Eigenvalues for sums of self-adjoint and skew-self-adjoint random matrices

Brian C. Hall

University of Notre Dame

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• Joint work with Ching Wei Ho (Indiana University)
• Builds on work of Ho and Zhong [arXiv:1908.08150]
• We use PDE method developed by Driver–Hall–Kemp [arXiv:1903.11015]
• Some of these results were previously obtained by Jarosz and Nowak by a different method [J. Phys. A 2006 and arXiv:math-ph/0402057]
Part 1: The random matrix problem
Take $S$ to be GUE (Gaussian-distributed Hermitian random matrix)
Take $X_0$ Hermitian and independent of $S$
Can take $X_0$ to be diagonal with fixed eigenvalue distribution
Consider
\[ Z := X_0 + i \sqrt{t} S, \quad t > 0. \]
Study eigenvalue distribution of $Z$ in $\mathbb{C}$
Example: Bernoulli case, $t = 1$
Bernoulli case, $t = 1.05$
Density of eigenvalues is *constant in the vertical direction* inside the domain.
Uniform case, $t = 0.1$
Quadratic case, $t = 1/4$
Part 2: Girko’s method and the Brown measure
Girko's formula

- Girko [1984] worked on general circular law: entries are i.i.d. but not Gaussian
- Matrix is not Hermitian (or normal); method of moments is not applicable
- No explicit formulas as in Gaussian case
- How to compute eigenvalues in this case?
Girko’s formula

For matrix $A$, define

$$S(\lambda) = \frac{1}{N} \text{Trace}[\log((A - \lambda)^*(A - \lambda))]$$

Then

$$S(\lambda) = \frac{2}{N} \sum_{j=1}^{N} \log |\lambda - \lambda_j|$$

This function is harmonic except at $\lambda = \lambda_j$ and

$$\frac{1}{4\pi} \Delta S = \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j}$$
The function $S(\lambda)$

- Plot of $-S(\lambda)$ with 5 eigenvalues
For random matrix $X^N$, define $\ast$-moments as

$$\mathbb{E} \left\{ \frac{1}{N} \text{Trace}[\text{word in } X^N \text{ and } (X^N)^\ast] \right\}$$

Take large-$N$ limit of these $\ast$-moments

Seek operator algebra $(\mathcal{A}, \tau)$ with trace $\tau : \mathcal{A} \to \mathbb{C}$ AND operator $a \in \mathcal{A}$ such that

$$\lim_{N \to \infty} \mathbb{E} \left\{ \frac{1}{N} \text{Trace}[\text{word in } X^N \text{ and } (X^N)^\ast] \right\} = \tau[\text{word in } a \text{ and } a^\ast]$$
Operator algebra formalism

- a not an operator-valued random variable—just one fixed operator in $A$
- Limits of this sort constructed using methods of free probability
Definition of Brown measure [Brown 1986]

- Given $a \in (\mathcal{A}, \tau)$, want something like “empirical eigenvalue distribution”
- Don’t assume $a$ is normal, so can’t use spectral theorem
- Imitate Girko’s formula!
- Define

$$S(\lambda) = \tau[\log((a - \lambda)^*(a - \lambda))]$$

then take Laplacian
Definition of Brown measure

- To be sure $S$ is well defined, introduce regularization $\varepsilon > 0$:

$$S(\lambda, \varepsilon) = \tau[\log((a - \lambda)^*(a - \lambda) + \varepsilon)]$$

- Then

$$s(\lambda) := \lim_{\varepsilon \to 0} S(\lambda, \varepsilon)$$

exists as a subharmonic function.

- Then **Brown measure** $\text{Brown}(a)$ defined as

$$\text{Brown}(a) = \frac{1}{4\pi} \Delta s$$

- Then $\text{Brown}(a)$ is probability measure supported on spectrum of $a$
The goal: Brown measure for the sum

- Large-\(N\) limit of \(\sqrt{tS}\): “semicircular Brownian motion” \(\sigma_t\)
- Large-\(N\) limit of \(X_0\): s.a. element \(x_0\) that is “freely independent” of \(\sigma_t\)
- Let \(\mu = \text{“law of } x_0\text{”} = \text{large-}\(N\) limit of eigenvalue distribution of } X_0

**Problem**

*Compute Brown measure of \(x_0 + i\sigma_t\) (in terms of the law \(\mu\) of \(x_0\)).*

- We **believe** that \(\text{Brown}(x_0 + i\sigma_t)\) is large-\(N\) limit of eigenvalue distribution of \(X_0 + i\sqrt{tS}\)
Biane: computed the distribution of $x_0 + \sigma_t$ (without the “$i$”)

If $x_0 = y_0 + \tilde{\sigma}_t$ for another semicircular Brownian motion $\tilde{\sigma}_t$, results of Ho and Zhong apply

Jarosz and Nowak have general algorithm for computing Brown measures for $x + iy$;

Their results agree with ours in case of $x_0 + i\sigma_t$, but we get additional information
The PDE method

- Apply definition of Brown measure to $x_0 + i\sigma_t$
- Define

$$S(t, \lambda, \varepsilon) = \tau[\log((x_0 + i\sigma_t - \lambda)^*(x_0 + i\sigma_t - \lambda) + \varepsilon)]$$

**Theorem**

*The function $S$ satisfies the PDE*

$$\frac{\partial S}{\partial t} = \varepsilon \left(\frac{\partial S}{\partial \varepsilon}\right)^2 + \frac{1}{4} \left(\left(\frac{\partial S}{\partial a}\right)^2 - \left(\frac{\partial S}{\partial b}\right)^2\right), \quad \lambda = a + ib$$

*with initial condition*

$$S(0, \lambda, \varepsilon) = \tau[\log(x_0^*x_0 + \varepsilon)].$$
The PDE method

- $\epsilon$ is a variable in the PDE
- Want to solve PDE, then evaluate at $\epsilon = 0$
- Take $\Delta$ with respect to $\lambda$ of $S(t, \lambda, 0)$
The Hamilton–Jacobi method

- Define “Hamiltonian” function $H$ from PDE, replacing derivatives on RHS with “momenta” (and overall sign change):

$$H(a, b, \epsilon, p_a, p_b, p_\epsilon) = -\epsilon p_\epsilon^2 - \frac{1}{4}(p_a^2 - p_b^2)$$

- Then introduce auxiliary system of ODE’s (Hamilton’s equations):

$$\frac{du}{dt} = \frac{\partial H}{\partial p_u}; \quad \frac{dp_u}{dt} = -\frac{\partial H}{\partial u}; \quad u \in \{a, b, \epsilon\}.$$

- These ODE’s can be solved explicitly
The Hamilton–Jacobi formula

- **Hamilton–Jacobi formula:**

\[
S(t, \lambda(t), \varepsilon(t)) = S(0, \lambda_0, \varepsilon_0) + tH_0
\]

where \(H_0 = H(a_0, b_0, \varepsilon_0, p_{a,0}, p_{b,0}, p_{\varepsilon,0})\)

- Initial momenta are \(p_{u,0} = \frac{\partial}{\partial u} S(0, \lambda_0, \varepsilon_0)\).

- Get \(S\) along solutions to ODE’s—in terms of \(\lambda_0\) and \(\varepsilon_0\)
Part 4: Where the Brown measure is zero
H-J method tells us $S(t, \lambda(t), \varepsilon(t))$

To get $S(t, \lambda, 0)$, choose $(\lambda_0, \varepsilon_0)$ so that $\lambda(t) = \lambda$ and $\varepsilon(t) = 0$

**Simple strategy:** to get $\varepsilon(t) \approx 0$, try $\varepsilon_0 \approx 0
Small-$\epsilon_0$ existence time

- **Problem**: If $\epsilon_0 \approx 0$, solution may cease to exist before time $t$
- Define

$$T(\lambda_0) = \text{existence time of solution with } \lambda(0) = \lambda_0, \epsilon(0) \approx 0$$

- Explicitly,

$$T(a_0 + ib_0) = \left( \int_{\mathbb{R}} \frac{1}{(a_0 - x)^2 + b_0^2} \, d\mu(x) \right)^{-1},$$

where $\mu$ is the distribution of $x_0$
Small-$\epsilon_0$ existence time

- Taking $\epsilon_0 \approx 0$ only allowed if $T(\lambda_0) > t$
Results of simple strategy

- If $T(\lambda_0) > t$, taking $\varepsilon_0 \approx 0$ works
- Find that $\lambda(t)$ is a holomorphic function of $\lambda_0$
- Find that $S(t, \lambda, 0)$ is harmonic
- Conclude that Brown measure is zero
"Good Set" in $\lambda_0$-plane

$$\{ \lambda_0 \in \mathbb{C} \mid T(\lambda_0) > t \}$$

[This set also appears in work of Biane and Ho–Zhong]

“Good Set” in $\lambda$-plane

$$\{ \lambda(t) \mid \lambda_0 \in \text{Good Set} \} ,$$

**Theorem**

*The Brown measure of $x_0 + i\sigma_t$ is zero in Good Set in $\lambda$-plane.*
Good Sets in Bernoulli case

Good set in $\lambda_0$-plane

Good set in $\lambda$-plane
With $\lambda_0$ in Good Set and $\varepsilon_0 = 0$, get

$$\lambda(t) = J_t(\lambda_0)$$

where

$$J_t(\lambda_0) = \lambda_0 - t \int \frac{1}{\lambda_0 - x} \, d\mu(x)$$

and $\mu = \text{Law}(x_0)$

$J_t$ is a conformal map of Good Set in $\lambda_0$-plane to Good Set in $\lambda$-plane
Part 5: Where the Brown measure is not zero
Part 4 was “ground assault”: Run along with $\varepsilon(t) \approx 0$

Where the Brown measure is not zero, need “air assault”!

Find $\lambda_0$ and $\varepsilon_0 > 0$ so ODE’s will “land” with $\varepsilon(t) = 0$, $\lambda(t) = \lambda$
Landing the cannonballs

- For each $\lambda_0$ in Bad Set in $\lambda_0$-plane, $\exists \varepsilon_0 > 0$ making $\varepsilon(t) = 0$
- Compute $\lambda(t)$ with this $\varepsilon_0$
- **Surjectivity**: each point in Bad Set in $\lambda$-plane gets hit this way
- **Decoupling**:
  
  \[
  b(t) = 2b_0 \\
  a(t) = f_t(a_0)
  \]
Decoupling visualized

\[ \lambda_0 \mapsto \lambda(t) \]
Main result

- Get $S(t, \lambda, 0)$ as a function of $\lambda_0$
- $\lambda_0$ and $\lambda$ are related by map on previous slide
- Then take Laplacian in $\lambda$

**Theorem (H–Ho 2020)**

*In the Bad Set in $\lambda$-plane, Brown measure has a strictly positive density that depends only on $a = \text{Re} \lambda$. Specifically, the density is given by*

\[
\frac{1}{2\pi t} \left( \frac{d a^t_0(a)}{da} - \frac{1}{2} \right),
\]

*where $a^t_0$ is the real part of the $\lambda_0$ that maps to $\lambda = a + ib$.***
Bernoulli case, $t = 1.05$
Uniform case
Part 6: Connection to the law of $x_0 + \sigma_t$
Relating $x_0 + \sigma_t$ and $x_0 + i\sigma_t$

- Biane [1997] computed law of $x_0 + \sigma_t$ using subordination method
- Connection between $x_0 + \sigma_t$ and $x_0 + i\sigma_t$ suggested by work of Janik, Nowak, Papp, Wambach, and Zahed [1997]
- We obtain a direct relationship between Brown measure of $x_0 + i\sigma_t$ and law of $x_0 + \sigma_t$
Define a map $Q_t$ from Bad Set in $\lambda$-plane to $\mathbb{R}$ by:

$$Q_t(a + ib) = 2a_0^t(a) - a$$
The connection

Theorem

The push-forward of the Brown measure of $x_0 + i\sigma_t$ under $Q_t$ is the law of $x_0 + \sigma_t$, as computed by Biane.

- Compute eigenvalues of $X_0 + i\sqrt{t}S$ in $\mathbb{C}$
- Map to real line by $Q_t$
- Distribution of points in $\mathbb{R}$ will be just like eigenvalues of $X_0 + \sqrt{t}S$
- A mysterious connection between $X_0 + i\sqrt{t}S$ and $X_0 + \sqrt{t}S$
Results of Driver–H–Kemp

- Studied Brownian motion in $GL(N; \mathbb{C})$ as $N \to \infty$ by similar method
THANK YOU FOR YOUR ATTENTION!