such that there is with high probability an estimator $\hat{u}_k$ of $u_k$ with

$$|\langle \hat{u}_k, u_k \rangle| \geqslant \gamma.$$

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⋆ Fix $0 < \gamma < 1$. Find the smallest $d$ such that there is with high probability an estimator $\hat{P}$ of $P$ with

$$\mathrm{MSE}(\hat{P}) = \|P - \hat{P}\|_F^2 \geqslant \gamma \|P\|_F^2.$$

# Applications

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*Guess what a user likes even before she knows it.* The Netflix prize launched in 2006 consisted in minimizing the MSE (on a sample) with respect to the matrix:

$$P_{ij} = \text{ mark given by user } i \text{ on movie } j.$$

# Estimation of a symmetric matrix

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Let $P \in M_n(\mathbb{R})$ be a symmetric matrix.

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We have $\mathbb{E}A = \mathbb{E}H = P$.

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Benefits of asymmetry: in some situations, the spectrum of a matrix $P$ is much less perturbed by a random asymmetric noise than by a random symmetric noise.

# Detection threshold

We set

$$Q_{ij} = n|P_{ij}|^2 \qquad \text{and} \qquad \rho = \|Q\|.$$

The detection threshold is defined as

$$\theta = \max\left(\sqrt{\frac{\rho}{d}}, \frac{L}{d}\right),$$

with

$$L = n \max_{i,j} |P_{ij}|.$$

# Incoherence

We order the real eigenvalues of $P$,

$$|\mu_1| \geqslant \cdots \geqslant |\mu_{r_0}| > \theta \geqslant |\mu_{r_0+1}| \geqslant \cdots \geqslant |\mu_n|.$$

In a ON basis of eigenvectors of $P$, for all $1 \leqslant k \leqslant r_0$,

$$\|\varphi_k\|_\infty = \max_i |\varphi_k(i)| \leqslant \frac{b}{\sqrt{n}}.$$

# Stable numerical rank

The stable numerical rank is

$$r = \frac{\sum_k \mu_k^2}{\mu_1^2} \leqslant \mathrm{rank}(P).$$

In this talk, we assume that $r, b, L, d, r_0$ are $O\big(n^{o(1)}\big)$. All results are quantitative but will be stated in an asymptotic way.

# Estimation of eigenvalues

Recall:

$$A_{ij} = \frac{n}{d} P_{ij} M_{ij} \qquad \text{and} \qquad \theta = \max\left(\sqrt{\frac{\rho}{d}}, \frac{L}{d}\right).$$

Eigenvalues of $P$ :

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## ESTIMATION OF EIGENVALUES

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Eigenvalues of $P$ :

$$|\mu_1| \geqslant \cdots \geqslant |\mu_{r_0}| > \theta \geqslant |\mu_{r_0+1}| \geqslant \cdots \geqslant |\mu_n|.$$

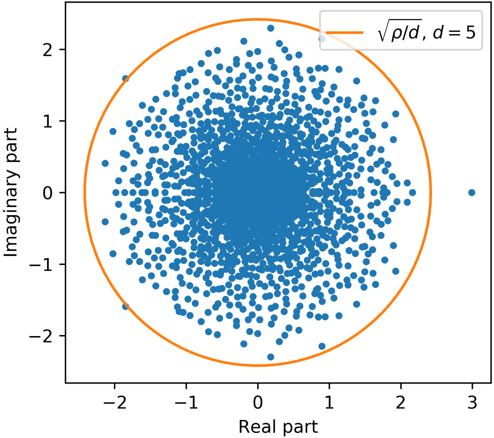
*Theorem*

*With high probability, there exists an ordering of the eigenvalues* $\lambda_1, \dots, \lambda_n$ *of* $A$ *such that*

$$\max_{1 \leqslant k \leqslant r_0} |\lambda_k - \mu_k| = o(1) \qquad \textit{and} \qquad \max_{r_0+1 \leqslant k \leqslant n} |\lambda_k| \leqslant (1 + o(1))\theta.$$

# Simulation

For $n = 2000$ and $P = 3\varphi_1\varphi_1^* + 2\varphi_2\varphi_2^* + \varphi_3\varphi_3^*$ with $\varphi_k$ uniform.



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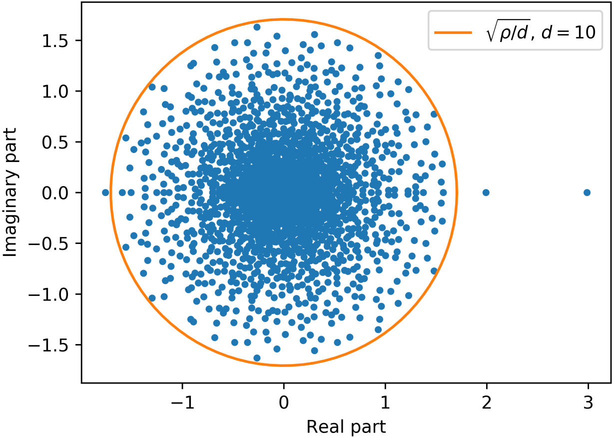

# Estimation of eigenvectors

Recall:

$$A_{ij} = \frac{n}{d} P_{ij} M_{ij} \qquad \text{and} \qquad \theta = \max\left(\sqrt{\frac{\rho}{d}}, \frac{L}{d}\right).$$

We assume that the large eigenvalues of

$$P = \sum_k \mu_k \varphi_k \varphi_k^*$$

are well separated:

$$\left|1 - \frac{\mu_k}{\mu_l}\right| \geqslant \frac{\log d}{\log n} \qquad \text{for all} \quad 1 \leqslant k \neq l \leqslant r_0.$$

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*Theorem*

*Let $\psi_k$ be a unit eigenvector associated to $k$-th eignevalue of $A$. There exists $\gamma_k > 0$ such that, with high probability, for $1 \leqslant k \leqslant r_0$,*

$$|\langle \psi_k, \varphi_k \rangle| = \gamma_k + o(1).$$

# Estimation of eigenvectors

The asymptotic scalar product $\gamma_k = |\langle \psi_k, \phi_k \rangle| + o(1)$ has an explicit formula:

$$\gamma_k = \frac{1}{\sqrt{\Gamma_{k,k}}}$$

with, for $1 \leqslant k, l \leqslant r_0$,

$$\Gamma_{k,l} = \sum_{i=1}^{n} w_{k,l}(i) \varphi_k(i) \varphi_l(i),$$

and

$$w_{k,l}(i) = \sum_j \left( I - \frac{Q}{\mu_k \mu_l d} \right)^{-1}_{i,j}.$$

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*Remark*: $|\langle \psi_k, \psi_l \rangle| = |\Gamma_{k,l}| / \sqrt{\Gamma_{k,k}\Gamma_{l,l}} + o(1)$ is non-zero for $k \neq l$ if $\mathbf{1}$ is not an eigenvector of $Q$.

# Rank one projector

If $P = \varphi\varphi^*$, we find

$$\theta = \sqrt{\frac{n \sum_i |\varphi(i)|^4}{d}}$$

$$\gamma = \sqrt{1 - \frac{n \sum_i |\varphi(i)|^4}{d}}.$$

# Estimation of eigenvectors

It is also possible to compute the scalar between the left $\psi'_k$ and right $\psi_k$ unit eigenvectors of the $k$-th eigenvalue of $A$:

$$\langle \psi'_k, \psi_k \rangle = \gamma_k^2 + o(1) = \frac{1}{\Gamma_{k,k}} + o(1).$$

We get an estimator

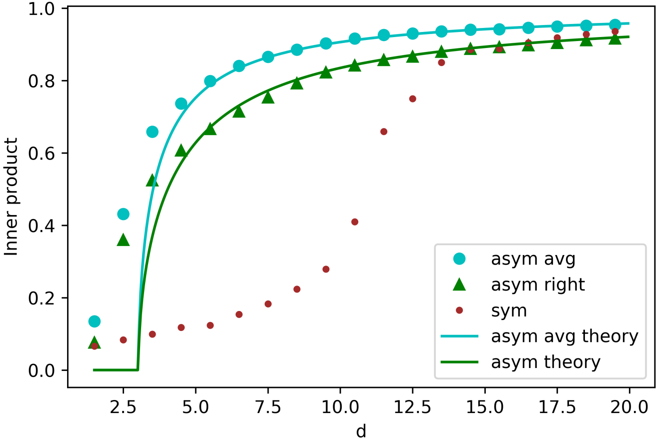
$$\hat{\varphi}_k = \frac{\psi_k + \psi'_k}{\|\psi_k + \psi'_k\|_2}$$

such that

$$|\langle \varphi_k, \hat{\varphi}_k \rangle| = \sqrt{\frac{2\gamma_k^2}{1 + \gamma_k^2}} + o(1).$$

# Simulation

For $n = 6000$ and $P = \varphi\varphi^*$ avec $\varphi$ uniform on the sphere.



# Improved estimation with non-backtracking matrices

# Put some symmetry back

We can improve the factor $d$ in $2d$ in the detection threshold:

$$\theta = \max\left(\sqrt{\frac{\rho}{d}}, \frac{L}{d}\right).$$

We have not taken into account the information

$$P_{ij} = P_{ji}.$$

There is in fact an average of $2d$ observed entries per row.

# Non-backtracking matrix

The set of symmetric observed entries is

$$E = \{(i,j) : (i,j) \text{ or } (j,i) \text{ is observed}\}.$$

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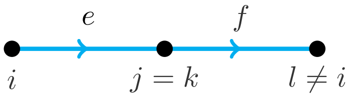
We consider the non-symmetric matrix $B \in M_E(\mathbb{C})$ defined for all $(i,j),(k,l)$ in $E$ by

$$B_{(i,j),(k,l)} = \frac{nP_{kl}}{2d}\mathbf{1}(j = k, l \neq i).$$



# Non-backtracking matrix

A vector $\varphi \in \mathbb{C}^n$ is lifted in $\mathbb{C}^E$ as

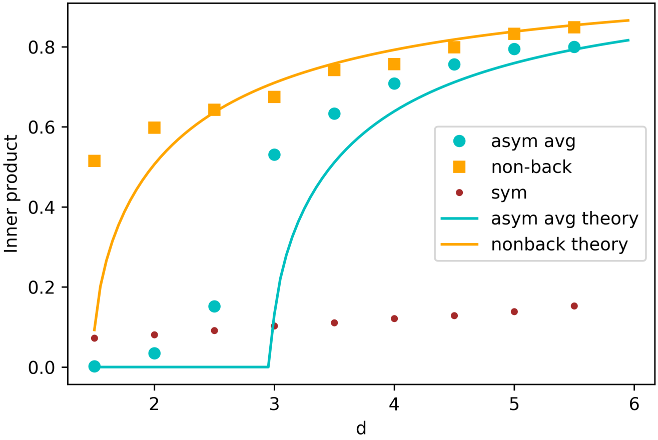
$$\varphi^+(i,j) = \varphi(j).$$

*Theorem*

*The preceding results on $A$ are true for the matrix $B$ (with minor extra changes) with $d$ replaced by $2d$.*

# Simulation

For $n = 5000$ and $P = \varphi\varphi^*$ with $\varphi$ uniform on the sphere.



## The detection threshold

$$\theta = \max\left(\sqrt{\frac{\rho}{d}}, \frac{L}{d}\right).$$

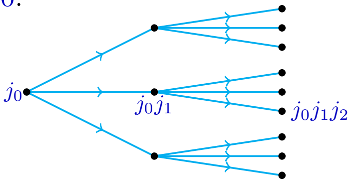
# Lift of a matrix

Fix $j_0 \in [n] = \{1, \dots, n\}$. Let $V$ be the set of finite integer sequences in $[n]$, $(j_0, j_1, \dots, j_k)$ starting with $j_0$.

We build an infinite matrix $\mathcal{P} = (\mathcal{P}_{uv})_{u,v \in V}$ by setting $u = (j_0, \dots, j_k) \in V$ and $j \in [n]$

$$\mathcal{P}_{u,(u,j)} = P_{j_k,j}.$$

Otherwise $\mathcal{P}_{uv} = 0$.



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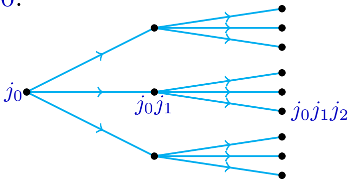



This defines a non-symmetric bounded operator on $\ell^2(V)$ build on an infinite $n$-ary tree.

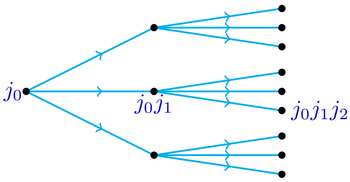
# Lift of a matrix

If $P\varphi = \mu\varphi$ then $\Phi$ defined on $V$ as

$$\Phi(j_0 \cdots j_k) = \varphi(j_k)$$

satisfies

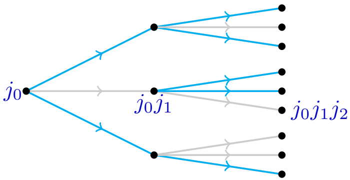
$$\mathcal{P}\Phi = \mu\Phi.$$




The function $\Phi$ is not in $\ell^2(V)$.

# Percolation on the lift

We keep each edge with probability $d/n$.




We denote by $\mathcal{P}_{\text{perc}}$ the corresponding operator and set

$$\mathcal{A} = \frac{n}{d}\mathcal{P}_{\text{perc}}.$$

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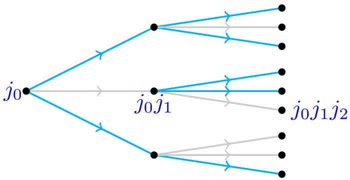


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The operator $\mathcal{A}$ is a local approximation of the matrix $A = (n/d)P \odot M$.

# Percolation on the lift

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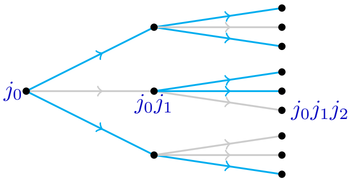


Since $\mathcal{P}\Phi = \lambda\Phi$, for all $v \in V$, the process in $t \in \mathbb{N}$,

$$\Psi_t(v) = \mu^{-t}(\mathcal{A}^t\Phi)(v)$$

is a discrete martingale for the filtration of the successive generations in the tree.

# Percolation on the lift

The bracket of the martingale can be computed and we find:

$$\mathbb{E}|\Psi_{t+1}(v) - \Psi_t(v)|^2 = \frac{Q^t(\varphi^2)(v)}{(\mu^2 d)^t}.$$

Recall: $Q_{ij} = n|P_{ij}|^2$ and $\rho = \|Q\|$.

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Recall: $Q_{ij} = n|P_{ij}|^2$ and $\rho = \|Q\|$.

Hence $\Psi_t(v) = \mu^{-t}(\mathcal{A}^t\Phi)(v)$ converges a.s. and in $L^2$ toward $\Psi(v)$ if

$$|\mu| > \sqrt{\frac{\rho}{d}}.$$

This is called the Kesten-Stigum threshold.

# Percolation on the lift

If $|\mu| > \sqrt{\rho/d}$, then $\Psi_t(v) = \mu^{-t}(\mathcal{A}^t\Phi)(v)$ converges a.s. and in $L^2$ toward $\Psi(v)$.

Since

$$\mathcal{A}\Psi_t = \mu\Psi_{t+1},$$

we can define a.s. a random eigenwave $\Psi$ on $V$ which satisfies

$$\mathcal{A}\Psi = \mu\Psi.$$

# Back to finite dimension

This analysis and concentration inequalities allow to show that if $t \gg 1$ but not too large,

$$\|A^{t+1}\varphi_k - \mu_k A^t \varphi_k\|_2 = o(\|A^t \varphi_k\|_2).$$

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We decompose $A^t$ in

$$A^t = \sum_{k=1}^{r_0} \mu_k^t u_k v_k^* + R_t,$$

with $u_k = A^t \varphi_k / \mu_k^t$ and $v_k = (A^t)^* \varphi_k / \mu_k^t$. We have

$$\langle u_k, v_l \rangle = \delta_{kl} + o(1).$$

# Proof strategy

For $t = c \ln n / \log d$ well chosen,

$$A^t = \sum_{k=1}^{r_0} \mu_k^t u_k v_k^* + R_t.$$

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- Compute the inner products between these $r_0^2$ vectors;
- Show that the Gram matrix is well-conditioned;
- Show that $\|R_t\| \leqslant (\log n)^c \theta^t$;
- Use an ad-hoc spectral perturbation theorem of a non-symmetric matrix of Bauer-Fike type.

*B-Lelarge-Massoulié 18.*

# Rectangular matrices

# Linearization trick

If $P \in M_{m,n}(\mathbb{C})$, the matrix

$$\widetilde{P} = \begin{pmatrix} 0 & P \\ P^* & 0 \end{pmatrix}$$

is of size $(m+n) \times (m+n)$ and is Hermitian.

The singular value decomposition of $P = \sum_k s_k u_k v_k^*$ is equivalent to the diagonalization of $\widetilde{P}$:

$$\widetilde{P} = \sum_k s_k w_k^+ (w_k^+)^* - s_k w_k^- (w_k^-)^*,$$

with $w_k^\pm = (u_k, \pm v_k)'/\sqrt{2}$.

# A randomized asymmetric SVD

Recall

$$P = \sum_k s_k u_k v_k^*.$$

Consider $Z = (Z_{ij}) \in M_{m,n}(\mathbb{R})$ with iid $\{0,1\}$-Bernoulli entries with parameter $1/2$ and define

$$P_1 P_2^* \quad \text{with} \quad P_1 = P \odot Z\,,\; P_2 = P - P_1.$$

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$$P_1 P_2^* \quad \text{with} \quad P_1 = P \odot Z\,,\ P_2 = P - P_1.$$

The $k$-th largest eigenvalue, say $\lambda_k$, of $P_1 P_2^*$ is a proxy for $s_k^2/4$.

The average of the left and right eigenvectors associated to $\lambda_k$ is a proxy for the left singular vector $u_k$.

# Matrix completion

Let $M = (M_{ij}) \in M_{m,n}(\mathbb{R})$ with iid $\{0,1\}$-Bernoulli entries with parameter $d/n$.

The observed matrix is

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At a higher computational cost, we may also consider the non-backtracking matrix associated to the linearized matrix $\widetilde{A}$.

In either case, if $n \asymp m$, we have explicit detection thresholds and formulas for the asymptotic inner products.

Recall

$$P = \sum_k s_k u_k v_k^*.$$

Once we have estimators $\hat{u}_k, \hat{v}_k$ of $u_k$ and $v_k$, it is possible to design an estimator of $P$:

$$\hat{P} = \sum_{k=1}^{r_0} x_k \hat{u}_k \hat{v}_k^*$$

for some vector $x = (x_k) \in \mathbb{R}^{r_0}$ which asymptotically minimizes

$$\|\hat{P} - P\|_F$$

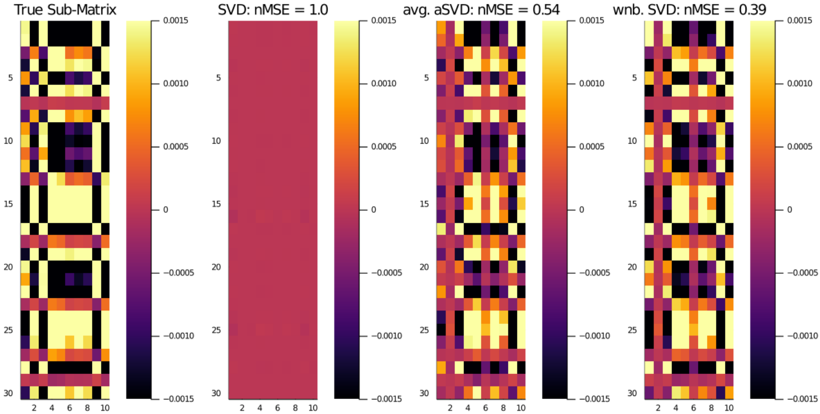
and compute an explicit asymptotic formula for

$$\text{MSE}(\hat{P}) = \|\hat{P} - P\|_F^2.$$

*Nadakuditi 14.*

# Simulation

We take $d = 9.7$, $(m, n) = (2000, 3000)$ and $P = uv^*$ with $u, v$ independent standard Gaussian vectors.



# Concluding words

# Conclusion

Spectral analysis methods on random non-symmetric matrices can be very efficient, *Chen-Cheng-Fan 18*.

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There is nowadays a lot of activities on tensor completion.

Thank you for your attention!