# Norm estimates for polynomials in random matrices: new results

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# Overview

Joint work with Alice Guionnet (ENS Lyon) and Felix Parraud (ENS Lyon & Kyoto)

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**Plan:**

1. Motivation.
2. Results (GUE).
3. Further results and applications.

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- We will mostly work with class of (random) matrices closed under addition, *-operation and product. Since $||X||^2 = ||XX^*||$, it is enough to work with selfadjoint $X$.
- So far, nothing random...

# Long known results

- (Wigner's Theorem): If $X^N$ is an $N \times N$ Wigner matrix (with appropriate assumptions on the entries), its *empirical measure* $\mu_{X^N} = N^{-1} \sum_i \delta_{\lambda_i}$ converges to the semi-circle distribution.

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(almost surely or in expectation).
- Similarly we can show $\liminf_N ||f(X^N)|| \geq ||f(x)||$ for any *continuous* function (functional calculus), where $x$ is a semi-circular distributed rrv. Here, some kind of smoothness for $f$ is unavoidable.

# Long known results

- A similar result can be proved if $f(X^N)$ is replaced by $P(X_1^N, \ldots, X_d^N)$, where $P$ is a non-commuting polynomial in $d$ free abstract variables and $X_1^N, \ldots, X_d^N$ are $d$ iid copies of of GUEs.

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- Indeed, Voiculescu's *asymptotic freeness* (1991) implies that $\lim_N tr(P(X_1^N, \ldots, X_d^N))$ converges to $\tau(P(x_1, \ldots, x_d))$ where $x_1, \ldots, x_d$ are *free* semi-circular variables.

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- Similar results hold true with many other matrix models (more general iid Wigner matrices, i.i.d random unitary matrices, etc).

# Less trivial bounds

- **Fundamental problem**: can we replace $\liminf_N ||X^N|| \geq XXX$ by $\limsup_N ||X^N|| \leq XXX$?? (i.e. can we get $\lim_N ||X^N|| = XXX$?).
- **Yes** for Wigner (under appropriate assumptions). Cf Füredi Komlós, etc

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- **Yes** for Wigner (under appropriate assumptions). Cf Füredi Komlós, etc
- (big breakthrough – Haagerup-Thorbjørnsen 2005): **Yes** for a NC polynomial $P$ in iid copies $X_1^N, \ldots, X_d^N$ of GUEs.

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- Many generalizations of Haagerup-Thorbjørnsen to more general Wigner setups (Anderson, Capitaine...)
- (Male, Pisier): **Yes** if one adds *tame* constant matrices & tensors to the models.
- (C & Male): Same as above if one replaces iid GUEs by iid unitaries.

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- (C, Bordenave): **Yes** for a polynomial in iid copies of random permutations or involutions without fixed points (acting on mean zero vectors).
- (C, Bordenave – in preparation): **Yes** for finite tensors of random unitaries.

# Operator norm or spectrum convergence?

- **A non obvious fact:** Actually, all the above results actually imply that if $P$ is self-adjoint, the spectrum of $P(X_i^N)$ converges to the spectrum of $P(x_i)$ in the sense of *Hausdorff distance*.

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- The proof of this fact in the case of *GUE*'s or random unitaries is simple modulo a hard result in OA: the limiting $C^*$-algebra has *no non-trival projection*.

# Operator norm or spectrum convergence?

- The proof of this fact in general relies on linearization – namely, understanding the spectrum of any *NC polynomial* in $X_i$ is equivalent to understanding the spectrum of any *linear equation* in $1_N, X_i^N, \ldots$ with matrix coefficients of arbitrary size.

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- The stability under addition, multiplication and conjugation of our families of models is also important (folding trick).

# Strong convergence

- **Definition:** a $d$-tuple of $n \times n$ matrices $X_1^N, \ldots, X_d^N$ *converges strongly* to $x_1, \ldots, x_d \in (A, \tau)$ ($\tau$ faithful tracial state) iff, for any NC $*$-polynomial $w$,

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- All the previously quoted results (HT, M, CM, BC, etc...) can be restated as strong convergence results. All proofs start by proving weak convergence...

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- But surprisingly, quite often, it does.
- It relies on the uniqueness of trace on the limiting object (reduced $C^*$-algebra).
- *Question*: Could this be used directly in RMT...?

# Motivation for our work: strong convergence sometimes fails

- C-Male's results imply $||U_1^N + \ldots + U_d^N|| \to 2\sqrt{d-1}$ (here, $U_i^N$ are iid random unitary matrices).
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  For $d > 2$, this is less than $d$ (the trivial bound).
- *However*, although this estimate holds for $||S_1^N + \ldots + S_d^N||$ (where $S_i^N$ are random permutations) when restricted to mean zero vectors (Friedman, Bordenave), it fails when applied to general vectors (Perron-Frobenius).

# Initial motivation: strong convergence sometimes fails.

- Likewise, $||O_1^N \otimes O_1^N + \ldots + O_d^N \otimes O_d^N||$ has an eigenvalue $d$ ($O_N^i$ are iid random orthogonal) although the collection $O_i^N \otimes O_i^N$ is asymptotically free.

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- However, restricted to the ortogonal subspace of the eigenvector of the eigenvalue $d$ (the Bell state / Jones projection), the operator norm tends to its usual candidate $2\sqrt{d-1}$ (Pisier, Hastings).

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- However, restricted to the ortogonal subspace of the eigenvector of the eigenvalue $d$ (the Bell state / Jones projection), the operator norm tends to its usual candidate $2\sqrt{d-1}$ (Pisier, Hastings).
- **Bottom line**: there is a strong motivation to understand better the norm situation for tensors, and this is hard.

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- Pisier: if one allows a relaxation by a constant, strong convergence holds with coefficient whose matrix dimension satisfies $M \leq \mathit{exponential}(N)$.
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- Pisier: if one allows a relaxation by a constant, strong convergence holds with coefficient whose matrix dimension satisfies $M \leq exponential(N)$.
- Hayes: What happens when $M = N$ and the coefficients are random and independent from $X_i^N$? Remark: the example of $||O_1^N \otimes O_1^N + \ldots + O_d^N \otimes O_d^N||$ shows that $M = N$ can be tricky (although here, the coefficients and the matrices are correlated).

# Main result (arXiv:1912.04588, C, Guionnet, Parraud)

**Ingredients:**

- $X^N = (X_1^N, \dots, X_d^N)$ are iid $GUE$, $x = (x_1, \dots, x_d)$ a system of free semicircular variable,
- $Z^{NM} = (Z_1^{NM}, \dots, Z_q^{NM})$ are deterministic matrices in $M_M \otimes M_N$,
- $P$ is a self-adjoint non-commuting polynomial in $d + 2q$ variables.

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- $P$ is a self-adjoint non-commuting polynomial in $d + 2q$ variables.
- Let $f : \mathbb{R} \mapsto \mathbb{R}$ smooth enough (in a Fourier sense): [Technically, there exists $\mu = \mu(f)$ with $\int (1 + y^4)\ d|\mu|(y)\ < +\infty$ and $f(x) = \int_{\mathbb{R}} e^{ixy}\ d\mu(y)$].

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**Then**: There exists a polynomial $L_P$ which only depends on $P$ such that for any $N, M$,

$$\left| \mathbb{E}\left[ \frac{1}{MN} Tr\Big( f\left( P\left( X^N \otimes I_M, Z^{NM}, Z^{NM^*} \right)\right) \Big) \right] - \right.$$

$$\left. \tau_N \otimes \tau_M \Big( f\left( P\left( x \otimes I_M, Z^{NM}, Z^{NM^*} \right)\right) \Big) \right|$$

$$\leqslant \frac{M^2}{N^2} L_P \left( \left\| Z^{NM} \right\| \right) \int_{\mathbb{R}} (|y| + y^4) \, d|\mu|(y) \, .$$

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**Comments**: The dependence $P \to L_P$ is somewhat explicit. For example it can be made uniform on a compact set of bounded degree.

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- More recently for some multimatrix models with non-backtracking techniques.
- **Analysis** – matrix valued Stieltjes transform + master equation / Schwinger-Dyson. (also: folding trick for the unitary case, integrable systems (TW), etc).
- **Important point**: all multimatrix-type results so far basically rely on linearization.

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- If $f \circ P$ is applied to a random matrix, it counts the random number of eigenvalues in an interval.
- **Idea:** take $f$ with support away from the support of $P(x)$.
- **Problem:** we know how to study $Tr(f \circ P(X_i^N))$ if $f$ is a polynomial (with, e.g. second order freeness) but not if $f$ is an indicator function.

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- **Problem:** how to estimate without combinatorics?
- **Solution:** expand in a "neighborhood of $N = \infty$".
- In practice, work on a free product of matrices $X_i^N$ and free semi-circulars $x_i$. Interpolate between matrices and their limit by writing $X_t^N = e^{-t/2} X_0^N + (1 - e^{-t})^{1/2} x$ (for each $X_i^N$).

# Under the hood

- Recall that $X_t^N = e^{-t/2} X_0^N + (1 - e^{-t})^{1/2} x$, ...and write

$$\mathbb{E}\left[\frac{1}{N} tr_N\Big(Q\left(X^N\right)\Big)\right] - \tau\Big(Q\left(x\right)\Big) = -\int_0^\infty \mathbb{E}\left[\frac{d}{dt}\Big(\tau_N\left(Q(X_t^N)\right)\Big)\right]$$

Then, show that $\frac{d}{dt}\tau_N(Q(X_t^N))$ is small.

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  Then, show that $\frac{d}{dt}\tau_N(Q(X_t^N))$ is small.
- This is to be expected because $X_t^N$ interpolates between our model and the limit (so, a priori, little knowledge about the limit is required).
- This can be done with free and classical stochastic calculus, Schwinger-Dyson type equations and semigroup theory.

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- Parraud: Unitary version of the above result (arXiv:2005.13834). Additive interpolation with a semi-circular system must be replaced by multiplicative interpolation with free Haar unitaries (Voiculescu's liberation).

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- Thank you for your attention!