Norm estimates for polynomials in random matrices: new results

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Overview

Joint work with Alice Guionnet (ENS Lyon) and Felix Parraud (ENS Lyon & Kyoto)
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Plan:

1. Motivation.
2. Results (GUE).
3. Further results and applications.
The operator norm of matrix $X \in M_{\mathbb{C}}^{N \times N}$. Its operator norm is $||X|| = \sup_{v \neq 0} \frac{||Xv||}{||v||^2}$.

If $X$ is selfadjoint with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_N$, then $||X||_{\infty} = \sup_i |\lambda_i|$.

We will mostly work with class of (random) matrices closed under addition, *-operation and product. Since $||X||_2 = ||XX^*||$, it is enough to work with selfadjoint $X$. So far, nothing random...
The operator norm of matrix

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Long known results

- (Wigner’s Theorem): If $X^N$ is an $N \times N$ Wigner matrix (with appropriate assumptions on the entries), its empirical measure $\mu_{X^N} = N^{-1} \sum_i \delta_{\lambda_i}$ converges to the semi-circle distribution.

This implies that $\lim \inf N \|X^N\| \geq 2$ (almost surely or in expectation).

Similarly we can show $\lim \inf N \|f(X^N)\| \geq \|f(x)\|$ for any continuous function (functional calculus), where $x$ is a semi-circular distributed rrv.

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- A similar result can be proved if $f(X^N)$ is replaced by $P(X_1^N, \ldots, X_d^N)$, where $P$ is a non-commuting polynomial in $d$ free abstract variables and $X_1^N, \ldots, X_d^N$ are $d$ iid copies of GUEs.
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Indeed, Voiculescu’s *asymptotic freeness* (1991) implies that $\lim_N tr(P(X_1^N, \ldots, X_d^N))$ converges to $\tau(P(x_1, \ldots, x_d))$ where $x_1, \ldots, x_d$ are free semi-circular variables.
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- Similar results hold true with many other matrix models (more general iid Wigner matrices, i.i.d random unitary matrices, etc).
Less trivial bounds

- **Fundamental problem**: can we replace \(\liminf_N \|X^N\| \geq XXX\) by \(\limsup_N \|X^N\| \leq XXX??\) (i.e. can we get \(\lim_N \|X^N\| = XXX??\)).

- **Yes** for Wigner (under appropriate assumptions). Cf Füredi Komlós, etc.
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- (big breakthrough – Haagerup-Thorbjørnsen 2005): **Yes** for a NC polynomial \( P \) in iid copies \( X_1^N, \ldots, X_d^N \) of GUEs.
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- (Male, Pisier): **Yes** if one adds *tame* constant matrices & tensors to the models.
- (C & Male): Same as above if one replaces iid GUEs by iid unitaries.
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- Fundamental problem: can we replace \( \lim \inf \| X^N \| \geq XXX \) by \( \lim \sup \| X^N \| \leq XXX \)?
- (C, Bordenave): \textbf{Yes} for a polynomial in iid copies of random permutations or involutions without fixed points (acting on mean zero vectors).
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- (C, Bordenave – in preparation): **Yes** for finite tensors of random unitaries.
A non obvious fact: Actually, all the above results actually imply that if $P$ is self-adjoint, the spectrum of $P(X_i^N)$ converges to the spectrum of $P(x_i)$ in the sense of Hausdorff distance.
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The proof of this fact in the case of GUE’s or random unitaries is simple modulo a hard result in OA: the limiting $C^*$-algebra has no non-trival projection.
The proof of this fact in general relies on linearization – namely, understanding the spectrum of any \textit{NC polynomial} in $X_i$ is equivalent to understanding the spectrum of any \textit{linear equation} in $1_N, X_i^N, \ldots$ with matrix coefficients of arbitrary size.
The proof of this fact in general relies on linearization – namely, understanding the spectrum of any \textit{NC polynomial} in $X_i$ is equivalent to understanding the spectrum of any \textit{linear equation} in $1_N, X_i^N, \ldots$ with matrix coefficients of arbitrary size.

The stability under addition, multiplication and conjugation of our families of models is also important (folding trick).
Strong convergence

Definition: a $d$-tuple of $n \times n$ matrices $X_1^N, \ldots, X_d^N$ converges strongly to $x_1, \ldots, x_d \in (A, \tau)$ ($\tau$ faithful tracial state) iff, for any NC $*$-polynomial $w$,
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All the previously quoted results (HT, M, CM, BC, etc...) can be restated as strong convergence results. All proofs start by proving weak convergence...
An intriguing remark: sometimes, norm convergence implies strong convergence

- A priori, knowing the asymptotic behavior of $\|P(X_n^i)\|$ for all $P$ does not imply knowing the asymptotic behavior of the eigenvalue counting measure of $P(X_n^i)$. 

Question: Could this be used directly in RMT...?
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- Question: Could this be used directly in RMT...?
Motivation for our work: strong convergence sometimes fails

C-Male’s results imply $\|U_1^N + \ldots + U_d^N\| \to 2\sqrt{d - 1}$ (here, $U_i^N$ are iid random unitary matrices).
For $d > 2$, this is less than $d$ (the trivial bound).
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- However, although this estimate holds for $\|S_1^N + \ldots + S_d^N\|$ (where $S_i^N$ are random permutations) when restricted to mean zero vectors (Friedman, Bordenave), it fails when applied to general vectors (Perron-Frobenius).
Initial motivation: strong convergence sometimes fails.

- Likewise, \( \|O_1^N \otimes O_1^N + \ldots + O_d^N \otimes O_d^N\| \) has an eigenvalue \( d \) (\( O_i^N \) are iid random orthogonal) although the collection \( O_i^N \otimes O_i^N \) is asymptotically free.

Bottom line: there is a strong motivation to understand better the norm situation for tensors, and this is hard.
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- However, restricted to the orthogonal subspace of the eigenvector of the eigenvalue $d$ (the Bell state / Jones projection), the operator norm tends to its usual candidate $2\sqrt{d} - 1$ (Pisier, Hastings).

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- Male, Pisier: strong convergence holds with coefficient whose matrix dimension satisfies $M \ll N^{1/4}$.

- Hayes: What happens when $M = N$ and the coefficients are random and independent from $X_N$?

Remark: the example of $||O_{N_1} \otimes O_{N_1} + ... + O_{N_d} \otimes O_{N_d}||$ shows that $M = N$ can be tricky (although here, the coefficients and the matrices are correlated).
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Main result (arXiv:1912.04588, C, Guionnet, Parraud)

Ingredients:

- $X^N = (X_1^N, \ldots, X_d^N)$ are iid $GUE$, $x = (x_1, \ldots, x_d)$ a system of free semicircular variable,
- $Z^{NM} = (Z_1^{NM}, \ldots, Z_q^{NM})$ are deterministic matrices in $M_M \otimes M_N$,
- $P$ is a self-adjoint non-commuting polynomial in $d + 2q$ variables.
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- Let $f : \mathbb{R} \mapsto \mathbb{R}$ smooth enough (in a Fourier sense):
  [Technically, there exists $\mu = \mu(f)$ with $\int (1 + y^4) \; d|\mu|(y) < +\infty$ and $f(x) = \int_{\mathbb{R}} e^{i xy} \; d\mu(y)$].
Then there exists a polynomial $L_P$ which only depends on $P$ such that for any $N, M$,
\[ \left| \left| E \left[ 1_{MN} \text{Tr} \left( f \left( P \left( X_N \otimes I_M, Z_{NM}, Z_{NM}^* \right) \right) \right) - \tau_N \otimes \tau_M \left( f \left( P \left( x \otimes I_M, Z_{NM}, Z_{NM}^* \right) \right) \right) \right] \right| \leq M^2 N^2 L_P \left( \left\| Z_{NM} \right\| \int R \left( |y| + y^4 \right) d|\mu|(y) \right). \]

Comments: The dependence $P \rightarrow L_P$ is somewhat explicit. For example it can be made uniform on a compact set of bounded degree.
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\left| \mathbb{E} \left[ \frac{1}{MN} \text{Tr} \left( f \left( P \left( X^N \otimes I_M, Z^{NM}, Z^{NM*} \right) \right) \right) \right] - \tau_N \otimes \tau_M \left( f \left( P \left( x \otimes I_M, Z^{NM}, Z^{NM*} \right) \right) \right) \right| \leq \frac{M^2}{N^2} L_P \left( \| Z^{NM} \| \right) \int_{\mathbb{R}} (|y| + y^4) \, d|\mu|(y).
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Under the hood

- Techniques for norm convergence / largest eigenvalue so far:

  - Moment methods: mainly for single matrix models (Füredi, Komlós, Soshnikov, etc)...
  - More recently for some multimatrix models with non-backtracking techniques.
  - Analysis – matrix valued Stieltjes transform + master equation / Schwinger-Dyson. (also: folding trick for the unitary case, integrable systems (TW), etc).
  - Important point: all multimatrix-type results so far basically rely on linearization.
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Under the hood

Our method does not rely on linearization.

Idea: if \( f \) is an indicator function (taking values \([0, 1]\)), \( \text{Tr}(f \circ P) \) is an integer (the number of eigenvalues in the set where \( f \) takes the value 1).

If \( f \circ P \) is applied to a random matrix, it counts the random number of eigenvalues in an interval.

Idea: take \( f \) with support away from the support of \( P(X_N) \).

Problem: we know how to study \( \text{Tr}(f \circ P(X_N)) \) if \( f \) is a polynomial (with, e.g. second order freeness) but not if \( f \) is an indicator function.
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- **Solution:** (1) extend the study of $\text{Tr}(f \circ P(X_i^N))$ to smooth functions, and (2) estimate indicator functions with smooth functions.
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- In practice, work on a free product of matrices $X_i^N$ and free semi-circulars $x_i$. Interpolate between matrices and their limit by writing $X_t^N = e^{-t/2}X_0^N + (1 - e^{-t})^{1/2}x$ (for each $X_i^N$).
Under the hood

- Recall that $X_t^N = e^{-t/2}X_0^N + (1 - e^{-t})^{1/2}x$, ...and write

$$
\mathbb{E} \left[ \frac{1}{N} tr_N \left( Q \left( X^N_t \right) \right) \right] - \tau \left( Q(x) \right) = - \int_0^\infty \mathbb{E} \left[ \frac{d}{dt} \left( \tau_N \left( Q(X_t^N) \right) \right) \right] dt
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Then, show that $\frac{d}{dt} \tau_N(Q(X_t^N))$ is small.
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Then, show that $\frac{d}{dt} \tau_N(Q(X_t^N))$ is small.

This is to be expected because $X_t^N$ interpolates between our model and the limit (so, a priori, little knowledge about the limit is required).
Under the hood

Recall that $X_t^N = e^{-t/2}X_0^N + (1 - e^{-t})^{1/2}x$, ...and write

$$\mathbb{E} \left[ \frac{1}{N} \text{tr}_N(Q(X^N)) \right] - \tau(Q(x)) = - \int_0^\infty \mathbb{E} \left[ \frac{d}{dt}(\tau_N(Q(X_t^N))) \right] dt.$$ 

Then, show that $\frac{d}{dt}\tau_N(Q(X_t^N))$ is small.

This is to be expected because $X_t^N$ interpolates between our model and the limit (so, a priori, little knowledge about the limit is required).

This can be done with free and classical stochastic calculus, Schwinger-Dyson type equations and semigroup theory.
Perspective and further reading


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Application of Parraud’s unitary results to QIT (work in progress)

Thank you for your attention!
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