Joint Moments of Characteristic Polynomials of Random Unitary Matrices

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Joint work with Theodoros Assiotis & Jon Warren.

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Moments

Let A be an $N \times N$ unitary matrix. Denote the eigenvalues of A by $e^{\mathrm{i}\theta_n}$, $1 \leq n \leq N$, and the characteristic polynomial of A on the unit circle in the complex plane by

$$P_N(A, \theta) = \det(I - Ae^{-\mathrm{i}\theta}) = \prod_n (1 - e^{\mathrm{i}\theta_n - \mathrm{i}\theta}).$$

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c.f. Moments of the Riemann-Zeta-Function

$$\frac{1}{T} \int_0^T |\zeta(1/2 + \mathrm{i}t)|^{2\beta} \mathrm{d}t$$

[Hardy & Littlewood (1918), Ingham (1926), ...]

Calculating RMT Moments

$$M_N(\beta) = \mathbb{E}_{A\in U(N)}|P_N(A,\theta)|^{2\beta} = \mathbb{E}_{A\in U(N)} \prod_{n=1}^{N} |1 - e^{\mathrm{i}(\theta_n-\theta)}|^{2\beta}$$

Using Weyl's integration formula

$$M_N(\beta) = \frac{1}{(2\pi)^N N!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{n=1}^{N} |1 - e^{\mathrm{i}(\theta_n-\theta)}|^{2\beta}.$$
$$\times \prod_{1\le j<k\le N} |e^{\mathrm{i}\theta_j} - e^{\mathrm{i}\theta_k}|^2 \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_N.$$

This integral can then be evaluated by relating it to one computed by Selberg (Selberg 1941, 1944), giving, for $\mathrm{Re}\beta > -1/2$

$$M_N(\beta) = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2\beta)}{\Gamma(j+\beta)^2}.$$

Hence (c.f. Keating & Snaith 2000), for $\mathrm{Re}\beta > -1/2$,

$$\lim_{N\to\infty} \frac{M_N(\beta)}{N^{\beta^2}} = \frac{G(1+\beta)^2}{G(1+2\beta)},$$

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where $G(s)$ is the Barnes G-function, and for $k \in \mathbb{N}$

$$\lim_{N\to\infty} \frac{M_N(k)}{N^{k^2}} = \left(\prod_{m=0}^{k-1} \frac{m!}{(m+k)!}\right) = \frac{g_k}{k^2!}$$

where $g(k)$ is an integer.

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where $g(k)$ is an integer.

N.B. g_k is also the number of ways of filling a $k \times k$ array with the integers $1, 2, \ldots, k^2$ in such a way that the numbers increase along each row and down each column (i.e. the number of $k \times k$ Young tableaux) – c.f. Bump & Gamburd (2006)

Moments of the zeta-function: RMT-inspired conjectures

Conjecture (Keating & Snaith 2000)

For $\mathrm{Re}\beta > -1/2$, as $T \to \infty$

$$\frac{1}{T}\int_0^T |\zeta(1/2+\mathrm{i}t)|^{2\beta}\mathrm{d}t \sim a(\beta)\frac{G(1+\beta)^2}{G(1+2\beta)}(\log T)^{\beta^2}$$

and for $k \in \mathbb{N}$, as $T \to \infty$

$$\frac{1}{T}\int_0^T |\zeta(1/2+\mathrm{i}t)|^{2k}\mathrm{d}t \sim a(k)\prod_{m=0}^{k-1}\frac{m!}{(m+k)!}(\log T)^{k^2}$$

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Set

$$V_N(A,\theta) := \exp\left(\mathrm{i}N\frac{(\theta+\pi)}{2} - \mathrm{i}\sum_{n=1}^{N}\frac{\theta_n}{2}\right) P_N(A,\theta),$$

($V_N(A,\theta)$ is real-valued for $\theta \in [0, 2\pi)$).

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($V_N(A,\theta)$ is real-valued for $\theta \in [0,2\pi)$).
The joint moments of the function $V_U(\theta)$ and its derivative are

$$F_N(k,h) := \mathbb{E}_{A\in U(N)}|V_N(A,0)|^{2k-2h}|V_N'(A,0)|^{2h},$$

where it is assumed that

$$h > -\frac{1}{2} \quad \text{and} \quad k > h - \frac{1}{2}.$$

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$$h > -\frac{1}{2} \quad \text{and} \quad k > h - \frac{1}{2}.$$

These joint moments have been studied by many authors, including Hughes (2001), Conrey Rubinstein & Snaith (2006), Dehaye (2008, 2010), Winn (2012), Riedtmann (2018), Basor *et al.* (2018), Bailey *et al.* (2019).

Asymptotics

Conjecture (Hughes 2001)

When $N \to \infty$, for $k > -1/2$ and $0 \leq h < k + 1/2$

$$F_N(k,h) \sim F(k,h) N^{k^2+2h}$$

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i.e.

$$F(k,h) := \lim_{N\to\infty} \frac{F_N(k,h)}{N^{k^2+2h}}$$

exists and is non-zero for $k > -1/2$ and $0 \leq h < k + 1/2$

Motivation

Set $\Theta(t) = \mathrm{Im} \log \Gamma\left(\frac{1}{4} + \mathrm{i}\frac{t}{2}\right) - \frac{t}{2}\log \pi$. Then $Z(t) = \mathrm{e}^{\mathrm{i}\Theta(t)}\zeta(1/2 + \mathrm{i}t)$ is real-valued for $t \in \mathbb{R}$

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For integer values of h and k, these formulae can be used to prove the conjectured asymptotics, with, for a given $h \in \mathbb{N}$, $F(k, h)$ equal to a product of $G(1 + k)^2/G(1 + 2k)$ and a rational function of k.

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It is far from straightforward to prove that these formulae for $F(k,h)$ extend, for a given $h \in \mathbb{N}$, to $k > h - 1/2$.

Previous Results

For integer values of h and k, there is a formula for $F_N(k, h)$ in terms of a (rational) solution of the σ-Painlevé V equation [Basor, Bleher, Buckingham, Grava, Its, Its, & Keating (2018)].

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Taking the limit $N \to \infty$, this yields a separate proof of Hughes's conjecture for integer values of h and k, with a formula for $F(k, h)$ in terms of a solution of the σ-Painlevé III equation [Basor, Bleher, Buckingham, Grava, Its, Its, & Keating (2018)]; c.f. also [Bailey, Bettin, Blower, Conrey, Prokhorov, Rubinstein & Snaith (2019)].

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Finally, for integer values of h and k, there are also (conjectural) conformal block expansions for $F_N(k,h)$ and $F(k,h)$ [Basor, Bleher, Buckingham, Grava, Its, Its, & Keating (2018)].

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Let $s > -\frac{1}{2}$. We consider the determinantal point process $\mathrm{C}^{(s)}$ on $\mathbb{R}^* = (-\infty, 0) \cup (0, \infty)$ with correlation kernel $\mathrm{K}^{(s)}(x, y)$given in integrable form:

$$\mathrm{K}^{(s)}(x,y) = \frac{1}{2\pi} \frac{\Gamma(s+1)^2}{\Gamma(2s+1)\Gamma(2s+2)} \frac{P^{(s)}(x)Q^{(s)}(y) - P^{(s)}(y)Q^{(s)}(x)}{x-y},$$

with

$$P^{(s)}(x) = 2^{2s-\frac{1}{2}} \Gamma\left(s + \frac{1}{2}\right) |x|^{-\frac{1}{2}} J_{s-\frac{1}{2}}\left(\frac{1}{|x|}\right),$$

$$Q^{(s)}(x) = \mathrm{sgn}(x) 2^{2s+\frac{1}{2}} \Gamma\left(s + \frac{3}{2}\right) |x|^{-\frac{1}{2}} J_{s+\frac{1}{2}}\left(\frac{1}{|x|}\right),$$

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where $J_s(\cdot)$ is the Bessel function.
Let $-\alpha_1^- < -\alpha_2^- < -\alpha_3^- < \cdots < 0$ and $\alpha_1^+ > \alpha_2^+ > \alpha_3^+ > \cdots > 0$ be the corresponding random points of $\mathrm{C}^{(s)}$.

Then the random variable

$$X(s) = \lim_{m\to\infty} \left[\sum_{i=1}^{\infty} \alpha_i^{+} \mathbf{1}\left(\alpha_i^{+} > \frac{1}{m^2}\right) - \sum_{i=1}^{\infty} \alpha_i^{-} \mathbf{1}\left(\alpha_i^{-} > \frac{1}{m^2}\right)\right]$$

is well defined [Qiu (2017)].

$X(s)$ can be thought of as a kind of principal value sum of the points in $C^{(s)}$.

This random variable will play a central role in our results.

Context

The ergodic measures for the action of the infinite dimensional unitary group $\mathbb{U}(\infty) = \varinjlim \mathbb{U}(N)$ on the space of infinite Hermitian matrices $\mathbb{H} = \varprojlim \mathbb{H}(N)$ were classified by Pickrell (1991), and by Olshanski & Vershik (1996).

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They are in bijection with a certain infinite dimensional space Ω of parameters $\left(\{\alpha_i^+\}_{i=1}^{\infty}, \{\alpha_i^-\}_{i=1}^{\infty}, \gamma_1, \gamma_2\right) \subset \mathbb{R}^{2\infty+2}$.

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Borodin and Olshanski proved that for any $\mathbb{U}(\infty)$-invariant probability measure $\mathfrak{M}$ on $\mathbb{H}$, there exists a unique probability measure $\nu^{\mathfrak{M}}$ on Ω such that $\mathfrak{M}\,(d\mathbf{H}) = \int_\Omega \nu^{\mathfrak{M}}(d\omega)\mathfrak{N}_\omega\,(d\mathbf{H})$

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[This is a special case of the more general problem of classifying ergodic measures for actions of inductively compact groups $\mathbb{G}(1) \subset \mathbb{G}(2) \subset \cdots \subset \mathbb{G}(N) \subset \cdots$.]

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The law of the parameters (γ_1, γ_2) was an open question until recent work of Qiu, who proved that almost surely $\gamma_2 \equiv 0$ and

$$\gamma_1 \stackrel{\mathrm{d}}{=} \mathsf{X}(s).$$

The Hua-Pickrell Measures

Let $\mathbb{W}_N$ denote the Weyl chamber:

$$\mathbb{W}_N = \{\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_1 \geq x_2 \geq \cdots \geq x_N\}.$$

For $N \geq 1$ and $s > -\frac{1}{2}$, the Hua-Pickrell probability measure $\mathrm{M}_N^{(s)}$ on $\mathbb{W}_N$ is

$$\mathrm{M}_N^{(s)}(d\mathbf{x}) = \frac{1}{\mathrm{c}_N^{(s)}} \prod_{j=1}^{N} \frac{1}{(1+x_j^2)^{N+s}} \Delta_N(\mathbf{x})^2 dx_1 \cdots dx_N$$

where $\Delta_N(\mathbf{x}) = \prod_{1 \leq i < j \leq N}(x_j - x_i)$ and

$$\mathrm{c}_N^{(s)} = (2\pi)^N 2^{-N^2-2sN} G(N+1) \prod_{j=1}^{N} \frac{\Gamma(2s+N-j+1)}{\Gamma(s+N-j+1)^2}$$

Let $s > -\frac{1}{2}$. Then,

$$\frac{1}{N}\sum_{i=1}^{N} \mathrm{x}_i^{(N)} \xrightarrow{\mathrm{d}} \mathrm{X}(s), \quad \text{as } N \to \infty,$$

where $(\mathrm{x}_1^{(N)}, \ldots, \mathrm{x}_N^{(N)})$ has law $\mathrm{M}_N^{(s)}$ and $\mathrm{X}(s)$ is the random variable defined earlier [Borodin & Olshanski (2001), Qiu (2017)].

This plays an important role in our calculations.

Results [Assiotis, Keating & Warren (2020)]

Theorem Let $s > -\frac{1}{2}$ and $0 \leq h < s + \frac{1}{2}$. Then,

$$\lim_{N \to \infty} \frac{1}{N^{s^2+2h}} F_N(s,h) \stackrel{\text{def}}{=} F(s,h) = F(s,0) 2^{-2h} \mathbb{E}\left[|\mathsf{X}(s)|^{2h} \right]$$

with the limit $F(s,h)$ satisfying $0 < F(s,h) < \infty$. The function $F(s,0)$ is given by

$$F(s,0) = \frac{G(s+1)^2}{G(2s+1)},$$

where G is the Barnes G-function.

Outline of Proof

The first key ingredient is a representation of $F_N(s,h)$ in terms of $F_N(s,0)$ and the moments $\mathbb{E}\left[\left|\sum_{i=1}^{N}\frac{\mathsf{x}_i^{(N)}}{N}\right|^{2h}\right]$, where $(\mathsf{x}_1^{(N)},\ldots,\mathsf{x}_N^{(N)})$ have the same distribution as the non-increasing eigenvalues of a random Hermitian matrix with law $\mathfrak{M}_N^{(s)}$.

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To prove convergence of the moments:

$$\mathbb{E}\left[\left|\sum_{i=1}^{N}\frac{\mathsf{x}_i^{(N)}}{N}\right|^{2h}\right]\longrightarrow\mathbb{E}\left[\left|\mathsf{X}(s)\right|^{2h}\right],\quad \text{as } N\to\infty,$$

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one needs to prove uniform integrability or, as we do, show uniform boundedness for some higher moment.

The averages that we want to control uniformly in N do not converge if we bring the absolute values inside, and it is essential that a cancellation due to symmetry around the origin of the points in $\mathsf{C}_N^{(s)}$ is taken into account.

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Extending this to the range $-\frac{1}{2} < s \leq 0$ takes more work.

Moments of $\mathsf{X}(s)$

Let $h \in \mathbb{N}$. Denote by $\mathbb{E}_N^{(s)}$ the expectation with respect to the probability measure $\mathsf{M}_N^{(s)}$ on ordered configurations in $\mathbb{R}^N$. Then, for $s > h - \frac{1}{2}$ we have:

$$\mathbb{E}\left[\mathsf{X}(s)^{2h}\right] = \frac{1}{(2h)!}\sum_{k=1}^{2h}(-1)^{2h-k}\binom{2h}{k}\mathbb{E}_k^{(s)}\left[\left(\mathsf{x}_1^{(k)} + \cdots + \mathsf{x}_k^{(k)}\right)^{2h}\right].$$

Moreover, for any $N \geq 1$ and $s > h - \frac{1}{2}$, the expectation in the summand is a rational function of s.

For example

$$\mathbb{E}\left[\mathsf{X}(s)^2\right] = \frac{1}{4s^2 - 1}, \quad s > \frac{1}{2}.$$

Let $s, h \in \mathbb{N}$ with $h \leq s$. Then,

$$\mathbb{E}\left[\mathrm{X}(s)^{2h}\right] = 2^{2h}(-1)^h \frac{d^{2h}}{dt^{2h}}\left[\exp \int_0^t \frac{\tau(x)}{x} dx\right]\Bigg|_{t=0},$$

where $\tau(x)$ is a non-trivial solution to a special case of the σ-Painlevé III' equation with two parameters:

$$\left(x \frac{d^2\tau}{dx^2}\right)^2 = -4x\left(\frac{d\tau}{dx}\right)^3 + (4s^2 + 4\tau)\left(\frac{d\tau}{dx}\right)^2 + x\frac{d\tau}{dx} - \tau,$$

with initial conditions:

$$\tau(0) = 0,\ \tau'(0) = 0.$$

Moreover, $\mathbb{E}\left[\mathrm{X}(s)^{2h}\right]$ can also be expressed as a determinant involving the modified Bessel function I_r.

Let $s \in \mathbb{N}_{>1} = \{2, 3, 4, \dots\}$ and define:

$$\Xi^{(s)}(t) = t \frac{d}{dt} \log \mathbb{E}\left[e^{\mathrm{i}\frac{t}{2}\mathsf{X}(s)}\right].$$

Then, there exists $T > 0$ such that $\Xi^{(s)}$ is C^2 in $[0, T]$ and $\Xi^{(s)}$ satisfies a special case of the σ-Painlevé III' equation with two parameters:

$$\left(t \frac{d^2 \Xi^{(s)}}{dt^2}\right)^2 = -4t \left(\frac{d\Xi^{(s)}}{dt}\right)^3 + (4s^2 + 4\Xi^{(s)}) \left(\frac{d\Xi^{(s)}}{dt}\right)^2 + t \frac{d\Xi^{(s)}}{dt} - \Xi^{(s)},$$

with initial conditions:

$$\Xi^{(s)}(0) = 0, \ \frac{d}{dt}\Xi^{(s)}(t)\Big|_{t=0} = 0.$$

Using the connection between the joint moments of the characteristic polynomials and the moments of $\mathrm{X}(s)$ allows one to prove various special values in the latter case using results previously obtained in the former case that it is hard to see how to derive directly.

For example, it follows from a result of Winn (2012) that

$$\mathbb{E}\left[|\mathrm{X}(1)|\right] = \frac{e^2 - 5}{2\pi}.$$

Conjecture for the Riemann zeta-function

It is natural to conjecture that for $s > -\frac{1}{2}$ and $0 \leq h < s + \frac{1}{2}$ as $T \to \infty$

$$\frac{1}{T}\int_0^T |Z(t)|^{2k-2h}|Z'(t)|^{2h}\mathrm{d}t \sim a(s)F(s,0)2^{-2h}\mathbb{E}\left[|\mathrm{X}(s)|^{2h}\right](\log T)^{s^2+2h},$$

and it would be interesting to see whether this could be justified directly using number-theoretic techniques.

Last words . . .

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June 2 – Paul Bourgade

June 9 – Nick Simm

June 16 – Christian Webb