Statistics of extremes in eigenvalue-counting staircases

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Based on: **YVF** & **P Le Doussal** *arXiv:* 2001.04135 (accepted to PRL)

Unitary $\beta-$ ensembles and interacting fermions :

Consider a unitary $N \times N$ matrix U and denote the corresponding unimodular eigenvalues as $z_j = e^{i\theta_j}$, $j = 1, \ldots, N$, with phases $-\pi < \theta_i \leq \pi$. For any given $\beta > 0$ one can construct the so-called Circular β-Ensemble CUE$_\beta(N)$ in such a way that the expectation of a function $F \equiv F(\theta_1, \ldots, \theta_n)$ is given by

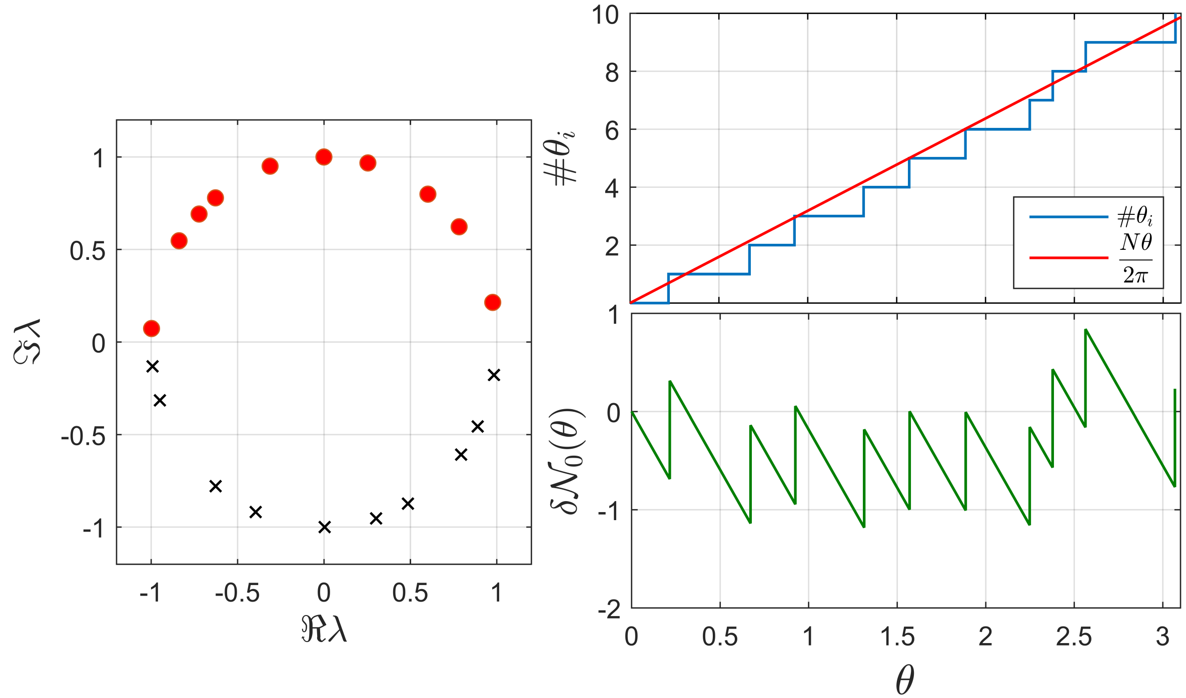
$$\mathbb{E}(F) = c_N \prod_{j=1}^N \int_{-\pi}^{\pi} d\theta_i \prod_{1\leq j<k\leq N} |e^{i\theta_j} - e^{i\theta_k}|^\beta F$$

For $\beta = 2$ such matrices can be thought of as drawn uniformly according to the corresponding Haar's measure on $U(N)$, whereas for a generic $\beta > 0$ the explicit construction is more involved, see **Killip-Nenciu** '04. Such eigenvalues essentially behave as classical particles with 1-d logarithmic repulsion at inverse temperature $\beta > 0$. On the other hand, the r.h.s can be interpreted as the **quantum expectation** value of F in the ground state of N spinless fermions, of coordinates θ_i on the unit circle, described by the **Sutherland** Hamiltonian:

$$H = -\sum_i \frac{\partial^2}{\partial\theta_i^2} + \sum_{i<j} \frac{\beta(\beta-2)}{8\sin^2\left(\frac{\theta_i-\theta_j}{2}\right)}$$

Thus, for $\beta = 2$ the eigenvalues behave as **non-interacting** fermions, while for $\beta \neq 2$ the fermions interact, via an inverse square distance pairwise potential.

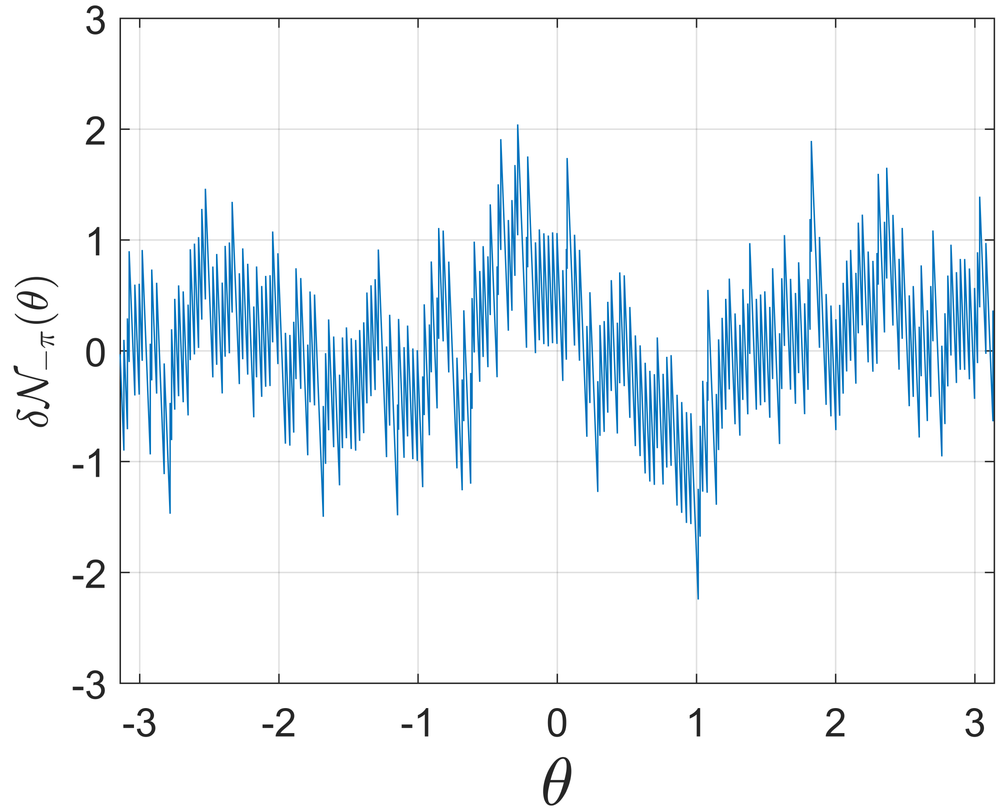
Define the number $\mathcal{N}_{\theta_A}(\theta)$ of eigenvalues $e^{i\theta_j}$ of a random unitary $N \times N$ matrix, drawn from CUE$_\beta(N)$, in the interval $\theta_j \in [\theta_A, \theta]$. As a function of θ this is a staircase with unit jumps upwards at random positions $\theta_j \in [\theta_A, \theta]$. The mean profile is $\mathbb{E}(\mathcal{N}_{\theta_A}(\theta)) = \frac{N(\theta-\theta_A)}{2\pi}$.



Constructing an instance of $\delta N_0(\theta)$ for $\theta \in [0, \pi]$ for $\beta = 2$ and $N = 20$. Left: eigenvalues $\lambda = e^{i\theta_i}$. Right: counting staircase (top), with mean subtracted (bottom).

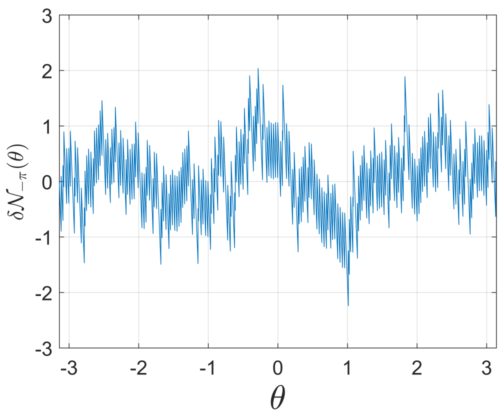
Staircase-deviation process:

In a given random matrix realization/sample one can define the deviation to the mean, $\delta\mathcal{N}_{\theta_A}(\theta) = \mathcal{N}_{\theta_A}(\theta) - \mathbb{E}(\mathcal{N}_{\theta_A}(\theta))$, and study it as a random process as a function of variable θ for a fixed θ_A.



A single realization of $\delta N_{-\pi}(\theta)$ for the full circle $\theta \in [-\pi, \pi]$ for $\beta = 2$ and $N = 200$.

Staircase-deviation process, non-local properties:



Outstanding: **Kolmogorov-Smirnov**-type statistics

$$\max_{\theta\in[\theta_A,\theta_B]} \left|\delta\mathcal{N}_{\theta_A}(\theta) := \mathcal{N}_{\theta_A}(\theta) - \mathbb{E}(\mathcal{N}_{\theta_A}(\theta))\right|$$

Some recent results for $\beta = 2$ in: **Clayes et al** '19.

We are able to shed some light on 'half' of the problem by calculating **cumulants** of the distribution of the **one-sided maximum value** for the process $\delta\mathcal{N}_{\theta_A}(\theta)$, i.e

$$\delta\mathcal{N}_m := \max_{\theta\in[\theta_A,\theta_B]} \left[\delta\mathcal{N}_{\theta_A}(\theta)\right]$$

for any $\beta > 0$ and $N \gg 1$ at fixed $\ell = |\theta_A - \theta_B|$.

Note: If we change above **max** $\Longrightarrow$ **min**, the distribution remains the same, but two extremal values are expected to be highly **correlated**, cf. **Cao-Le Doussal** '16

We also can characterize the **location** of the maximum:

$$\theta_m := argmax_{\in[\theta_A,\theta_B]} \left[\delta\mathcal{N}_{\theta_A}(\theta)\right] \quad .$$

Relation to 'log-correlated' processes:

Naively, the process $\delta N_{\theta_A}(\theta)$ is given by the difference

$$\delta\mathcal{N}_{\theta_A}(\theta) = \frac{1}{\pi}\mathrm{Im}\log\xi_N(\theta) - \frac{1}{\pi}\mathrm{Im}\log\xi_N(\theta_A)$$

where $\xi_N(\theta)$ is the **characteristic polynomial** defined as $\xi_N(\theta) = \det(1 - e^{-i\theta}U)$.

As has been shown in **Hughes-Keating-O'Connell** '01 for $\beta = 2$ (see **Chhaibi-Madaule-Najnudel** '18 for general $\beta > 0$) the joint probability density of $\mathrm{Im}\log\xi_N(\theta)$ at any fixed distinct points $\theta_1 \neq \theta_2 \neq \ldots \neq \theta_k$ converges (in a suitable sense) as $N \to \infty$ to that of a mean-zero Gaussian process $W_\beta(\theta)$ with the covariance structure

$$\mathbb{E}(W_\beta(\theta_1)W_\beta(\theta_2)) = -\frac{1}{2\beta}\log\left[4\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)\right]$$

which is a particular instance of the 1D **log-correlated** Gaussian field. A large but finite N provides a natural small-scale **regularization**, augmenting $W_\beta(\theta)$ with the finite variance: $\mathbb{E}(W_{N,\beta}(\theta)^2) = \beta^{-1}\log N + O(1)$.

Note that $\delta\mathcal{N}_{\theta_A}(\theta = \theta_A) = 0$ in any realization, hence the relevant object is the **pinned** log-process closely related to fBm0. We shall however see that naively replacing $\delta\mathcal{N}_{\theta_A}(\theta) \to \frac{1}{\pi}[W_{N,\beta}(\theta) - W_{N,\beta}(\theta_A)]$ (the procedure closely related to the **bosonization** approach to fermionic problems) is not sufficient for characterizing the maximum of the process: it misses local 'fermionic' contributions.

Summary of the main results:

We predict that for any interval of the **fixed** length $\ell = \theta_B - \theta_A$ the mean value of the maximum deviation $\delta\mathcal{N}_m = \max_{\theta \in [\theta_A, \theta_B]} \left[\delta\mathcal{N}_{\theta_A}(\theta)\right]$ should exhibit, for $N \to \infty$, the universal behavior of the **log-correlated processes**

$$2\pi\sqrt{\tfrac{\beta}{2}}\mathbb{E}(\delta\mathcal{N}_m) \simeq 2\log N - \tfrac{\mathbf{3}}{2}\log\log N + c_\ell^{(\beta)}$$

where $c_\ell^{(\beta)} = O(1)$ is an unknown ℓ-dependent constant.
The variance for the maximum $\delta\mathcal{N}_m$ exhibits to the leading order the extensive universal logarithmic growth typical for **pinned** log-correlated field, on top of which we can evaluate the corrections of the order of unity:

$$\mathbb{E}^c(\delta\mathcal{N}_m^2) \simeq \tfrac{2}{\beta(2\pi)^2}(2\log N + \tilde{C}_2^{(\beta)} + C_2(\ell))$$

Finally, the higher cumulants converge to a finite limit as $N \to \infty$:

$$\mathbb{E}^c(\delta\mathcal{N}_m^k) \simeq \tfrac{2^{k/2}}{\beta^{k/2}(2\pi)^k}(\tilde{C}_k^{(\beta)} + C_k(\ell)),$$

The constants $\tilde{C}_k^{(\beta)}$ depend on β but not on ℓ, and reflect **local fermion number** statistics, whereas $C_k(\ell)$ depend on the length ℓ, reflect **log-correlated** statistics and are known only for the **full circle** $\ell = 2\pi$ and **mesoscopic** $\Delta \ll \ell \ll 2\pi$.

Local ‘fermionic’ contribution to max-cumulants:

All odd cumulants $\tilde{C}^{(\beta)}_{2p+1}$ vanish. All even cumulants for any $\beta > 0$ can be expressed in terms of functions $\psi^{(k)}(x) = \frac{d^{k+1}}{dx^{k+1}} \log \Gamma(x)$ as convergent infinite “dual” series. For the lowest cumulant we obtain:

$$\begin{aligned}\tilde{C}^{(\beta)}_2 &= 2\gamma_E + 2\sum_{k=0}^{\infty}[\tfrac{\beta}{2}\psi^{(1)}(1+\tfrac{\beta k}{2}) - \tfrac{1}{1+k}] \\ &= 2\gamma_E + 2\log\tfrac{\beta}{2} + 2\sum_{m=1}^{\infty}(\tfrac{2}{\beta}\psi^{(1)}(\tfrac{2m}{\beta}) - \tfrac{1}{m})\end{aligned}$$

where $\gamma_E = \lim_{n\to\infty}\left(-\ln n + \sum_{k=1}^{n}\frac{1}{k}\right)$ is the Euler-Mascheroni constant.

For higher even cumulants $\tilde{C}^{(\beta)}_{2p}$ with $p \geq 2$ we find

$$\tilde{C}^{(\beta)}_{2p} = (-2)^{1-p}\beta^p\sum_{k=0}^{\infty}\psi^{(2p-1)}(1+\tfrac{\beta k}{2}) = (-2)^{p+1}\tfrac{1}{\beta^p}\sum_{m=1}^{\infty}\psi^{(2p-1)}(\tfrac{2m}{\beta})$$

For **rational** $\beta = 2s/r$, with s, r mutually prime and $k \geq 2$ we have an alternative representation: $\tilde{C}^{(\beta)}_k = \frac{d^k}{dt^k}|_{t=0}\log\left(A_\beta(t)A_\beta(-t)\right)$, with

$$A_\beta(t) = r^{-t^2/2}\prod_{\nu=0}^{r-1}\prod_{p=0}^{s-1}\frac{G\left(1-\frac{p}{s}+\frac{\nu+it\sqrt{2/\beta}}{r}\right)}{G\left(1-\frac{p}{s}+\frac{\nu}{r}\right)}$$

These expressions are intimately related to **fermion number** statistics.

Global 'log-correlated' contributions to max-cumulants:

By contrast the constants $C_k(\ell)$ are β-independent and can be directly related to the maximum of Fractional Brownian Motion with **vanishing Hurst exponent** (fBm0) ('**pinned**' at one end for the interval or at both ends in '**bridge**' version for the full circle) studied in **YVF-Le Doussal** '16 and **Cao-YVF-Le Doussal** '18. We find:

(i) maximum over the full circle $\ell = 2\pi$. In that case we find:

$$C_{k\geq 2}(2\pi) = (-1)^k \frac{d^k}{dt^k}|_{t=0} \log\left[\frac{\Gamma(1+t)^2 G(2-2t)}{G(2-t)^3 G(2+t)}\right]$$

(ii) maximum over a mesoscopic interval $\Delta \ll \ell \ll 1$, where we obtain

$$C_k(\ell) \simeq 2\log\ell\,\delta_{k,2} + (-1)^k \frac{d^k}{dt^k}|_{t=0}\left[\frac{2\Gamma(1+t)^2 G(2-2t)}{G(2+t)^2 G(2-t) G(4-t)}\right]$$

Note that $\ell \to 0$ limit is expected to provide the $L \gg 1$ asymptotic for statistics of the maximum of $\delta\mathcal{N}_{\theta_A}(\theta)$ in intervals of the order $L\Delta$, comparable with the mean eigenvalue spacing. In particular, our mean $\mathbb{E}(\delta\mathcal{N}_m)$ agrees with one found in **Holcomb-Paquette**'18 for the large-L asymptotics of *Sine*$_\beta$ process.

Distribution of the absolute maximum via Statistical Mechanics approach:

Given a **random sequence** $\{V_i,\, i = 1, \ldots, M\}$ we are interested in finding the distribution of $V_{(m)} = \max(V_1, \ldots, V_M)$ that is

$$P(v) = \text{Prob}(V_{(m)} < v) = \text{Prob}(V_i < v,\, \forall i) = \mathbb{E}\left\{\prod_{i=1}^{M} \chi(v - V_i)\right\}$$

where $\chi(v) = \begin{cases} 1\,, & v > 0 \\ 0\,, & v < 0 \end{cases}$ is the indicator function.

Next we use: $$\lim_{b\to\infty} \exp\left[-e^{-b(v-V_i)}\right] = \begin{cases} 1 & v > V_i \\ 0 & v < V_i \end{cases} \equiv \chi(v - V_i)$$

which immediately shows that:

$$P(v) = \text{Prob}(V_{(m)} < v) = \lim_{b\to\infty} \mathbb{E}\left\{\exp\left[-e^{-bv} Z(b)\right]\right\}$$

where $Z(b) = \sum_{i=1} e^{bV_i}$ is a kind of **Partition Function** associated with the problem, with $b = 1/T$ playing the role of **inverse temperature**.

For a **random process** $V(\theta)$, $\theta \in I$ similar method works with $Z(b) = \int_I e^{bV(\theta)}\, d\theta$.

Sketch of the Calculation I:

Method: introduce the following "partition sum":

$$Z(b) = \frac{N}{2\pi} \int_{\theta_A}^{\theta_B} d\phi \, e^{2\pi b \sqrt{\beta/2}\, \delta\mathcal{N}_{\theta_A}(\phi)},$$

thus mapping the search of the maximum to a **statistical mechanics** problem, with the "**inverse temperature**" equal to $-2\pi b\sqrt{\beta/2}$. The **maximum** is retrieved from the "**free energy**" $\mathcal{F}$ for $b \to +\infty$ as

$$\delta\mathcal{N}_m = \lim_{b\to+\infty} \mathcal{F} \quad , \quad \mathcal{F} = \frac{1}{2\pi b\sqrt{\beta/2}} \log Z(b)$$

To study the statistics of the associated free energy we start with considering the integer moments of $Z(b)$. Using the representation of the counting function given by

$$\mathcal{N}_{\theta_A}(\theta) = \sum_{j=1}^{N} \left(\chi(\theta-\theta_j) - \chi(\theta_A - \theta_j)\right) \ , \ \chi(u) = \begin{cases} 1 \ , \ u > 0 \\ 0 \ , \ u < 0 \end{cases}$$

we get $\mathbb{E}[Z^n(b)] = \left(\frac{N}{2\pi}\right)^n \int_{\theta_A}^{\theta_B} e^{-b\sqrt{\beta/2}\sum_{a=1}^{n} N(\phi_a - \theta_A)}\, \mathbb{E}[\prod_{j=1}^{N} g(\theta_j)] \prod_{a=1}^{n} d\phi_a$

where we defined

$$\log g(\theta) = 2\pi b\sqrt{\beta/2} \sum_{a=1}^{n} (\chi(\phi_a - \theta) - \chi(\theta_A - \theta))$$

Sketch of the Calculation II:

For $\beta = 2$

$$\mathbb{E}[\prod_{j=1}^{N} g(\theta_j)] = \det_{1\le j,k\le N}[g_{j-k}]$$ - **Toeplitz determinant**

where $g_p = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-ip\theta} g(\theta)$ is the associated symbol, and $g(\theta)$ has n **jump singularities**. The corresponding asymptotics as $N \to \infty$ is given by the famous **Fisher-Hartwig** formula proved rigorously by **Deift-Its-Krasovsky** '09 -'11. For a general rational β the extension of **FH** formula has been conjectured by **Forrester** $\&$ **Frenkel** '04. Applying their formula to our case gives for $N \to +\infty$ and $nb^2 < 1$:

$$\mathbb{E}[Z^n(b)] \simeq \left(\frac{N}{2\pi}\right)^n N^{b^2(n+n^2)} |A_\beta(b)|^{2n} |A_\beta(bn)|^2 \, I_C(n, b^2)$$

where $I_C(n, b^2)$ is the so-called '**Coulomb integral**' given by

$$I_C := \int_{\theta_A}^{\theta_B} \prod_{1\le a<c\le n} |1 - e^{i(\phi_a - \phi_c)}|^{-2b^2} \prod_{1\le a\le n} |1 - e^{i(\phi_a - \theta_A)}|^{2nb^2} \prod_{a=1}^{n} d\phi_a$$

Note: Had we used instead in our calculation an approximation replacing the difference $\delta\mathcal{N}_{\theta_A}(\theta)$ in the large$-N$ limit with the logarithmically correlated Gaussian process $W_\beta(\theta)$ we would reproduce the Coulomb integral factor but completely missed the factors $A_\beta(b)$. Hence, this product encapsulates the residual non-Gaussianity of the process due to microscopic 'descrete' fermionic nature.

Sketch of the Calculation III:

Further progress is possible in the two cases (**full circle** case and **mesoscopic** interval case) when the Coulomb integrals $I_C(n, b^2)$ can be explicitly calculated by reducing them to **Selberg integrals**, for **integer** n and $nb^2 < 1$. The same integrals appeared in the problem of maximum in logarithmically correlated fBm0 (Hurst index $H = 0$) **Cao-YVF-Le Doussal**'18.

Combining those results with analytical continuation of factors $|A_\beta(b)|^{2n}|A_\beta(bn)|^2$ we find that appropriately continued partition function moments depend on b in the whole high-temperature phase $b < 1$ only via the combination $Q = b + \frac{1}{b}$, hence satisfy the conditions of the **Freezing-Duality Conjecture** (FDC).

This allows to perform $b \to \infty$ limit arriving at a generation function for cumulants for probabilty density of the maximum value, e.g. for the full circle:

$$\mathbb{E}(e^{-2\pi\sqrt{\frac{\beta}{2}}\delta\mathcal{N}_m t}) \simeq N^{-2t+t^2}\, e^{(\frac{3}{2}\ln\ln N + c)t} A_\beta(t)A_\beta(-t)\, \frac{\Gamma(1+t)^2 G(2-2t)}{G(2-t)^3 G(2+t)}$$

Finally, addressing the question of the **location** of the maximum $\theta_m \in [\theta_A, \theta_B]$ of $\delta\mathcal{N}_{\theta_A}(\theta)$, let us define $y_m = (\theta_m - \theta_A)/\ell$. For the mesoscopic interval, we predict the PDF of y_m to be symmetric around $\frac{1}{2}$, with $\mathbb{E}(y_m^2) = \frac{17}{50}$ and $\mathbb{E}(y_m^4) = \frac{311}{1470}$, thus deviating from the uniform distribution.