Eigenvalues for sums of self-adjoint and skew-self-adjoint random matrices

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Credits and references

- Joint work with Ching Wei Ho (Indiana University)
- Builds on work of Ho and Zhong [arXiv:1908.08150]
- We use PDE method developed by Driver–Hall–Kemp [arXiv:1903.11015]
- Some of these results were previously obtained by Jarosz and Nowak by a different method [*J. Phys. A* 2006 and arXiv:math-ph/0402057]

Part 1: The random matrix problem

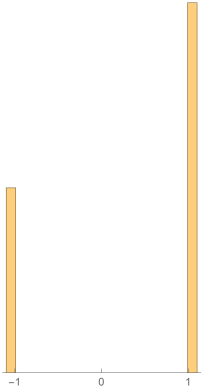
Sum of Hermitian random matrix imaginary multiple of GUE

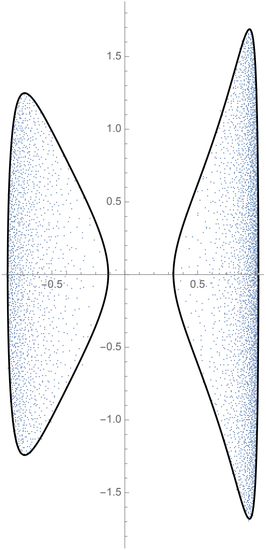
- Take S to be GUE (Gaussian-distributed Hermitian random matrix)
- Take X_0 Hermitian and independent of S
- Can take X_0 to be diagonal with fixed eigenvalue distribution
- Consider

$$Z := X_0 + i\sqrt{t}S, \quad t > 0.$$

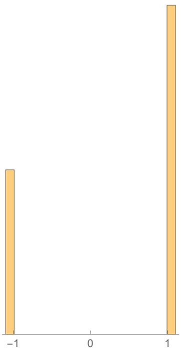
- Study eigenvalue distribution of Z in $\mathbb{C}$

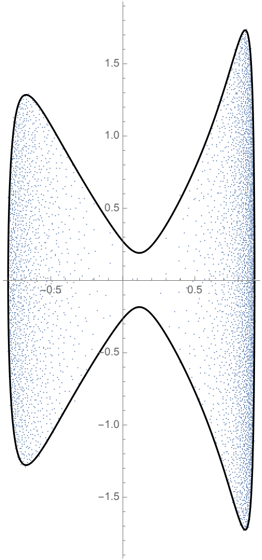
Example: Bernoulli case, $t = 1$





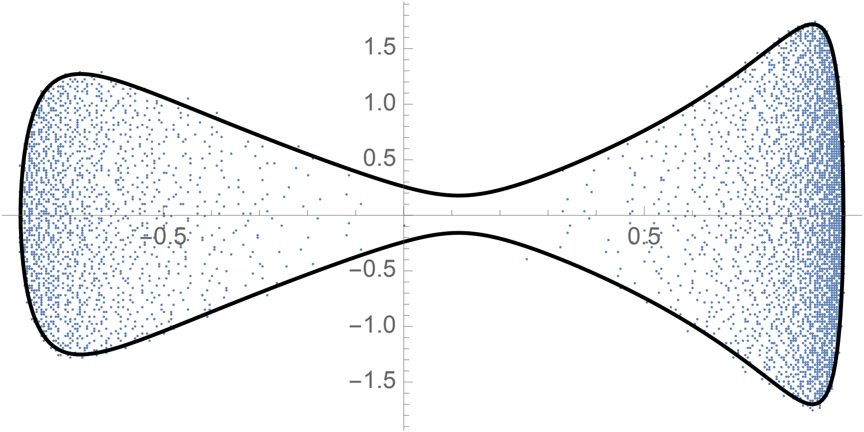
Bernoulli case, $t = 1.05$



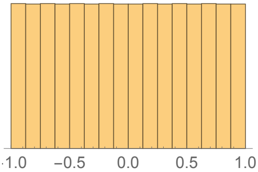


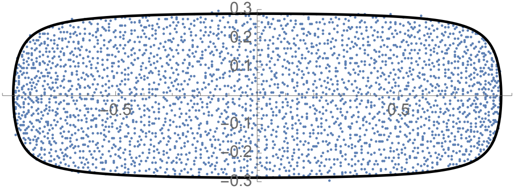
Bernoulli with $t = 1.05$, stretched

- Density of eigenvalues is *constant in the vertical direction* inside the domain

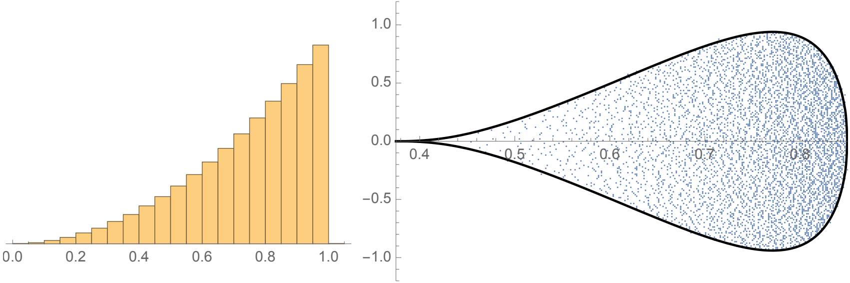


Uniform case, $t = 0.1$





Quadratic case, $t = 1/4$



Part 2: Girko's method and the Brown measure

Girko's formula

- Girko [1984] worked on general circular law: entries are i.i.d. but not Gaussian
- Matrix is not Hermitian (or normal); method of moments is not applicable
- No explicit formulas as in Gaussian case
- How to compute eigenvalues in this case?

Girko's formula

- For matrix A, define

$$S(\lambda) = \frac{1}{N}\mathrm{Trace}[\log((A-\lambda)^*(A-\lambda))]$$

- Then

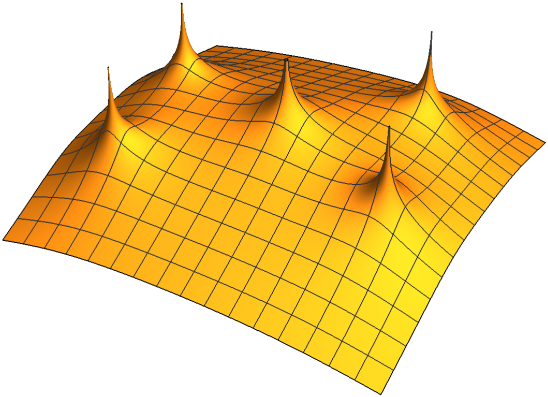
$$S(\lambda) = \frac{2}{N}\sum_{j=1}^{N}\log|\lambda - \lambda_j|$$

- This function is harmonic except at $\lambda = \lambda_j$ and

$$\frac{1}{4\pi}\Delta S = \frac{1}{N}\sum_{j=1}^{N}\delta_{\lambda_j}$$

The function $S(\lambda)$

- Plot of $-S(\lambda)$ with 5 eigenvalues



Operator algebra formalism

- For random matrix X^N, define $*$-**moments** as

$$\mathbb{E}\left\{\frac{1}{N}\mathrm{Trace}[\text{word in } X^N \text{ and } (X^N)^*]\right\}$$

- Take large-N limit of these $*$-moments
- Seek operator algebra $(\mathcal{A}, \tau)$ with trace $\tau : \mathcal{A} \to \mathbb{C}$ AND operator $a \in \mathcal{A}$ such that

$$\lim_{N\to\infty} \mathbb{E}\left\{\frac{1}{N}\mathrm{Trace}[\text{word in } X^N \text{ and } (X^N)^*]\right\}$$
$$= \tau[\text{word in } a \text{ and } a^*]$$

Operator algebra formalism

- a not an operator-valued random variable—just *one fixed* operator in $\mathcal{A}$
- Limits of this sort constructed using methods of free probability

Definition of Brown measure [Brown 1986]

- Given $a \in (\mathcal{A}, \tau)$, want something like "empirical eigenvalue distribution"
- Don't assume a is normal, so can't use spectral theorem
- Imitate Girko's formula!
- Define

$$S(\lambda) = \tau[\log((a-\lambda)^*(a-\lambda))]$$

  then take Laplacian

Definition of Brown measure

- To be sure S is well defined, introduce regularization $\varepsilon > 0$:

$$S(\lambda, \varepsilon) = \tau[\log((a-\lambda)^*(a-\lambda)+\varepsilon)]$$

- Then

$$s(\lambda) := \lim_{\varepsilon \to 0} S(\lambda, \varepsilon)$$

  exists as a subharmonic function
- Then **Brown measure** $\mathrm{Brown}(a)$ defined as

$$\mathrm{Brown}(a) = \frac{1}{4\pi}\Delta s$$

- Then $\mathrm{Brown}(a)$ is probability measure supported on spectrum of a

Part 3: The PDE method

The goal: Brown measure for the sum

- Large-N limit of $\sqrt{t}S$: "semicircular Brownian motion" σ_t
- Large-N limit of X_0: s.a. element x_0 that is "freely independent" of σ_t
- Let μ = "law of x_0" = large-N limit of eigenvalue distribution of X_0

Problem

Compute Brown measure of $x_0 + i\sigma_t$ (in terms of the law μ of x_0).

- We **believe** that $\mathrm{Brown}(x_0 + i\sigma_t)$ is large-N limit of eigenvalue distribution of $X_0 + i\sqrt{t}S$

Work of Biane, Ho–Zhong, and Jarosz–Nowak

- Biane: computed the distribution of $x_0 + \sigma_t$ (without the "i")
- If $x_0 = y_0 + \tilde{\sigma}_t$ for another semicircular Brownian motion $\tilde{\sigma}_t$, results of Ho and Zhong apply
- Jarosz and Nowak have general algorithm for computing Brown measures for $x + iy$;
- Their results agree with ours in case of $x_0 + i\sigma_t$, but we get additional information

The PDE method

- Apply definition of Brown measure to $x_0 + i\sigma_t$
- Define

$$S(t, \lambda, \varepsilon) = \tau[\log((x_0 + i\sigma_t - \lambda)^*(x_0 + i\sigma_t - \lambda) + \varepsilon)]$$

Theorem

The function S satisfies the PDE

$$\frac{\partial S}{\partial t} = \varepsilon \left(\frac{\partial S}{\partial \varepsilon}\right)^2 + \frac{1}{4}\left(\left(\frac{\partial S}{\partial a}\right)^2 - \left(\frac{\partial S}{\partial b}\right)^2\right), \quad \lambda = a + ib$$

with initial condition

$$S(0, \lambda, \varepsilon) = \tau[\log(x_0^* x_0 + \varepsilon)].$$

The PDE method

- ε is a *variable* in the PDE
- Want to solve PDE, *then* evaluate at $\varepsilon = 0$
- Take Δ with respect to λ of $S(t, \lambda, 0)$

The Hamilton–Jacobi method

- Define "Hamiltonian" function H from PDE, replacing derivatives on RHS with "momenta" (and overall sign change):

$$H(a, b, \varepsilon, p_a, p_b, p_\varepsilon) = -\varepsilon p_\varepsilon^2 - \frac{1}{4}(p_a^2 - p_b^2)$$

- Then introduce auxiliary system of ODE's (Hamilton's equations):

$$\frac{du}{dt} = \frac{\partial H}{\partial p_u}; \quad \frac{dp_u}{dt} = -\frac{\partial H}{\partial u}; \quad u \in \{a, b, \varepsilon\}.$$

- These ODE's can be solved explicitly

The Hamilton–Jacobi formula

- **Hamilton–Jacobi formula**:

$$S(t, \lambda(t), \varepsilon(t)) = S(0, \lambda_0, \varepsilon_0) + tH_0$$

where

$$H_0 = H(a_0, b_0, \varepsilon_0, p_{a,0}, p_{b,0}, p_{\varepsilon,0})$$

- Initial momenta are

$$p_{u,0} = \frac{\partial}{\partial u} S(0, \lambda_0, \varepsilon_0).$$

- Get S along solutions to ODE's—in terms of λ_0 and ε_0

Part 4: Where the Brown measure is zero

Simple strategy for applying Hamilton–Jacobi method

- H-J method tells us $S(t, \lambda(t), \varepsilon(t))$
- To get $S(t, \lambda, 0)$, choose $(\lambda_0, \varepsilon_0)$ so that $\lambda(t) = \lambda$ and $\varepsilon(t) = 0$
- **Simple strategy**: to get $\varepsilon(t) \approx 0$, try $\varepsilon_0 \approx 0$

Small-ϵ_0 existence time

- **Problem**: If $\varepsilon_0 \approx 0$, solution may cease to exist before time t
- Define

$$T(\lambda_0) = \text{existence time of solution with } \lambda(0) = \lambda_0,\ \varepsilon(0) \approx 0$$

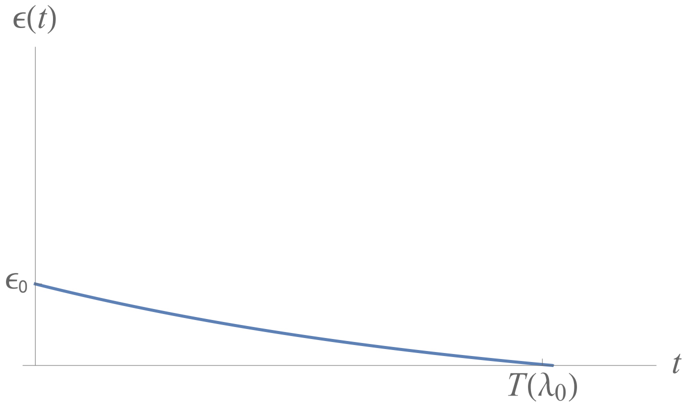
- Explicitly,

$$T(a_0 + ib_0) = \left(\int_{\mathbb{R}} \frac{1}{(a_0 - x)^2 + b_0^2} \, d\mu(x) \right)^{-1},$$

where μ is the distribution of x_0

Small-ϵ_0 existence time

- Taking $\varepsilon_0 \approx 0$ only allowed if $T(\lambda_0) > t$



Results of simple strategy

- If $T(\lambda_0) > t$, taking $\varepsilon_0 \approx 0$ works
- Find that $\lambda(t)$ is a holomorphic function of λ_0
- Find that $S(t, \lambda, 0)$ is harmonic
- Conclude that Brown measure is zero

Good Sets

- "Good Set" in λ_0-plane

$$\{\lambda_0 \in \mathbb{C} |\, T(\lambda_0) > t\}$$

[This set also appears in work of Biane and Ho–Zhong]

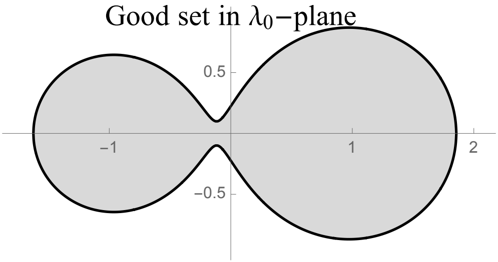
- "Good Set" in λ-plane

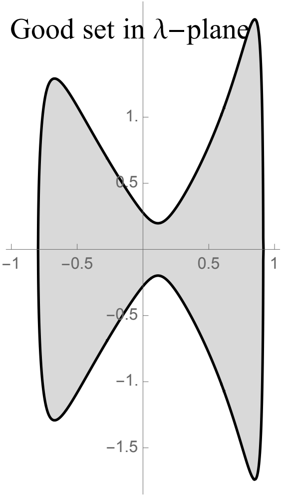
$$\{\lambda(t) |\, \lambda_0 \in \text{Good Set}\},$$

Theorem

The Brown measure of $x_0 + i\sigma_t$ is zero in Good Set in λ-plane.

Good Sets in Bernoulli case





Conformal map

- With λ_0 in Good Set and $\varepsilon_0 = 0$, get

$$\lambda(t) = J_t(\lambda_0)$$

where

$$J_t(\lambda_0) = \lambda_0 - t \int \frac{1}{\lambda_0 - x} \, d\mu(x)$$

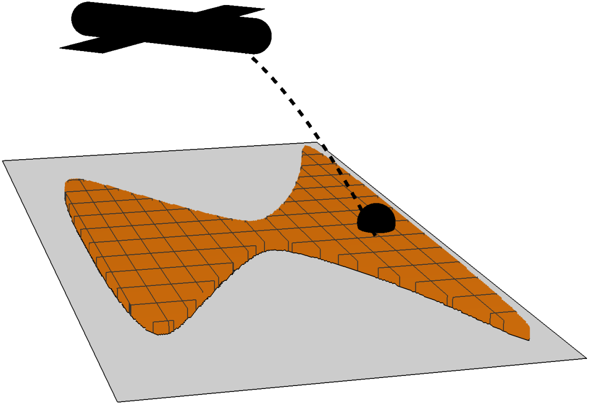
and $\mu = \mathrm{Law}(x_0)$

- J_t is a conformal map of Good Set in λ_0-plane to Good Set in λ-plane

Part 5: Where the Brown measure is not zero

Air assault!

- Part 4 was “ground assault”: Run along with $\varepsilon(t) \approx 0$
- Where the Brown measure is *not* zero, need “air assault”!
- Find λ_0 and $\varepsilon_0 > 0$ so ODE's will “land” with $\varepsilon(t) = 0$, $\lambda(t) = \lambda$

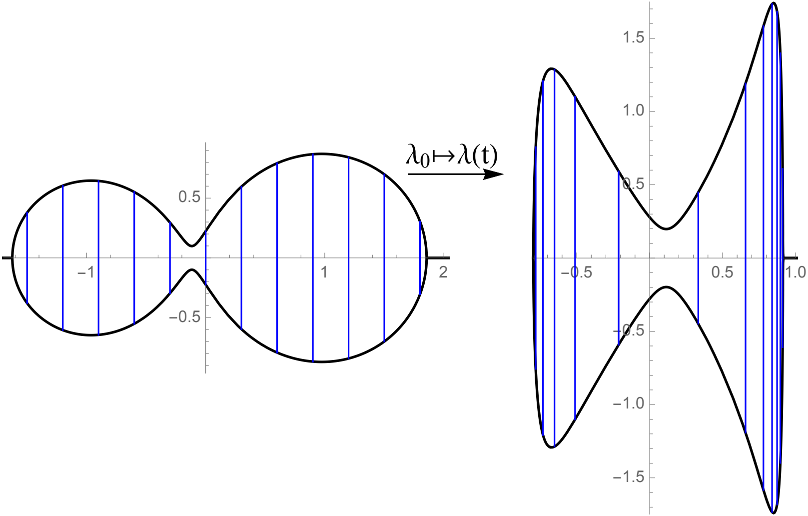


Landing the cannonballs

- For each λ_0 in Bad Set in λ_0-plane, $\exists\ \varepsilon_0 > 0$ making $\varepsilon(t) = 0$
- Compute $\lambda(t)$ with this ε_0
- **Surjectivity**: each point in Bad Set in λ-plane gets hit this way
- **Decoupling**:

$$b(t) = 2b_0$$
$$a(t) = f_t(a_0)$$

Decoupling visualized



Main result

- Get $S(t, \lambda, 0)$ as a function of λ_0
- λ_0 and λ are related by map on previous slide
- Then take Laplacian in λ

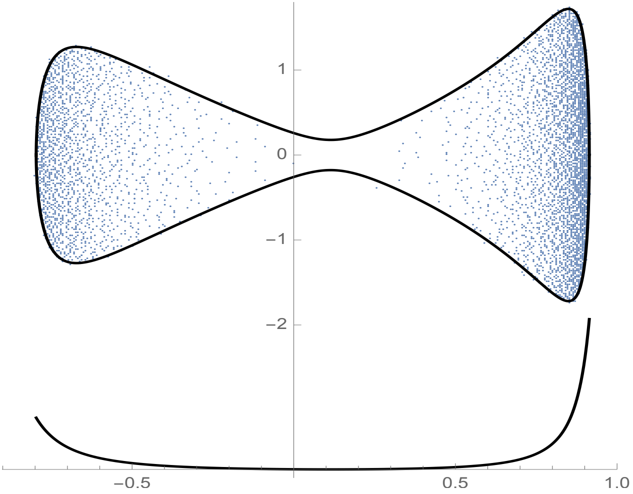
Theorem (H–Ho 2020)

In the Bad Set in λ-plane, Brown measure has a strictly positive density that **depends only on** $a = \operatorname{Re} \lambda$. *Specifically, the density is given by*

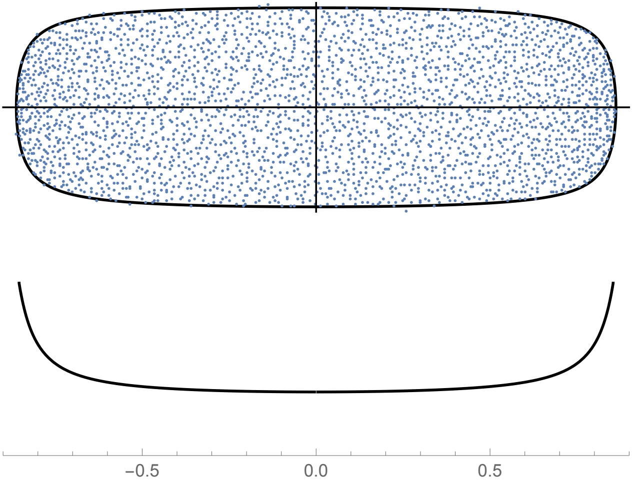
$$\frac{1}{2\pi t} \left(\frac{da_0^t(a)}{da} - \frac{1}{2} \right),$$

where a_0^t is the real part of the λ_0 that maps to $\lambda = a + ib$.

Bernoulli case, $t = 1.05$



Uniform case



Part 6: Connection to the law of $x_0 + \sigma_t$

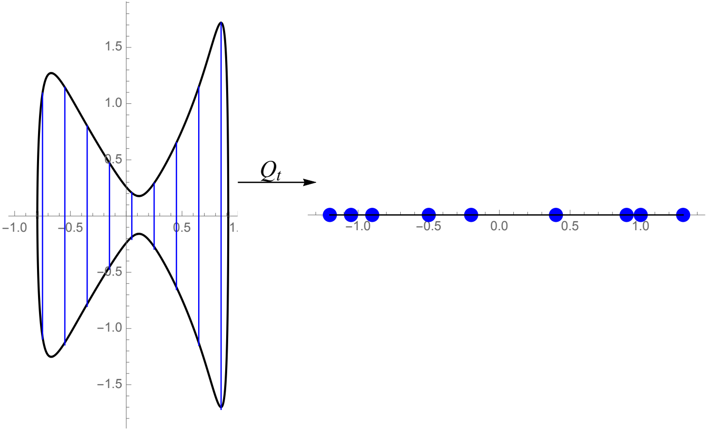
Relating $x_0 + \sigma_t$ and $x_0 + i\sigma_t$

- Biane [1997] computed law of $x_0 + \sigma_t$ using subordination method
- Connection between $x_0 + \sigma_t$ and $x_0 + i\sigma_t$ suggested by work of Janik, Nowak, Papp, Wambach, and Zahed [1997]
- We obtain a *direct relationship* between Brown measure of $x_0 + i\sigma_t$ and law of $x_0 + \sigma_t$

Defining the map

- Define a map Q_t from Bad Set in λ-plane to $\mathbb{R}$ by:

$$Q_t(a+ib) = 2a_0^t(a) - a$$



The connection

Theorem

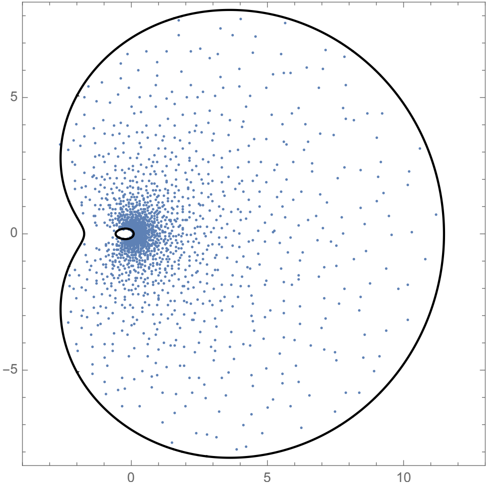
The push-forward of the Brown measure of $x_0 + i\sigma_t$ under Q_t is the law of $x_0 + \sigma_t$, as computed by Biane.

- Compute eigenvalues of $X_0 + i\sqrt{t}S$ in $\mathbb{C}$
- Map to real line by Q_t
- Distribution of points in $\mathbb{R}$ will be just like eigenvalues of $X_0 + \sqrt{t}S$
- A mysterious connection between $X_0 + i\sqrt{t}S$ and $X_0 + \sqrt{t}S$

Epilog

Results of Driver–H–Kemp

- Studied Brownian motion in $GL(N;\mathbb{C})$ as $N \to \infty$ by similar method



Conclusion

THANK YOU FOR YOUR ATTENTION!