Localization of the continuous Anderson Hamiltonian in 1-d and its transition towards delocalization

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Works in collaboration with Cyril Labbé

Schrödinger operators

Schrödinger operator in 1-d

For $u : [0, L] \to \mathbb{R}$

$$u \mapsto -u'' + V \cdot u\,.$$

$V : [0, L] \to \mathbb{R}$: potential, self-adjoint operator with Dirichlet boundary conditions.

Models disordered solids in physics where disorder $= V$.

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Discrete analog: tridiagonal matrix

$$\begin{pmatrix} V_1 & 1 & & \\ 1 & V_2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & V_N \end{pmatrix}$$

Case $V = 0$: Laplacian on $[0, L]$ with Dirichlet b.c.

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Eigenvalues $\lambda_1 < \lambda_2 < \cdots$ satisfy:

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And the associated eigenvectors are:

$$x \in [0, L] \mapsto \sin(\frac{\pi k}{L} x).$$

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Eigenvectors are completely **delocalized**!

Continuous Anderson Hamiltonian in 1-d

We choose $V = \xi$: white noise. For $u : [0, L] \to \mathbb{R}$,

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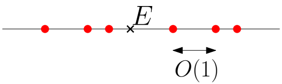
Fukushima, Nakao ('77) proved:

- Well-defined self-adjoint operator,
- discrete simple spectrum bounded from below: $\lambda_1 < \lambda_2 < \cdots$,
- associated eigenvectors $(\varphi_k)_k$ form an orthonormal basis of $L^2([0, L])$ and are $C^{3/2-}$.

Goal

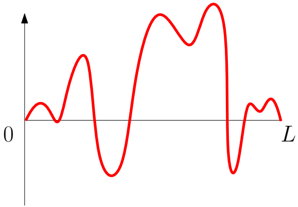
Study the **spectrum** of this operator when $L \to \infty$.

Eigenvalues



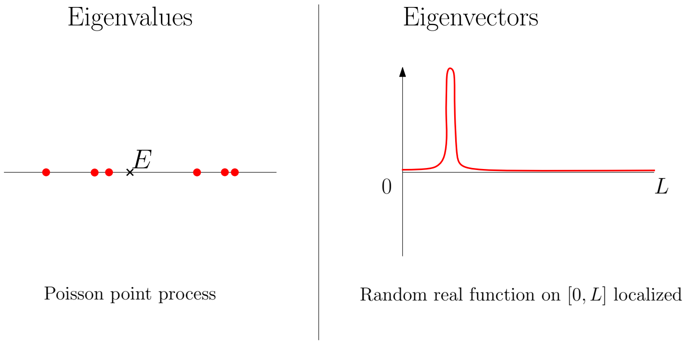
Random point process on the real line

Eigenvectors



Random real function on $[0, L]$

Study the **spectrum** of this operator when $L \to \infty$.



Poisson point process

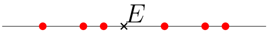
Random real function on $[0, L]$ localized

Usually for random operators, there is a dichotomy:

- Localization of the eigenvectors and Poisson distribution of eigenvalues.

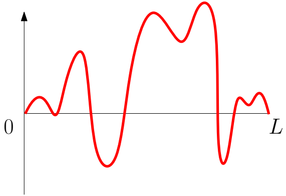
Study the **spectrum** of this operator when $L \to \infty$.

Eigenvalues



Random point process with repulsion

Eigenvectors



Random real function on $[0, L]$ delocalized

Usually for random operators, there is a dichotomy:

- Delocalization of the eigenvectors and repulsion of the eigenvalues.

Previous results on $\mathcal{H}_L$

Density of states

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$$n \ : \ E \mapsto \frac{d}{dE} \lim_{L\to\infty} \frac{1}{L} \#\{\text{eigenvalues } \ \leq E\}.$$

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For the Laplacian, its eigenvalues are:

$$\lambda_k = (\pi k/L)^2.$$

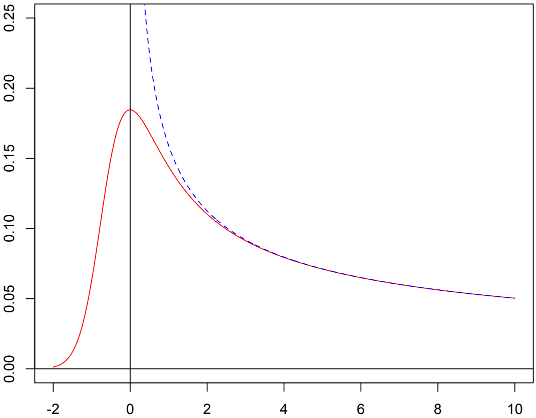
$\rightarrow$ Density of states:

$$E \in \mathbb{R}_+ \mapsto \frac{1}{2\pi\sqrt{E}}.$$

Density of states for $\mathcal{H}_L$

Frisch and Lloyd ('60), Halperin ('65) and then Fukushima, Nakao ('77): Explicit integral formula for the density of states of $\mathcal{H}_L$:

$$n(E) = \frac{d}{dE}\left(\sqrt{2\pi}\int_0^\infty u^{-1/2} e^{-\frac{1}{6}u^3 - 2Eu} du\right)^{-1} .$$



First eigenvalue

McKean ('94) : Convergence of the **smallest eigenvalue** λ_1 (recentred and rescaled) for Dirichlet, Neumann and periodic b.c.:

$$-4\sqrt{a_L}\,(\lambda_1 + a_L) \Rightarrow_{L\to\infty} e^{-e^{-x}}dx,$$

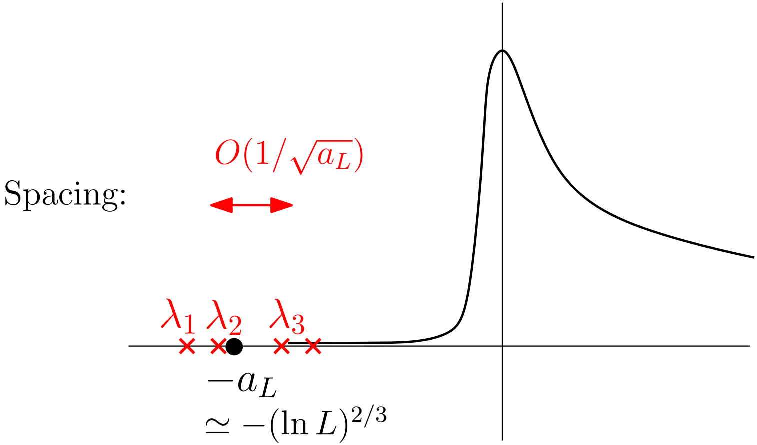
where $a_L \sim \left(\frac{3}{8}\ln L\right)^{2/3}$

OUR RESULTS ON $\mathcal{H}_L$

Localization of the smallest eigenvectors

Recall $a_L \sim \left(\frac{3}{8}\ln L\right)^{2/3}$. Denote by

$$\mathcal{Q}_L := \sum_{k\geq 1} \delta_{4\sqrt{a_L}(\lambda_k + a_L)},$$



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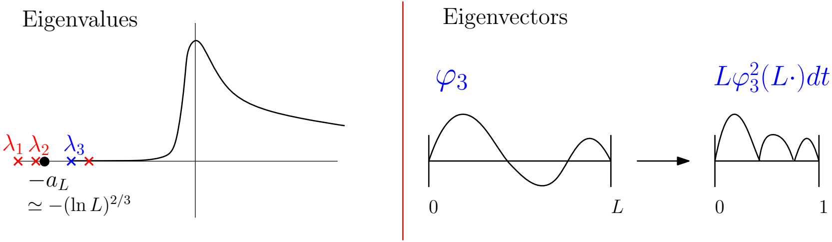

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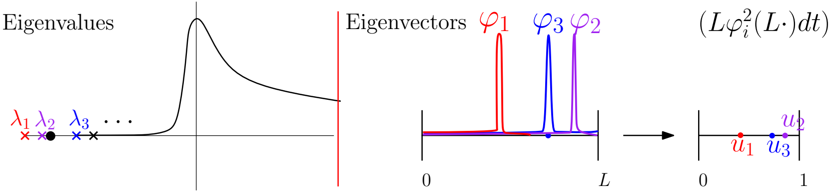
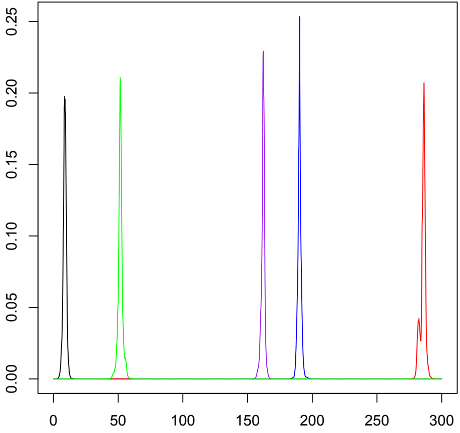
Theorem (D., Labbé ('17))

$(\mathcal{Q}_L, m_{L,k}(dt))$ converges in distribution towards $(\mathcal{Q}_\infty, m_\infty)$ where:

- *$\mathcal{Q}_\infty$: Poisson point process of intensity $e^x\,dx$,*
- *$m_\infty = (\delta_{u_k})_{k\geq 1}$: $(u_k)_{k\geq 1}$ i.i.d, uniform on $[0,1]$, independent of $\mathcal{Q}_\infty$.*

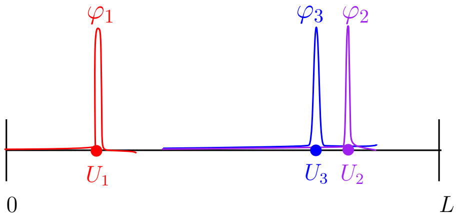


Simulation of the first eigenvectors



The first 5 eigenvectors φ_k^2 in order: black, blue, purple, red, green ($L = 300$).

Shape of the first eigenvectors



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- *For all fixed k,* φ_k *decays exponentially at rate* $\sqrt{a_L}$.

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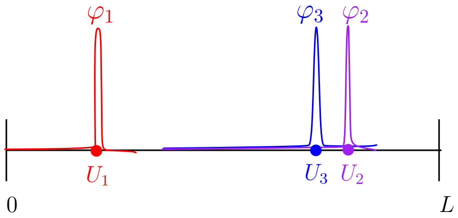


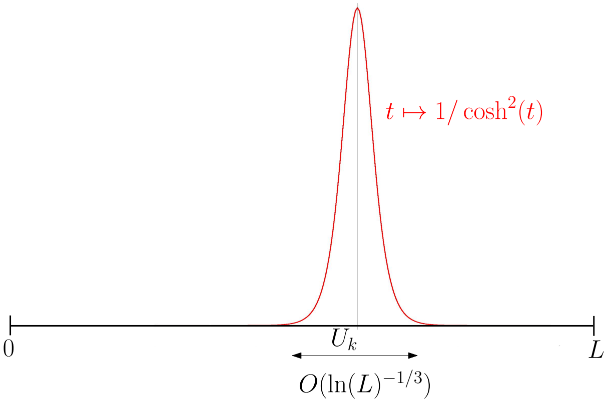
Theorem (D., Labbé ('17))

- *For all fixed k, φ_k decays exponentially at rate $\sqrt{a_L}$.*
- *Let U_k be the point where φ_k reaches its maximum.*

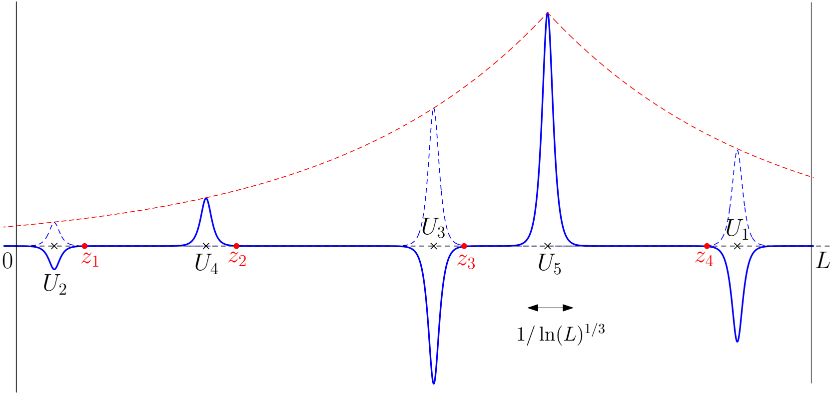
$$h_k(t) := \sqrt{a_L}\,\varphi_k^2(U_k + \sqrt{a_L}\,t) \to_{L\to\infty} 1/\cosh(t)^2$$

uniformly over compact subsets of $\mathbb{R}$.

Zoom around the maximum of φ_k^2

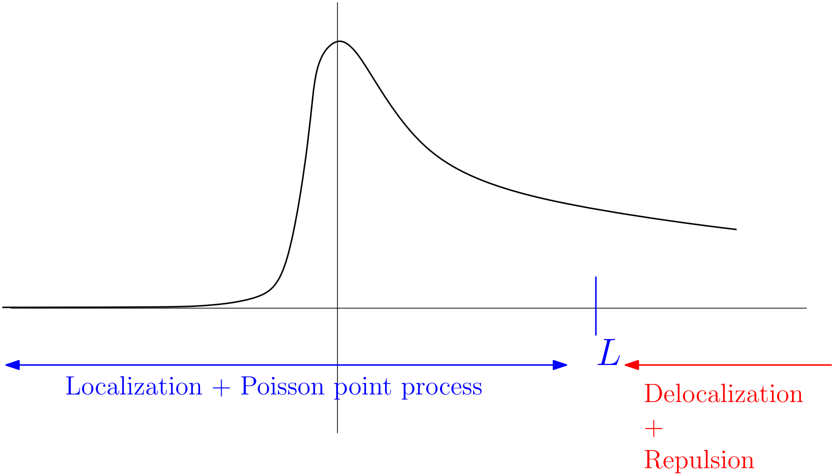


Schematic shape of the fifth eigenvector



Note that we know for example precisely the position of the $k-1$ zeros of φ_k

Localization and transition towards delocalization



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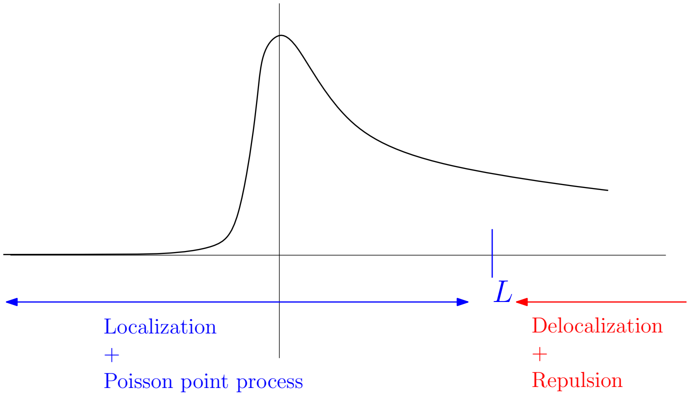
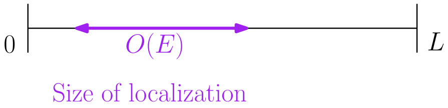


Eigenvector located around E:



Localization for $E \ll L$

Let $E = E(L)$ be the re-centering of the eigenvalues and define

$$\mathcal{Q}_E(dx) := \sum_{i \geq 1} \delta_{L\, n(E)(\lambda_i - E)}(dx)\ , \quad \text{where } n(E) = \text{density of states.}$$

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- ***Bulk regime:*** E ***fixed (independent of*** L***)***
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- ***Crossover regime:*** $1 \ll E \ll L$
  $\mathcal{Q}_E$ converges to a Poisson point process of intensity dx. Eigenvectors: exponentially decreasing at speed $c(E) = O(1/E)$. Moreover, a "typical" eigenvector chosen w.r.t. the "spectral measure" looks like the exponential of a Brownian motion plus a drift on a region of size E.

Transition towards delocalization

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- ***If*** $E \sim \alpha L$ ***with*** $\alpha \in (0,\infty)$ $\mathcal{Q}_E - ((2\sqrt{\alpha}L^{3/2}) \mod 2\pi)$ *converges towards a point process with repulsion between the points. It corresponds to the point process* $Sch_{1/\alpha}$*.*
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  Eigenvectors: exponential of a Brownian motion plus a drift (as conjectured by Rifking and Virág).
- ***If*** *$E \gg L$ then $\mathcal{Q}_E$ converges to the deterministic process of eigenvalues of $-d^2/dx^2$.*

Limiting operator

For this slide, let us define $\mathcal{H}_L$ on $L^2[-L, L]$ instead of $L^2[0, L]$, so that $[-L, L]$ converges to the whole line $\mathbb{R}$. We denote φ_i^L and λ_i^L its eigenvectors and eigenvalues.

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Define $\mathcal{H}f := -f'' + \xi f$ on a domain $\mathcal{D} := \{f \in L^2(\mathbb{R}),\ f \text{ AC},\ f' - Bf \text{ AC},\ \mathcal{H}f \in L^2(\mathbb{R})\}$ of $L^2(\mathbb{R})$. It is a self-adjoint operator, which is limit point at both sides.

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Theorem (D., Labbé ('20+))

The spectral measures $\sum_i (\varphi_i^L(0)^2 + \varphi_i'^L(0)^2)\delta_{\lambda_i^L}$ converge a.s. (for the topology of the vague convergence) towards the spectral measure of $\mathcal{H}$.
This spectral measure is pure point: the operator $\mathcal{H}$ is a ***pure point operator.***

Some ideas for the proofs

Eigenvalue equation

Eigenvalue equation for $\mathcal{H}_L$ defined on $[0, L]$:

$$-\varphi'' + \xi \cdot \varphi = \lambda \varphi$$

with $\varphi(0) = 0$ (without any condition on $\varphi(L)$).

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The couple $(\lambda, \varphi_\lambda)$ is an eigenvalue/eigenvector when

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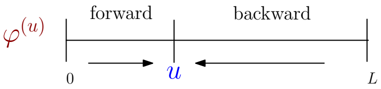
Concatenation forward/backward

Key idea: Use *forward solution* φ_λ on the time-interval $[0, u]$ and then *backward solution* $\hat{\varphi}_\lambda$ on $[u, L]$ for some well-chosen u.
$\to$ Concatenation is $\varphi^{(u)}$.



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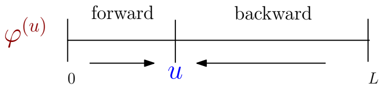

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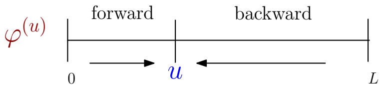

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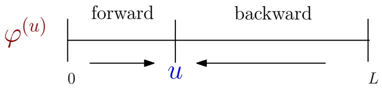

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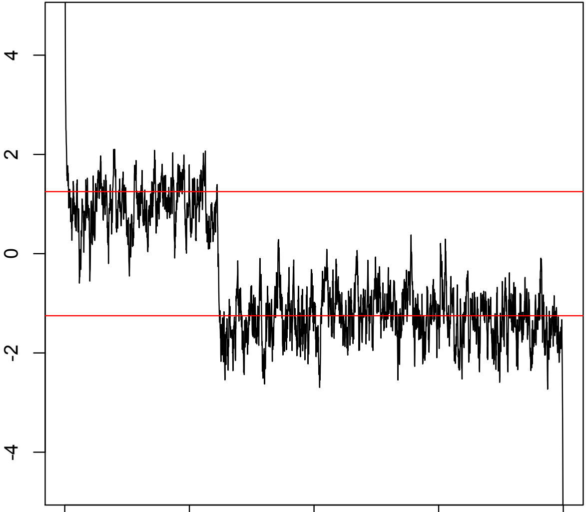
It helps A LOT because it is **much easier** to analyze the forward or backward solution of the ODE than the eigenvalue equation (when λ eigenvalue, λ is random and depends on the whole potential ξ!).

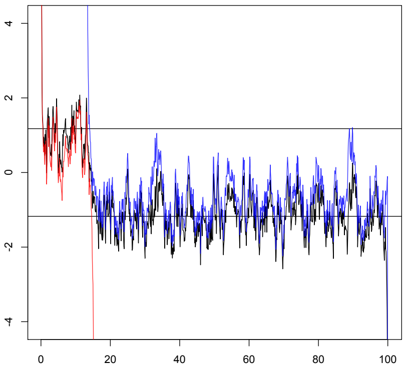
Localization of the first eigenvector

Simulation of $\varphi'_{\lambda_1}/\varphi_{\lambda_1}$:



Localization of the first eigenvector

One can approximate $\varphi'_{\lambda_1}/\varphi_{\lambda_1}$ by $\varphi'_\lambda/\varphi_\lambda$ on $[0,u]$ and then by $\hat\varphi'_\lambda/\hat\varphi_\lambda$ on $[u,L]$ for λ close to λ_1



$$\frac{\varphi_{\lambda_1}(t)}{\varphi_{\lambda_1}(t_0)} = \exp\Big(\int_{t_0}^{t} \frac{\varphi'_{\lambda_1}(s)}{\varphi_{\lambda_1}(s)} ds\Big)$$

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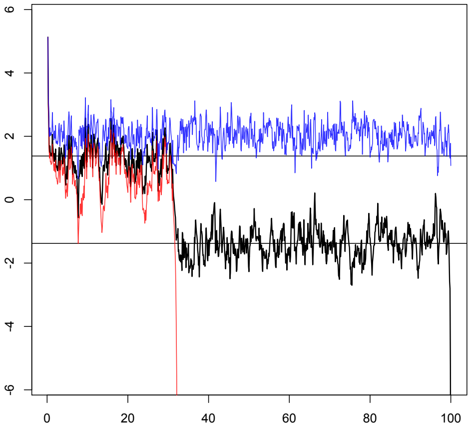

Localization in the bulk: A key formula

Proposition (Goldsheid Molchanov Pastur formula)

For all continuous and bounded G:

$$\mathbb{E}\Big[\sum_{\lambda \text{ eigenvalue}} G(\lambda, \varphi_\lambda)\Big]$$

$$= \int_{\lambda\in\mathbb{R}} \int_0^L \int_0^\pi \sin^2(\theta) p_\lambda(\theta) p_\lambda(\pi-\theta) \mathbb{E}\Big[G\Big(\lambda, \frac{\varphi_\lambda^{(u)}}{||\varphi_\lambda^{(u)}||_2}\Big)\Big] d\lambda du d\theta,$$

where

- $\varphi^{(u)}$ *is the concatenation of the forward process and backward process at time u.*
- $p_\lambda(\theta)$ *transition probability of θ_λ "phase function" (argument of $\varphi'_\lambda + i\varphi_\lambda$).*

Strategy to prove convergence towards a Poisson point process when you know localization

Let $\Delta = [E - h/(Ln(E)), E + h/(Ln(E))]$ (E fixed) and denote $N(\Delta) = \#$ eigenvalues in Δ.

Strategy to prove convergence towards a Poisson point process when you know localization

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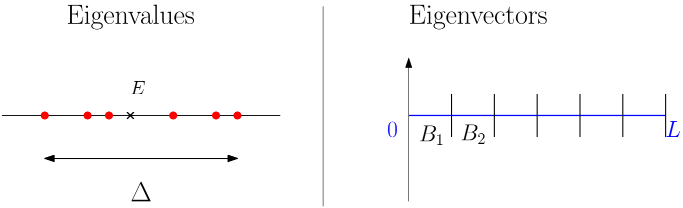

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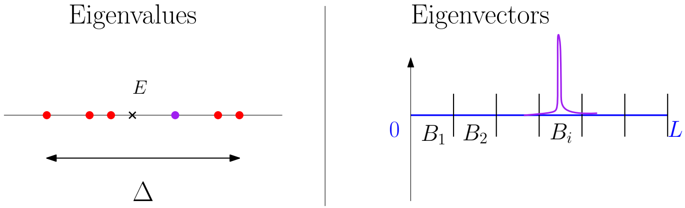

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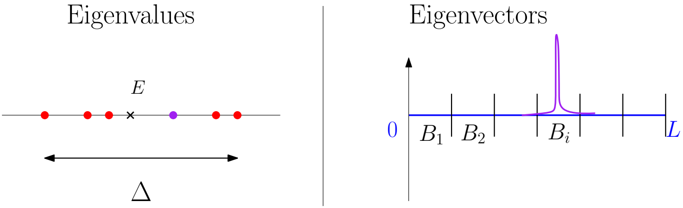

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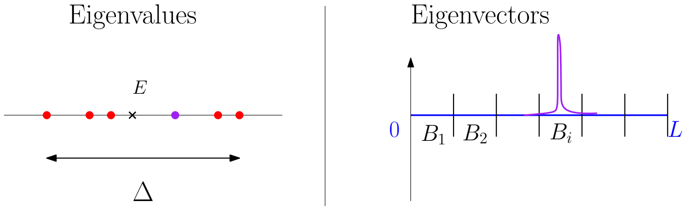

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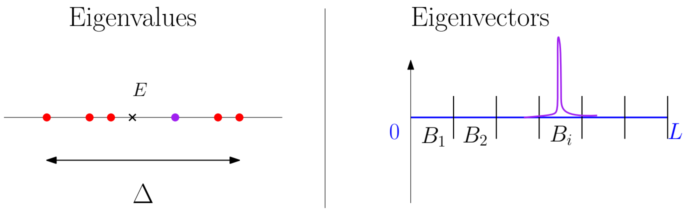

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(C) $\sum_i \mathbb{P}[N_i(\Delta) \geq 2] \rightarrow 0$.

THANK YOU!