Spectra of random regular hypergraphs

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Joint work with Yizhe Zhu

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1 Motivation: Hypergraphs

2 Perspectives on Regular Hypergraphs

3 A Key Bijection

4 Applications: unwrapping of the spectra of regular hypergraphs

5 Conclusions
Hypergraph: \( V = \text{vertex set}, E = \text{edge set} \)
Hypergraphs

Hypergraphs

- $d$-regular: the degree of each vertex is $d$. 
Hypergraphs

- \( H = (V, E) \), \( V \): vertex set, \( E \): hyperedge set.
- \( d \)-regular: the degree of each vertex is \( d \).
- \( k \)-uniform: each hyperedge is of size \( k \).
Motivation: Hypergraphs

Hypergraphs

- $d$-regular: the degree of each vertex is $d$.
- $k$-uniform: each hyperedge is of size $k$.
- $(d, k)$-regular: both $k$-uniform and $d$-regular.
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- $k$-uniform: each hyperedge is of size $k$.
- $(d, k)$-regular: both $k$-uniform and $d$-regular.
- $k = 2$: $d$-regular graphs.
Applications of Hypergraphs

- Introduced by Berge (1970)
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- Model data; recommender systems; pattern recognition, bioinformatics
- As with graphs, one main object of study is expansion (edge, vertex, spectral)
Recall that for regular/biregular bipartite graphs we know...
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(Finite degrees) ESD (Kesten-McKay, “transformed, finite” Marčenko-Pastur (Mojar et. al?))
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Eigenvalue statistics of random (regular) graphs

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- Local laws, eigenvectors, etc.
Two different perspectives: tensors

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- Applications in optimization, etc.
Two different perspectives: adjacency matrix

$A \in \mathbb{Z}^{n \times n}$ Introduced in Feng-Li (1996).

$\lambda_1 = d(k-1)$, since $A \vec{e} = d(k-1) \vec{e}$ with $\vec{e} = (1, \ldots, 1)$.

What about $\lambda_2$?

What about other properties?

Incidentally, general hypergraphs' eigenvalues have been connected to diameters, random walks, Ricci curvature (Banerjee '17).
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**Theorem (Feng-Li 1996)**

Let $G_n$ be any sequence of connected $(d, k)$-regular hypergraphs with $n$ vertices. Then

$$\lambda_2(A_n) \geq k - 2 + 2\sqrt{(d - 1)(k - 1)} - \epsilon_n.$$ 

with $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. 

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A Key Bijection

Bijection between hypergraphs and BBGs

Use a result in McKay (1981) to estimate the probability of seeing a forbidden subgraph in a random sample. Any event $F$ holds whp for random bipartite biregular graphs $\iff F$ holds whp for the uniform measure over $S_1$ $\iff$ corresponding $F'$ holds whp for random regular hypergraphs.
Bijection between hypergraphs and BBGs

- \( S_1 = \{ \text{bipartite biregular graphs without certain subgraphs} \} \) and 
- \( S_2 = \{ (d, k)\text{-regular hypergraphs} \} \).
S₁={bipartite biregular graphs without certain subgraphs} and S₂={(d, k)-regular hypergraphs}.

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Theorem (D.-Zhu 2020)

Let $G_n$ be a random $(d,k)$-regular hypergraphs with $n$ vertices. Then with high probability for any eigenvalue $\lambda \neq d(k-1)$,

$$|\lambda(A_n) - (k-2)| \leq 2\sqrt{(d-1)(k-1)} + \epsilon n$$

with $\epsilon n \to 0$.

A matching upper bound to Feng-Li (1996).


Almost all regular hypergraphs are almost Ramanujan.
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- Uses Brito, D., Harris ('20)
- What does \(\lambda_2\) tell us about \(H\)?
Theorem (D.-Zhu 2020)

Let $H$ be a $(d, k)$-regular hypergraph and $\lambda = \max\{\lambda_2, |\lambda_n|\}$. Then the following holds: for any subsets $V_1, V_2 \subset V$,

$$\left| e(V_1, V_2) - d(k-1)n^{|V_1|\cdot|V_2|} \right| \leq \lambda \sqrt{|V_1| \cdot |V_2|} \left(1 - \frac{|V_1|}{n}\right)\left(1 - \frac{|V_2|}{n}\right).$$

$e(V_1, V_2)$: number of hyperedges between $V_1, V_2$ with multiplicity $|e \cap V_1| \cdot |e \cap V_2|$ for any hyperedge $e$. 
Applications: unwrapping of the spectra of regular hypergraphs

Expander Mixing Lemma

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Non-backtracking Random Walks (NBRWs)

A non-backtracking walk of length $\ell$ in a hypergraph is a sequence $w = (v_0, e_1, v_1, e_2, \ldots, v_{\ell-1}, e_{\ell}, v_\ell)$ such that $v_i \neq v_{i+1}$, $\{v_i, v_{i+1}\} \subset e_i$, and $e_i \neq e_{i+1}$ for $1 \leq i \leq \ell - 1$.

A NBRW of length $\ell$ from $v_0$: a uniformly chosen member of all non-backtracking walks of length $\ell$ starting at $v_0$.

How fast does the NBRW converge to a stationary distribution?

Mixing rate: $\rho(H) := \limsup_{\ell \to \infty} \max_{i, j \in V} \left| \left( \frac{1}{n} \right)_{ij} - \frac{1}{\ell} \right|$. 

Ioana Dumitriu (UCSD)

Regular hypergraphs

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Mixing Rate

Theorem (D.-Zhu 2019)

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\[ \psi(x) := \begin{cases} 
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- NBRWs mix faster than simple random walks.
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Non-Hermitian, complex eigenvalues.

Theorem (D.-Zhu 2020) Let $H$ be a random $(d, k)$-regular hypergraph. Then any eigenvalue $\lambda$ of $B_H$ with $\lambda \neq (d - 1)(k - 1)$ satisfies $|\lambda| \leq \sqrt{(k - 1)(d - 1) + \epsilon n}$ asymptotically almost surely as $n \to \infty$ for some $\epsilon_n \to 0$.

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Empirical Spectral Distributions for Random Regular Hypergraphs
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For $M_n = \frac{A_n-(k-2)}{\sqrt{(d-1)(k-1)}}$:
Empirical Spectral Distributions for Random Regular Hypergraphs

For $M_n = \frac{A_n - (k-2)}{\sqrt{(d-1)(k-1)}}$:

<table>
<thead>
<tr>
<th>$d, k$ constant</th>
<th>$f(x) = \frac{1 + \frac{k-1}{q}}{(1 + \frac{1}{q} - \frac{x}{\sqrt{q}})(1 + \frac{(k-1)^2}{q} + \frac{(k-1)x}{\sqrt{q}})} \frac{1}{\pi} \sqrt{1 - \frac{x^2}{4}}$ with $q = (k-1)(d-1)$. $k = 2$: Kesten-McKay law</th>
</tr>
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<td>$d \to \infty$, $\frac{d}{k} \to \alpha &gt; 0$</td>
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<td>$d = o(n^\epsilon)$ for any $\epsilon &gt; 0$</td>
<td></td>
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Another reason to study RBBGs.

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Two ingredients: cycle counts (via switchings) and spectral gap ($\lambda_2 = O(\sqrt{\lambda_1})$).
Beyond ESDs, growing degrees

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- Another reason to study RBBGs.
- Can examine fluctuations from ESD
- Two ingredients: cycle counts (via switchings) and spectral gap ($\lambda_2 = O(\sqrt{\lambda_1})$).
- A cycle in a hypergraph is a cycle in the RBBG. A non-backtracking cycle in the hypergraph is a non-backtracking cycle in the RBBG.
Cyclically non-backtracking cycles/walks
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  then the connection is through the matrix $XX^T - d_1 I$, not $A$. 

Zhu, '20+: spectral gap ($\lambda_2 = O(\sqrt{d_1})$ for BBGs when $d_1 \geq d_2 = O(n^2/3)$, for a more refined, $(\lambda_2^2 - d_1) = O(\sqrt{d_1(d_2 - 1)})$ when $d_2 = O(1)$.)
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- Enough to calculate fluctuations for $A$ when $d_1/d_2$ bounded in both directions, but if $d_2/d_1 \to 0$, only good enough when $d_2$ constant. More work needed.
Conclusions

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- Reason to study and understand tensors