Fluctuations of extreme eigenvalues of sparse Erdős-Rényi graphs

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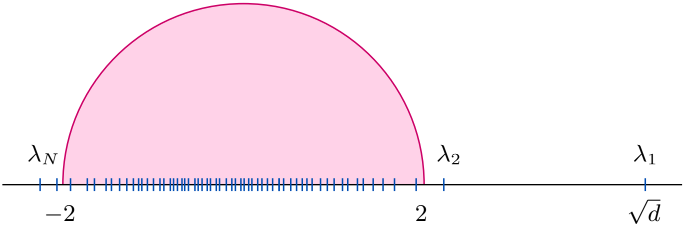
Erdős-Rényi graph

Let $A = (A_{ij})$ be the adjacency matrix of the Erdős-Rényi graph $\mathbb{G}(N, d/N)$.

$0 < d \leqslant N/2$ is the expected degree.

Denote by $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N$ the eigenvalues of $A/\sqrt{d(1-d/N)}$.

If $d \to \infty$ as $N \to \infty$ then the empirical measure $\frac{1}{N}\sum_i \delta_{\lambda_i}$ converges to the semicircle law.



Goal: fluctuations of individual eigenvalues near the edges, e.g. λ_2 and λ_N.

Previous results

Notation: write $X \ll Y$ to mean $X = O_\varepsilon(N^{-\varepsilon} Y)$ for some $\varepsilon > 0$.

- [Erdős, K, Yau, Yin; 2012]: If $d \gg N^{2/3}$ then λ_2 has GOE Tracy-Widom fluctuations:
$$N^{2/3}(\lambda_2 - \mathbb{E}\lambda_2) \xrightarrow{\mathrm{d}} \mathrm{TW}_1$$

- [Lee, Schnelli; 2016]: The same holds for $d \gg N^{1/3}$.

- [Huang, Landon, Yau; 2017]: If $N^{2/9} \ll d \ll N^{1/3}$ then λ_2 has Gaussian fluctuations:
$$\sqrt{Nd}\,(\lambda_2 - \mathbb{E}\lambda_2) \xrightarrow{\mathrm{d}} \mathcal{N}(0,2)\,.$$

Crossover from Tracy-Widom to Gaussian fluctuations at $d \asymp N^{1/3}$.

Remarks

- The expectation $\mathbb{E}\lambda_2$ undergoes a shift, first observed by [Lee, Schnelli; 2016] and refined by [Huang, Landon, Yau; 2017]

$$\mathbb{E}\lambda_2 = 2 + \frac{1}{d} - \frac{5}{4d^2} + \cdots .$$

- The Gaussian fluctuations are explicitly given by the random variable

$$\mathcal{Z} := \frac{\mathcal{D}}{d} - 1 , \qquad \mathcal{D} := \frac{1}{N} \sum_{i,j} A_{ij} \quad \text{(average degree)} ,$$

satisfying $\sqrt{Nd}\mathcal{Z} \xrightarrow{\mathrm{d}} \mathcal{N}(0,2)$.

Then [Huang, Landon, Yau; 2017] show with very high probability

$$|\lambda_2 - \mathbb{E}\lambda_2 - \mathcal{Z}| \leqslant N^{o(1)} \left(\frac{1}{N^{2/3}} + \frac{1}{d^3} \right) .$$

Result

Theorem [He, K; 2020]. If $1 \ll d \ll N^{1/3}$ then

$$\sqrt{Nd}\,(\lambda_2 - \mathbb{E}\lambda_2) \xrightarrow{\mathrm{d}} \mathcal{N}(0,2)\,.$$

In fact, we prove a rigidity result for all eigenvalues:

$$\lambda_i = (\mathbb{E}\lambda_i)\biggl(1 + \frac{\mathcal{Z}}{2}\biggr) + O\biggl(\frac{N^{-c}}{\sqrt{Nd}}\biggr)$$

with very high probability for $2 \leqslant i \leqslant N$. (Shown in bulk by [He; 2019].)

Interpretation: the Gaussian fluctuations vanish to leading order if one replaces the determinstic scaling $A/\sqrt{d(1-d/N)}$ with the random scaling $A/\sqrt{\mathcal{D}(1-d/N)}$:

$$\sqrt{\frac{d}{\mathcal{D}}}\lambda_i = \mathbb{E}\lambda_i + O\biggl(\frac{N^{-c}}{\sqrt{Nd}}\biggr)\,.$$

Heurstic picture of eigenvalue fluctuations of sparse matrices

The fluctuations of the eigenvalues of a sparse random matrix consists of two components:

- Random matrix component coinciding with fluctuations of GOE.

  Order $\sqrt{\log N}/N$ in bulk ([Gustavsson; 2005], [O'Rourke; 2010])

  Order $N^{-2/3}$ at edge.

- Sparseness component from $\mathcal{Z}$.

  Order $1/\sqrt{Nd}$ everywhere except origin.

Sparseness component dominates over random matrix component for $d \ll N$ in bulk and $d \ll N^{1/3}$ at edge.

Proof

Essence of proof: rigidity bounds for eigenvalue locations with accuracy

$$\frac{N^{-c}}{\sqrt{Nd}}\,.$$

Previously: accuracy

$$\frac{1}{N^{2/3}} + \frac{1}{d^a}\,,$$

with $a = 1$ [Erdős, K, Yau, Yin; 2012], $a = 2$ [Lee, Schnelli; 2016], $a = 3$ [Huang, Landon, Yau; 2017].

Need to go beyond a $1/d$-expansion to reach sparseness $d \gg 1$.

Main estimate: upper bound on spectrum of $H := (A - \mathbb{E}A)/\sqrt{d}$ with eigenvalues $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_N$.

Use the Green function and its normalized trace

$$G(z) := (H - z)^{-1}\,, \qquad \underline{G}(z) := \frac{1}{N} \operatorname{Tr} G(z)\,, \qquad z = E + \mathrm{i}\eta \in \mathbb{C}_+\,.$$

Simple observation: if

$$\operatorname{Im} \underline{G}(z) = \frac{1}{N} \sum_i \frac{\eta}{(E - \mu_i)^2 + \eta^2} \ll \frac{1}{N\eta}$$

then there is no eigenvalue in $[E - \eta, E + \eta]$.

Main work: estimate $\operatorname{Im} \underline{G}(z)$ just outside the spectrum.

Starting point: construction of self-consistent polynomial from [Huang, Landon, Yau; 2017], generalizing [Lee, Schnelli; 2016].

Lemma [Huang, Landon, Yau]. There exists a deterministic polynomial

$$P_0(z,x) = 1 + zx + x^2 + \frac{a_2}{d}x^4 + \frac{a_3}{d^2}x^6 + \cdots$$

such that

$$\left|\mathbb{E} P_0(z, \underline{G}(z))\right| \leqslant N^{o(1)}\left(\frac{\mathbb{E}\,\mathrm{Im}\,\underline{G}(z)}{N\eta} + \frac{1}{N}\right)$$

for $\eta \gg 1/N$. Here $a_2, a_3, \dots$ are universal constants.

Note: this says nothing about the size of $P_0(z, \underline{G}(z))$. In fact, $P_0(z, \underline{G}(z))$ is typically much bigger than the RHS.

Define the random polynomial $P(z, x) := P_0(z, x) + \mathcal{Z}x^2$.

Lemma [Huang, Landon, Yau]. There exists a random algebraic function $m : \mathbb{C}_+ \to \mathbb{C}_+$ satisfying

$$P(z, m(z)) = 0\,,$$

such that m is the Stieltjes transform of a random symmetric probability measure ϱ. We have $\operatorname{supp}\varrho = [-L, L]$, where

$$L = L_0 + \mathcal{Z} + O\left(\frac{N^{o(1)}}{d\sqrt{Nd}}\right)$$

and

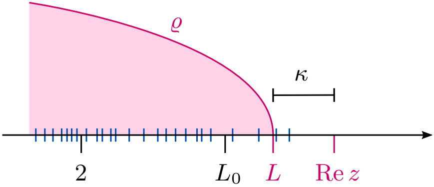
$$L_0 = 2 + \frac{b_1}{d} + \frac{b_2}{d^2} + \cdots\,.$$

Here $b_1, b_2, \ldots$ are universal constants (e.g. $b_1 = 1$, $b_2 = -5/4$).

Interpretation: L is the approximate location of the right spectral edge, and we have to prove that

$$\mu_1 \leqslant L_0 + \mathcal{Z} + \frac{N^{-c}}{\sqrt{Nd}}\,.$$

To that end, choose $z = L_0 + \mathcal{Z} + w$ with $w = \kappa + \mathrm{i}\eta$ deterministic.



The proof follows from the following result.

Proposition. There exists $\eta \gg N^{-1}$ such that for $\kappa = \frac{N^{-c}}{\sqrt{Nd}}$ we have

$$\operatorname{Im} \underline{G}(z) \ll \frac{1}{N\eta}\,.$$

All previous works on eigenvalue rigidity use the following steps:

(i) $|P(z, \underline{G}(z))| \ll 1$ with very high probability;

(ii) $|\underline{G}(z) - m(z)| \ll 1$ is small with very high probability, by inversion of self-consistent equation associated with P;

(iii) $\operatorname{Im} \underline{G}(z) \leqslant \operatorname{Im} m(z) + |\underline{G}(z) - m(z)| \ll \frac{1}{N\eta}$.

Problem: if d is small enough there is no choice of η such that both terms in (iii) are small enough.

Instead, in (iii) we estimate $|\operatorname{Im} \underline{G}(z) - \operatorname{Im} m(z)|$.

To that end, we estimate $|\operatorname{Im} P(z, \underline{G}(z))|$ instead of $|P(z, \underline{G}(z))|$. Exploits a crucial cancellation which ensures that $|\operatorname{Im} P(z, \underline{G}(z))| \ll |P(z, \underline{G}(z))|$.

How to invert self-consistent equation associated with $\operatorname{Im} P$?

Expand

$$P(z, \underline{G}) = \partial_2 P(z,m)(\underline{G} - m) + \frac{1}{2}\partial_2^2 P(z,m)(\underline{G} - m)^2 + \cdots .$$

As $\partial_2^2 P(z,m) \approx 2$, taking the imaginary part and rearranging terms yields

$$\begin{aligned}\operatorname{Re}\partial_2 P(z,m)\operatorname{Im}(\underline{G} - m) = \operatorname{Im} P(z,\underline{G}) - \operatorname{Im}\partial_2 P(z,m)\operatorname{Re}(\underline{G} - m)\\ - 2\operatorname{Re}(\underline{G} - m)\operatorname{Im}(\underline{G} - m) + \cdots .\end{aligned}$$

To solve this in $\operatorname{Im}(\underline{G} - m)$ we need

$$|\underline{G} - m| \ll |\operatorname{Re}\partial_2 P(z,m)| \asymp \sqrt{\kappa} .$$

Thus, the proof splits into two main steps:

(a) Show $|\underline{G} - m| \ll \sqrt{\kappa}$ by estimating $|P(z, \underline{G}(z))|$.

(b) Estimate $|\operatorname{Im} P(z, \underline{G})|$.

Step (a) is harder because we cannot exploit the cancellation from Im.

Basic strategy for proving (a) and (b): recursive high moment estimates using cumulant expansion.

Recall that

$$P(z,x) = 1 + zx + Q(x) + \mathcal{Z}\,, \qquad Q(x) := x^2 + \frac{a_2}{d}x^4 + \frac{a_3}{d^2}x^6 + \cdots\,.$$

Abbreviate $P = P(z, \underline{G}(z))$, $Q = Q(\underline{G}(z))$ and write

$$\mathbb{E}|P|^{2n} = \frac{1}{N}\sum_{i,j}\mathbb{E}H_{ij}G_{ji}P^{n-1}P^{*n} + \mathbb{E}(Q+\mathcal{Z})P^{n-1}P^{*n}\,,$$

since

$$1 + z\underline{G} = \underline{HG}\,.$$

Then use cumlant expansion (or generalized Stein lemma)

$$\mathbb{E}\big[h \cdot f(h)\big] = \sum_{k=0}^{\ell}\frac{1}{k!}\mathcal{C}_{k+1}(h)\mathbb{E}[f^{(k)}(h)] + \mathcal{R}_{\ell+1}\,,$$

where $\mathcal{C}_k(h)$ is the kth cumulant of h.

The cumulant expansion yields

$$\mathbb{E}|P|^{2n} = \frac{1}{N}\sum_{k=1}^{\ell}\frac{1}{k!}\sum_{s=1}^{k}\binom{k}{s}\sum_{i,j}\mathcal{C}_{k+1}(H_{ij})\mathbb{E}\left[\frac{\partial^s(P^{n-1}P^{*n})}{\partial H_{ij}^s}\frac{\partial^{k-s}G_{ij}}{\partial H_{ij}^{k-s}}\right]$$

$$+\frac{1}{N}\sum_{k=1}^{\ell}\frac{1}{k!}\sum_{i,j}\mathcal{C}_{k+1}(H_{ij})\mathbb{E}\left[\frac{\partial^k G_{ij}}{\partial H_{ij}^k}P^{n-1}P^{*n}\right]+\mathbb{E}(Q+\mathcal{Z})P^{n-1}P^{*n}+\text{error}\,.$$

The polynomial P is designed so that

$$\begin{aligned}\mathbb{E}P &= \frac{1}{N}\sum_{i,j}\mathbb{E}H_{ij}G_{ji}+\mathbb{E}(Q+\mathcal{Z})\\ &= \frac{1}{N}\sum_{k=1}^{\ell}\frac{1}{k!}\sum_{i,j}\mathcal{C}_{k+1}(H_{ij})\mathbb{E}\left[\frac{\partial^k G_{ij}}{\partial H_{ij}^k}\right]+\mathbb{E}(Q+\mathcal{Z})+\text{error}\approx 0\,,\end{aligned}$$

and for the same reason there is a near-exact cancellation between the blue terms above.

Dangerous terms: red ones, which capture the fluctuations of P.

Example of dangerous term ($k = 3$ and $s = 2$):

$$T := \frac{1}{N} \sum_{i,j} \mathcal{C}_4(H_{ij}) \mathbb{E}\Big[(\partial_2 P^*) N^{-1} (G^{*2})_{ii} G^*_{jj} G_{ii} G_{jj} |P|^{2n-2}\Big] .$$

Unlike in [Erdős, K, Yau, Yin], [Lee, Schnelli], [Huang, Landon, Yau]: taking absolute value inside expectation is not affordable.

Instead, we have to exploit higher-order cancellations arising from construction of P. Use key identity

$$(\partial_2 P^*) \underline{G^{*2}} = \partial_{w^*} P(z^*, \underline{G}^*) - \underline{G^*}$$

and approximations (to be justified)

$$(G^{*2})_{ii} \approx \underline{G^{*2}} , \qquad G^*_{jj} \approx \underline{G^*} , \qquad G_{ii}, G_{jj} \approx \underline{G}$$

to write

$$(\partial_2 P^*) N^{-1} (G^{*2})_{ii} G^*_{jj} G_{ii} G_{jj} = N^{-1} \partial_{w^*} P(z^*, \underline{G}^*) \underline{G^*}\, \underline{G}^2 + \text{error} .$$

Thus, up to error terms,

$$
\begin{aligned}
T &= \frac{1}{N^2}\sum_{i,j}\mathcal{C}_4(H_{ij})\mathbb{E}\Big[\partial_{w^*}P(z^*,\underline{G^*})\underline{G^*}\,\underline{G}^2|P|^{2n-2}\Big] \\
&= \frac{1}{N^3}\sum_{i,j,k,l}\mathcal{C}_4(H_{ij})\mathbb{E}\Big[\partial_{w^*}(H_{kl}G^*_{lk})\underline{G^*}\,\underline{G}^2|P|^{2n-2}\Big] \\
&\qquad + \frac{1}{N^2}\sum_{i,j}\mathcal{C}_4(H_{ij})\mathbb{E}\Big[\partial_{w^*}(Q^*+\mathcal{Z})\underline{G^*}\,\underline{G}^2|P|^{2n-2}\Big].
\end{aligned}
$$

Now apply cumulant expansion again, using that ∂_{w^*} and $\partial/\partial H_{ij}$ commute. Thus, we can exploit the cancellation built into P.

Need a systematic machinery to track the algebraic structure of all terms generated by this procedure and the corresponding higher-order cancellations. This is done by constructing a hierarchy of Schwinger-Dyson equations for a sufficiently large class of polynomials in $(G_{ij})_{i,j=1}^N$.