Non-selfadjoint random matrices: spectral statistics and applications

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Mesoscopic spectral statistics: selfadjoint and non-selfadjoint
Spectra of random matrices

Setup:
- Random matrix \( X = (x_{ij}) \in \mathbb{C}^{n \times n} \)
- Eigenvalues \( \lambda_1, \ldots, \lambda_n \) encoded by \( \mu_X = \frac{1}{n} \sum_i \delta_{\lambda_i} \)
- Analyse statistical behaviour as \( n \to \infty \)

Global law
- Is there a deterministic measure/density \( \rho \) such that
  \[ \mu_X \to \rho \text{ in probability?} \]

Basic representatives

<table>
<thead>
<tr>
<th>Selfadjoint</th>
<th>Non-selfadjoint</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wigner matrix with i.i.d. entries above diagonal ( (\mathbb{E}</td>
<td>x_{ij}</td>
</tr>
<tr>
<td>Spectrum in ( \mathbb{R} )</td>
<td>Spectrum in ( \mathbb{C} )</td>
</tr>
<tr>
<td>Semicircle law: ( \rho(\lambda) = \frac{1}{2\pi} \sqrt{4 - \lambda^2} \mathbb{1}(</td>
<td>\lambda</td>
</tr>
</tbody>
</table>
Spectra on local scales: selfadjoint models

**Local law**
- Down to which scales $n^{-\alpha} \ll 1$ is the approximation $\mu_X \approx \rho$ valid?
- Precisely: $\frac{1}{n} \sum_i f(\lambda_i) - \int f(\lambda)\rho(\lambda)d\lambda \to 0$ for local observable $f$?

**For selfadjoint models**
- Expect local law on all mesoscopic scales $\alpha \in (0, 1)$
- Local observable
  $f_{\alpha, \lambda_0}(\lambda) := n^{\alpha}f(n^{\alpha}(\lambda - \lambda_0))$
- Spectrally stable
  $\rightarrow$ Rigidity: $|\lambda_i - \mathbb{E}\lambda_i| \leq n^{-1+\varepsilon}$

**Local laws for Wigner matrices:**
[Erdős, Schlein, Yau ’10], [Erdős, Knowles, Yau, Yin ’13], [Tao, Vu ’13], . . .
For non-selfadjoint models

- Expect local law on all mesoscopic scales $n^{-\alpha}$ with $\alpha \in (0, 1/2)$

- Local observable
  
  \[ f_{\alpha, \lambda_0}(\lambda) := n^{2\alpha} f(n^{\alpha}(\lambda - \lambda_0)) \]

- Spectral instability
  
  $\rightarrow$ pseudospectrum, hermitization

Local law for i.i.d. model: [Bourgade, Yau, Yin '14], [Tao, Vu '14]

Two basic assumptions: **identical distribution and independence**

What happens if these assumptions are dropped?
Dropping basic assumptions

### Dropping identical distribution
- Density $\rho$ determined by variance profile $s_{ij} = \mathbb{E}|x_{ij}|^2$

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<th>Selfadjoint</th>
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<tr>
<td>$\rho$ several interval support</td>
<td>$\rho$ is radially symmetric</td>
</tr>
<tr>
<td>square root edges</td>
<td>supported on disk</td>
</tr>
<tr>
<td>Global law [Girko, Anderson, Zeitouni, Guionnet, Shlyakhtenko, ...]</td>
<td>Global law [Cook, Hachem, Najim, Renfrew'16]</td>
</tr>
<tr>
<td>Local law [Ajanki, Erdős, K.'16]</td>
<td>Local law [Alt, Erdős, K.'16]</td>
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</table>

### Dropping independence for general local correlations
- Density $\rho$ determined by covariances $\text{Cov}(x_{ij}, x_{lk})$

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<td>Regularity as indep. case [Alt, Erdős, K.'18]</td>
<td>Next</td>
</tr>
<tr>
<td>Global law [Girko, Pastur, Khorunzhy, Anderson, Zeitouni, Speicher, Banna, Merlevéde, Peligrad, Shcherbina, ...]</td>
<td></td>
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<tr>
<td>Local law [Ajanki, Erdős, K.'16], [Che’16], [Erdős, K., Schröder’17]</td>
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Results
Non-selfadjoint random matrix with decaying correlations

Index space

- $x_{ij}$ with indices in a discrete space $i, j \in \Omega$ with $|\Omega| = n$
- Metric gives notion of distance $(\Omega, d)$

Assumptions

- Centered, i.e. $\mathbb{E} x_{ij} = 0$
- Conditional bounded density
  $$\mathbb{P}[\sqrt{n}x_{ij} \in dz | X \setminus \{x_{ij}\}] = \psi_{ij}(z)dz$$
- Finite volume growth
  $$|\{j : d(i, j) \leq r\}| \leq C r^d$$
- Decaying correlations
  $$\text{Cov}(f_1(\sqrt{n}X), f_2(\sqrt{n}X)) \leq \frac{C_\nu \|f_1\|_2 \|f_2\|_2}{1 + d \times d(\text{supp } f_1, \text{supp } f_2)\nu} , \quad \nu \in \mathbb{N}$$
- Lower bound on variances
  $$\mathbb{E}|u \cdot Xv|^2 \geq \frac{c}{n} \|u\|^2 \|v\|^2$$
The local law

**Theorem (Local law for non-selfadjoint matrices [Alt, K. ’20])**

Let $X$ be a non-selfadjoint random matrix with decaying correlations. Then there is a deterministic density $\rho$ such that around any spectral parameter $\lambda_0$ inside the spectral bulk the local law holds on any scale $n^{-\alpha}$ with $\alpha \in (0, 1/2)$, i.e.

$$
P\left[ \left| \frac{1}{n} \sum_i f_{\alpha,\lambda_0}(\lambda_i) - \int f_{\alpha,\lambda_0}(\lambda)\rho(\lambda)d^2\lambda \right| \leq n^{-1+2\alpha+\epsilon} \right] \geq 1 - C_{\epsilon,\nu}n^{-\nu}$$

for any $\epsilon > 0$ and $\nu \in \mathbb{N}$. Recall: $f_{\alpha,\lambda_0}(\lambda) := n^{2\alpha}f(n^{\alpha}(\lambda - \lambda_0))$.

**Corollary (Isotropic eigenvector delocalization)**

The corresponding bulk eigenvectors $u$ are all delocalized, i.e.

$$
P\left[ |\langle v, u \rangle| \leq n^{-1/2+\epsilon}\|u\|\|v\| \right] \geq 1 - C_{\epsilon,\nu}n^{-\nu}$$

for any $v \in \mathbb{C}^n$, $\epsilon > 0$ and $\nu \in \mathbb{N}$.
The self-consistent density of states

What is the density $\rho$?

- The covariance of the entries of $X$ are encoded in
  $$SA := \mathbb{E}XAX^*, \quad S^* A := \mathbb{E}X^* AX.$$  
  Solve the coupled system of $n \times n$-matrix equations with $\text{Tr} V_1 = \text{Tr} V_2$

$$\frac{1}{V_1(\tau)} = SV_2(\tau) + \frac{\tau}{S^* V_1(\tau)}, \quad \frac{1}{V_2(\tau)} = S^* V_1(\tau) + \frac{\tau}{S V_2(\tau)}.$$  

Definition (Self-consistent density of states)

The self-consistent density of states (scDOS) of $X$ is defined as

$$\rho(\lambda) := \frac{1}{\pi n} \frac{d}{d\tau} \bigg|_{\tau = |\lambda|^2} \text{Tr} \left( \frac{\tau}{\tau + (S^* V_1(\tau))(SV_2(\tau))} \right) 1(|\lambda|^2 < r_{sp}(S)).$$

where $r_{sp}(S)$ is the spectral radius of $S$.

Theorem (Properties of the density of states [Alt, K. '20])

The scDOS is a probability density which is real analytic in $|\lambda|^2$ and bounded away from zero on the disk with radius $\sqrt{r_{sp}(S)}$. 
Connection to Brown measure and free probability

- Free circular elements $c_1, \ldots, c_K$ on non-commutative probability space $(\mathcal{A}, \tau)$
- Matrix valued linear combination $\mathcal{X} = \sum_k A_k \otimes c_k \in \mathcal{A}^{n \times n}$
- Brown measure $\mu_\mathcal{X}$ of non-normal operator $\mathcal{X}$ is defined by
  \[ \int_{\mathbb{C}} \log |\lambda - \zeta| \mu_\mathcal{X}(d\zeta) = \log D(\mathcal{X} - \lambda) \]
- Fuglede-Kadison determinant is
  \[ D(Y) := \lim_{\varepsilon \downarrow 0} \exp \left( \frac{1}{2n} \text{Tr} \otimes \tau \log (Y^*Y + \varepsilon) \right) \]

Corollary (Brown measure of $\mathcal{X}$)

The Brown measure of $\mathcal{X}$ has density $\rho = \rho_S$ with $SR := \sum_k A_k R A_k^*$, i.e.

\[ \mu_\mathcal{X}(d\lambda) = \rho(\lambda)d^2\lambda. \]

Some recent Brown measure results: [Haagerup, Larsen '00], [Biane, Lehner '01], [Guionnet, Wood, Zeitouni '14], [Belinschi, Śniady, Speicher '18], [Driver, Hall, Kemp '19], . . .
Edge behaviour

Lemma (Edge jump height [Alt, K. '20])

The jump height at the spectral edge is explicitly given by the formula

$$\lim_{|\lambda|^2 \uparrow r_{sp}(S)} \rho(\lambda) = \frac{\left(\frac{1}{n} \text{Tr} S_l S_r\right)^2}{r_{sp}(S) \frac{n}{\pi} \text{Tr}(S_l S_r)^2}$$

in terms of the right and left Perron-Frobenius eigenmatrices of $S$, i.e. $S S_r = r_{sp}(S) S_r$ and $S^* S_l = r_{sp}(S) S_l$.

Theorem (Spectral radius [Alt, Erdős, K. '19])

Let $X$ have independent entries and $s_{ij} := \mathbb{E}|x_{ij}|^2$. Then the spectral radius $r_{sp}(X) := \max_i |\lambda_i|$ of $X$ satisfies

$$r_{sp}(X) = \sqrt{r_{sp}(S)} + O(n^{-1/2+\varepsilon}),$$

for any $\varepsilon > 0$ with very high probability.

New even for i.i.d. case.

Edge universality: [Tao, Vu '15], [Cipolloni, Erdős, Schröder '19]
Example for inhomogeneous circular law

**Model with variance profile**

- Independent entries
- Variance profile $s_{ij} = \mathbb{E}|x_{ij}|^2$
- $4 \times 4$ - block matrix with $n \times n$ - blocks
- Normalization such that $r_{sp}(S) = 1$

$$S \propto \begin{pmatrix} 1 & 10 & 7 & 15 \\ 8 & 1 & 2 & 1 \\ 15 & 2 & 6 & 3 \\ 10 & 2 & 1 & 5 \end{pmatrix}$$

Global law

Eigenvalues of $4n \times 4n$-matrix

Limiting density $\rho(|\lambda|)$
Proof ideas
**Symmetrization**

**Symmetrization (Girko’s trick)**

- Use log-potential
  
  \[ \frac{1}{n} \sum_i f(\lambda_i) = \frac{1}{2\pi n} \sum_i \int_{\mathbb{C}} \Delta f(z) \log|z - \lambda_i| d^2 z \]

- Use \( z \)-dependent family of symmetrizations
  
  \[ H_z = \begin{pmatrix} 0 & X - z \\ (X - z)^* & 0 \end{pmatrix} \]

- Translate question about spectrum of \( X \) to question about \( H_z \) via
  
  \[ \sum_i \log|\lambda_i - z| = \log|\det(X - z)| = \frac{1}{2} \log \det H_z = - \int_0^\infty d\eta \ Tr \frac{1/2}{H_z - i\eta} \]
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The Dyson equation

Self-consistent equation for resolvents of hermitian random matrices

- Study resolvent $G = G_z(\eta) = (H_z - i\eta)^{-1}$
- Dyson equation $1 + (i\eta + Z + \Sigma G)G = D$, $|\langle x, Dy \rangle| \ll 1$

\[ Z = \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} SR_{22} & 0 \\ 0 & S^*R_{11} \end{pmatrix} \]

Matrix Dyson equation without error

- Asymptotically $G \approx M$ where $M$ solves $2n \times 2n$-matrix equation
  \[ \mathcal{J}[M] := \frac{1}{M} + i\eta + Z + \Sigma M = 0 \]
- Positivity structure $(\text{Im } M = \frac{1}{2i}(M - M^*) > 0)$ [Helton, Far, Speicher’15]
  \[ -\frac{1}{M} \text{Im } M \frac{1}{M^*} + \Sigma \text{Im } M = -\eta \to 0 \quad (\star) \]

Inherent instability

- Take the derivative in direction $R \in \mathbb{C}^{n\times n}$ of the Dyson equation
  \[ (\nabla \mathcal{J}[M])R = -\frac{1}{M} R \frac{1}{M} + \Sigma R \]
- Unfortunately $(\star)$ implies
  \[ (\nabla \mathcal{J}[M])R_0 = \mathcal{O}(\eta) \quad \text{for} \quad R_0 = E_- \text{Im } M \quad \text{with} \quad E_- := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
Treating instability

**Stable manifold of perturbations**
- Perturbed Dyson equation: \( 1 + (i\eta + Z + \Sigma \hat{G}(\hat{D}))\hat{G}(\hat{D}) = \hat{D} \)
- Observe that \( G, M \in E_\perp \). Define stable manifold \( \mathcal{M} := \hat{G}^{-1}[E_\perp] \)

**Parametrization of stable manifold**
- Find a parameterization of \( \mathcal{M} \) and perform \( 2n \times 2n - 1 \) dimensional stability analysis
  \[
  \mathcal{M} \ni \hat{D} \leftrightarrow (i\eta + \Sigma \hat{G}(\hat{D})) \frac{1}{i\eta + Z + \Sigma \hat{G}(\hat{D})} \hat{D} \in E_\perp
  \]
Applications
Application to randomly coupled ODEs

**Randomly coupled ODE**
- System of \( n \) coupled ODEs
  \[
  \partial_t u_t = (gX - 1)u_t
  \]

**Elliptic type correlations**
- \( x_{ij} \) and \( x_{ji} \) may be positively correlated, i.e. \( t_{ij} := \mathbb{E}x_{ij}x_{ji} > 0 \)
- otherwise independent entries with arbitrary distribution
- Truly non-selfadjoint if \( |\mathbb{E}x_{ij}x_{ji}|^2 \leq (1 - \varepsilon)\mathbb{E}|x_{ij}|^2\mathbb{E}|x_{ji}|^2 \)

**Theorem (Long-time asymptotics [Erdős, K., Renfrew ’19])**

Let \( X \) have elliptic type correlations and \( \lambda = \max \text{Re \ supp } \rho > 0 \) the point furthest to the right inside the asymptotic spectrum. Then for any \( 0 < g \leq \frac{1}{\lambda} \) the solution \( u_t \) has the long-time behaviour

\[
\mathbb{E}_{u_0} \| u_t \|^2 = \frac{\text{const}}{\sqrt{2\pi g t}} e^{-2(1-g\lambda)t} + \mathcal{O}(n^{-c}), \quad 1 \ll t \leq nc,
\]

for some \( c > 0 \) with very high probability, where \( u_0 \) is uniformly distributed on the sphere.

Gaussian i.i.d. elliptic case with \( t_{ij} = \text{const} \) by [Mehlig, Chalker ’00]
Translation to selfadjoint problem

Multi-resolvent problem

- Solve the ODE to get $u_t = e^{gX - 1}u_0$
- Take expectation with respect to initial data
  $$\mathbb{E}_{u_0} \|u_t\|^2 = \mathbb{E}_{u_0} \langle u_0, e^{gX^* - 1}e^{gX - 1}u_0 \rangle = \frac{1}{n} \text{Tr } e^{gX^* - 1}e^{gX - 1}$$
- Represent in terms of resolvents
  $$\frac{1}{n} \text{Tr } f(X^*)g(X) = \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} K(z, w)f(w)g(z), \quad K(z, w) := \frac{1}{n} \text{Tr } \frac{1}{X - z} \frac{1}{X^* - w}$$

Linearization technique

- Create resolvent product from self-adjoint model
  $$\frac{1}{X - z} \frac{1}{X^* - w} = \partial_\alpha G_{31}^\alpha(z, w) |_{\alpha = 0}$$
- The resolvent here is $G^\alpha(z, w) = \lim_{\eta \downarrow 0} (H^\alpha(z, w) - i\eta)^{-1}$ with
  $$H^\alpha(z, w) = \begin{pmatrix}
0 & 0 & 0 & X^* - w \\
0 & 0 & X - z & -\alpha \\
0 & X^* - \bar{z} & 0 & 0 \\
X - w & -\alpha & 0 & 0 \\
\end{pmatrix}$$
Thank you!