Non-selfadjoint random matrices: spectral statistics and applications

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Mesoscopic spectral statistics: selfadjoint and non-selfadjoint

Spectra of random matrices

Setup:

- Random matrix $X = (x_{ij}) \in \mathbb{C}^{n\times n}$
- Eigenvalues $\lambda_1, \dots, \lambda_n$ encoded by $\mu_X = \frac{1}{n}\sum_i \delta_{\lambda_i}$
- Analyse statistical behaviour as $n \to \infty$

Global law

- Is there a deterministic measure/density ρ such that

$$\mu_X = \frac{1}{n}\sum_i \delta_{\lambda_i} \;\to\; \rho \quad \text{in probability?}$$

Basic representatives

Selfadjoint	**Non-selfadjoint**
• Wigner matrix with i.i.d. entries above diagonal ($\mathbb{E}\lvert x_{ij}\rvert^2 = \frac{1}{n}$)	• i.i.d. entries without symmetry constraints ($\mathbb{E}\lvert x_{ij}\rvert^2 = \frac{1}{n}$)
• Spectrum in $\mathbb{R}$	• Spectrum in $\mathbb{C}$
• Semicircle law: $\rho(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}\,\mathbb{1}(\lvert\lambda\rvert \le 2)$	• Circular law: $\rho(\lambda) = \frac{1}{\pi}\mathbb{1}(\lvert\lambda\rvert \le 1)$

Spectra on local scales: selfadjoint models

Local law

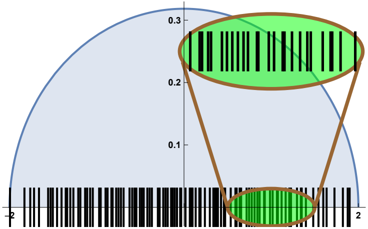
- Down to which scales $n^{-\alpha} \ll 1$ is the approximation $\mu_X \approx \rho$ valid?
- Precisely: $\frac{1}{n}\sum_i f(\lambda_i) - \int f(\lambda)\rho(\lambda)\mathrm{d}\lambda \to 0$ for local observable f?

For selfadjoint models

- Expect local law on all mesoscopic scales $\alpha \in (0,1)$
- Local observable

$$f_{\alpha,\lambda_0}(\lambda) := n^{\alpha} f(n^{\alpha}(\lambda - \lambda_0))$$

- Spectrally stable
  $\to$ Rigidity:$|\lambda_i - \mathbb{E}\lambda_i| \le n^{-1+\varepsilon}$



Local laws for Wigner matrices:

[Erdős, Schlein, Yau '10], [Erdős, Knowles, Yau, Yin '13], [Tao, Vu '13], ...

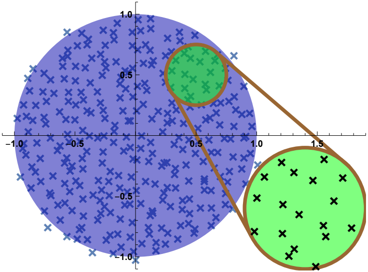
Spectra on local scales: non-selfadjoint models

For non-selfadjoint models

- Expect local law on all mesoscopic scales $n^{-\alpha}$ with $\alpha \in (0, 1/2)$
- Local observable

$$f_{\alpha,\lambda_0}(\lambda) := n^{2\alpha} f(n^{\alpha}(\lambda - \lambda_0))$$

- Spectral instability
  $\rightarrow$ pseudospectrum, hermitization



Local law for i.i.d. model: [Bourgade, Yau, Yin '14], [Tao, Vu '14]

Two basic assumptions: identical distribution and independence

What happens if these assumptions are dropped?

Dropping basic asssumptions

Dropping identical distribution

- Density ρ determined by variance profile $s_{ij} = \mathbb{E}|x_{ij}|^2$

Selfadjoint	Non-selfadjoint
• ρ several interval support	• ρ is radially symmetric
• square root edges	• supported on disk
• Global law [Girko, Anderson, Zeitouni, Guionnet, Shlyakhtenko, . . .]	• Global law [Cook, Hachem, Najim, Renfrew'16]
• Local law [Ajanki, Erdős, K.'16]	• Local law [Alt, Erdős, K.'16]

Dropping independence for general local correlations

- Density ρ determined by covariances $\mathrm{Cov}(x_{ij}, x_{lk})$

Selfadjoint	Non-selfadjoint
• Regularity as indep. case [Alt, Erdős, K.'18]	Next
• Global law [Girko, Pastur, Khorunzhy, Anderson, Zeitouni, Speicher, Banna, Merlevéde, Peligrad, Shcherbina, . . .]	
• Local law [Ajanki, Erdős, K.'16], [Che'16], [Erdős, K., Schröder'17]	

Results

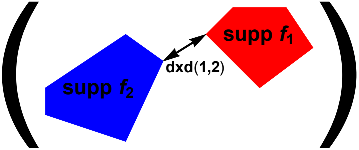
Non-selfadjoint random matrix with decaying correlations

Index space

- x_{ij} with indices in a discrete space $i, j \in \Omega$ with $|\Omega| = n$
- Metric gives notion of distance (Ω, d)

Assumptions

- Centered, i.e. $\mathbb{E}\, x_{ij} = 0$
- Conditional bounded density
$$\mathbb{P}[\sqrt{n} x_{ij} \in \mathrm{d}z | X \setminus \{x_{ij}\}] = \psi_{ij}(z)\mathrm{d}z$$
- Finite volume growth
$$|\{j : d(i,j) \le r\}| \le C\, r^d$$
- Decaying correlations
$$\mathrm{Cov}(f_1(\sqrt{n}X), f_2(\sqrt{n}X)) \le \frac{C_\nu \|f_1\|_2 \|f_2\|_2}{1 + d \times d(\mathrm{supp}\, f_1, \mathrm{supp}\, f_2)^\nu}\,, \quad \nu \in \mathbb{N}$$
- Lower bound on variances
$$\mathbb{E}|u \cdot Xv|^2 \ge \frac{c}{n}\|u\|^2\|v\|^2$$



The local law

Theorem (Local law for non-selfadjoint matrices [Alt, K. '20])

Let X be a non-selfadjoint random matrix with decaying correlations. Then there is a deterministic density ρ such that around any spectral parameter λ_0 inside the spectral bulk the local law holds on any scale $n^{-\alpha}$ with $\alpha \in (0, 1/2)$, i.e.

$$\mathbb{P}\Big[\Big|\frac{1}{n}\sum_i f_{\alpha,\lambda_0}(\lambda_i) - \int f_{\alpha,\lambda_0}(\lambda)\rho(\lambda)\mathrm{d}^2\lambda\Big| \le n^{-1+2\alpha+\varepsilon}\Big] \ge 1 - C_{\varepsilon,\nu} n^{-\nu}$$

for any $\varepsilon > 0$ and $\nu \in \mathbb{N}$. Recall: $f_{\alpha,\lambda_0}(\lambda) := n^{2\alpha} f(n^\alpha(\lambda - \lambda_0))$.

Corollary (Isotropic eigenvector delocalization)

The corresponding bulk eigenvectors u are all delocalized, i.e.

$$\mathbb{P}\Big[|\langle v, u\rangle| \le n^{-1/2+\varepsilon}\|u\|\|v\|\Big] \ge 1 - C_{\varepsilon,\nu} n^{-\nu}$$

for any $v \in \mathbb{C}^n$, $\varepsilon > 0$ and $\nu \in \mathbb{N}$.

The self-consistent density of states

What is the density ρ?

- The covariance of the entries of X are encoded in

$$\mathcal{S}A := \mathbb{E}XAX^* , \qquad \mathcal{S}^*A := \mathbb{E}X^*AX .$$

- Solve the coupled system of $n \times n$-matrix equations with $\operatorname{Tr} V_1 = \operatorname{Tr} V_2$

$$\frac{1}{V_1(\tau)} = \mathcal{S}V_2(\tau) + \frac{\tau}{\mathcal{S}^* V_1(\tau)} , \qquad \frac{1}{V_2(\tau)} = \mathcal{S}^* V_1(\tau) + \frac{\tau}{\mathcal{S} V_2(\tau)} .$$

Definition (Self-consistent density of states)

The self-consistent density of states (scDOS) of X is defined as

$$\rho(\lambda) := \frac{1}{\pi n} \frac{\mathrm{d}}{\mathrm{d}\tau}\Big|_{\tau = |\lambda|^2} \operatorname{Tr} \frac{\tau}{\tau + (\mathcal{S}^* V_1(\tau))(\mathcal{S} V_2(\tau))} \mathbb{1}\big(|\lambda|^2 < r_{\mathrm{sp}}(\mathcal{S})\big) ,$$

where $r_{\mathrm{sp}}(\mathcal{S})$ is the spectral radius of $\mathcal{S}$.

Theorem (Properties of the density of states [Alt, K. '20])

The scDOS is a probability density which is real analytic in $|\lambda|^2$ and bounded away from zero on the disk with radius $\sqrt{r_{\mathrm{sp}}(\mathcal{S})}$.

Brown measure for operator valued circular elements

Connection to Brown measure and free probability

- Free circular elements $\mathfrak{c}_1, \ldots, \mathfrak{c}_K$ on non-commutative probability space $(\mathcal{A}, \tau)$
- Matrix valued linear combination $\mathfrak{X} = \sum_k A_k \otimes \mathfrak{c}_k \in \mathcal{A}^{n \times n}$
- Brown measure $\mu_{\mathfrak{X}}$ of non-normal operator $\mathfrak{X}$ is defined by
$$\int_{\mathbb{C}} \log|\lambda - \zeta| \mu_{\mathfrak{X}}(\mathrm{d}\zeta) = \log D(\mathfrak{X} - \lambda)$$
- Fuglede-Kadison determinant is
$$D(\mathfrak{Y}) := \lim_{\varepsilon \downarrow 0} \exp\left(\frac{1}{2n} \operatorname{Tr} \otimes \tau \log(\mathfrak{Y}^* \mathfrak{Y} + \varepsilon)\right)$$

Corollary (Brown measure of $\mathfrak{X}$)

The Brown measure of $\mathfrak{X}$ has density $\rho = \rho_{\mathcal{S}}$ with $\mathcal{S}R := \sum_k A_k R A_k^*$, i.e.
$$\mu_{\mathfrak{X}}(\mathrm{d}\lambda) = \rho(\lambda) \mathrm{d}^2 \lambda \,.$$

Some recent Brown measure results: [Haagerup, Larsen '00], [Biane, Lehner '01], [Guionnet, Wood, Zeitouni '14], [Belinschi, Śniady, Speicher '18], [Driver, Hall, Kemp '19], ...

Edge behaviour

Lemma (Edge jump height [Alt, K. '20])

The jump height at the spectral edge is explicitly given by the formula

$$\lim_{|\lambda|^2 \uparrow r_{\mathrm{sp}}(\mathcal{S})} \rho(\lambda) = \frac{(\frac{1}{n}\operatorname{Tr} S_{\mathrm{l}} S_{\mathrm{r}})^2}{r_{\mathrm{sp}}(\mathcal{S}) \frac{\pi}{n} \operatorname{Tr}(S_{\mathrm{l}} S_{\mathrm{r}})^2}$$

in terms of the right and left Perron-Frobenius eigenmatrices of $\mathcal{S}$, *i.e.* $\mathcal{S} S_{\mathrm{r}} = r_{\mathrm{sp}}(\mathcal{S}) S_{\mathrm{r}}$ *and* $\mathcal{S}^* S_{\mathrm{l}} = r_{\mathrm{sp}}(\mathcal{S}) S_{\mathrm{l}}$.

Theorem (Spectral radius [Alt, Erdős, K. '19])

Let X *have independent entries and* $s_{ij} := \mathbb{E}|x_{ij}|^2$. *Then the spectral radius* $r_{\mathrm{sp}}(X) := \max_i |\lambda_i|$ *of* X *satisfies*

$$r_{\mathrm{sp}}(X) = \sqrt{r_{\mathrm{sp}}(S)} + \mathcal{O}(n^{-1/2+\varepsilon}),$$

for any $\varepsilon > 0$ *with very high probability.*

New even for i.i.d. case.

Edge universality: [Tao, Vu '15], [Cipolloni, Erdős, Schröder '19]

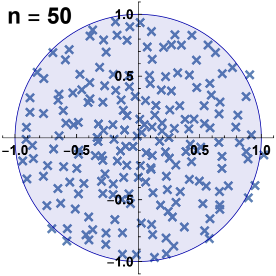
Example for inhomogeneous circular law

Model with variance profile

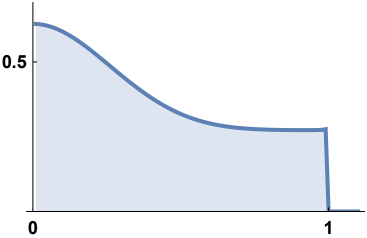
- Independent entries
- Variance profile $s_{ij} = \mathbb{E}|x_{ij}|^2$
- 4×4 - block matrix with $n \times n$ - blocks
- Normalization such that $r_{\mathrm{sp}}(S) = 1$

$$S \propto \begin{pmatrix} 1 & 10 & 7 & 15 \\ 8 & 1 & 2 & 1 \\ 15 & 2 & 6 & 3 \\ 10 & 2 & 1 & 5 \end{pmatrix}$$

Global law



Eigenvalues of $4n \times 4n$-matrix



Limiting density $\rho(|\lambda|)$

Proof ideas

Symmetrization (Girko's trick)

- Use log-potential

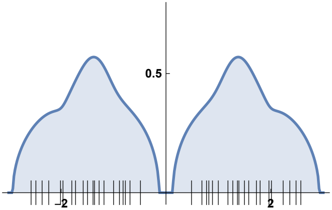
$$\frac{1}{n}\sum_i f(\lambda_i) = \frac{1}{2\pi n}\sum_i \int_{\mathbb{C}} \Delta f(z) \log|z-\lambda_i| \mathrm{d}^2 z$$

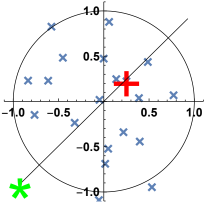
- Use z-dependent family of symmetrizations

$$H_z = \begin{pmatrix} 0 & X-z \\ (X-z)^* & 0 \end{pmatrix}$$

- Translate question about spectrum of X to question about H_z via

$$\sum_i \log|\lambda_i - z| = \log|\det(X-z)| = \frac{1}{2}\log\det H_z = -\int_0^\infty \mathrm{d}\eta \ \mathrm{Tr}\, \frac{1/2}{H_z - \mathrm{i}\eta}$$





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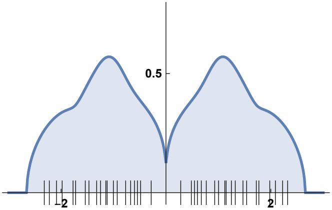

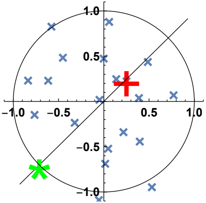

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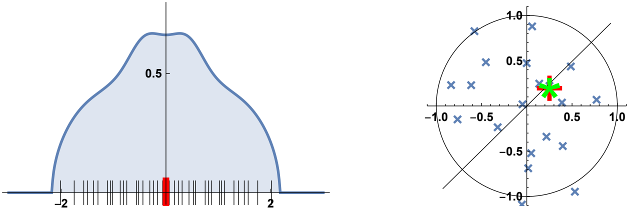

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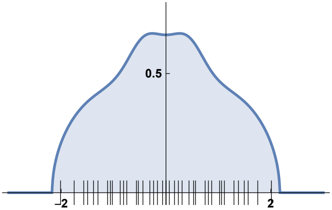

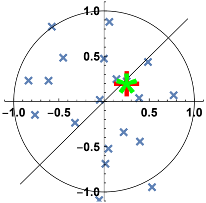

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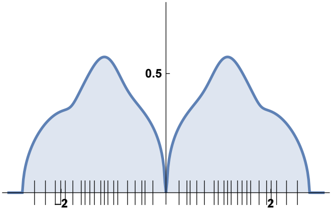
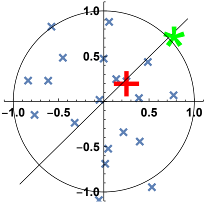




Self-consistent equation for resolvents of hermitian random matrices

- Study resolvent $G = G_z(\eta) = (H_z - \mathrm{i}\eta)^{-1}$
- Dyson equation $1 + (\mathrm{i}\eta + Z + \Sigma G)G = D\,, \qquad |\langle x\,, Dy\rangle| \ll 1$

with

$$Z = \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}, \qquad \Sigma \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} \mathcal{S}R_{22} & 0 \\ 0 & \mathcal{S}^* R_{11} \end{pmatrix}$$

Matrix Dyson equation without error

- Asymptotically $G \approx M$ where M solves $2n \times 2n$-matrix equation

$$\mathcal{J}[M] := \frac{1}{M} + \mathrm{i}\eta + Z + \Sigma M = 0$$

- Positivity structure ($\operatorname{Im} M = \frac{1}{2\mathrm{i}}(M - M^*) > 0$) [Helton, Far, Speicher'15]

$$-\frac{1}{M} \operatorname{Im} M \frac{1}{M^*} + \Sigma \operatorname{Im} M = -\eta \to 0 \qquad (\star)$$

Inherent instability

- Take the derivative in direction $R \in \mathbb{C}^{n\times n}$ of the Dyson equation

$$(\nabla \mathcal{J}[M])R = -\frac{1}{M} R \frac{1}{M} + \Sigma R$$

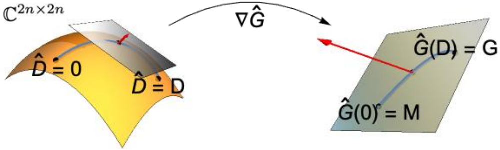
- Unfortunately $(\star)$ implies

$$(\nabla \mathcal{J}[M])R_0 = \mathcal{O}(\eta) \quad \text{for} \quad R_0 = E_- \operatorname{Im} M \ \text{ with } \ E_- := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Treating instability

Stable manifold of perturbations

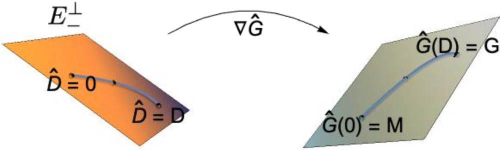
- Perturbed Dyson equation $\quad 1 + (\mathrm{i}\eta + Z + \Sigma\widehat{G}(\widehat{D}))\widehat{G}(\widehat{D}) = \widehat{D}$
- Observe that $G, M \in E_-^\perp$. Define stable manifold $\mathcal{M} := \widehat{G}^{-1}[E_-^\perp]$



Parametrizaton of stable manifold

- Find a paramerization of $\mathcal{M}$ and perform $2n \times 2n - 1$ dimensional stability analysis

$$\mathcal{M} \ni \widehat{D} \leftrightarrow (\mathrm{i}\eta + \Sigma\widehat{G}(\widehat{D}))\frac{1}{\mathrm{i}\eta + Z + \Sigma\widehat{G}(\widehat{D})}\widehat{D} \in E_-^\perp$$



Applications

Application to randomly coupled ODEs

Randomly coupled ODE

- System of n coupled ODEs

$$\partial_t u_t = (gX - 1)u_t$$

Elliptic type correlations

- x_{ij} and x_{ji} may be positively correlated, i.e. $t_{ij} := \mathbb{E} x_{ij} x_{ji} > 0$
- otherwise independent entries with arbitrary distribution
- Truly non-selfadjoint if $|\mathbb{E} x_{ij} x_{ji}|^2 \le (1-\varepsilon)\mathbb{E}|x_{ij}|^2 \mathbb{E}|x_{ji}|^2$

Theorem (Long-time asymptotics [Erdős, K., Renfrew '19])

Let X have elliptic type correlations and $\lambda = \max \operatorname{Re} \operatorname{supp} \rho > 0$ the point furthest to the right inside the asymptotic spectrum. Then for any $0 < g \le \frac{1}{\lambda}$ the solution u_t has the long-time behaviour

$$\mathbb{E}_{u_0} \|u_t\|^2 = \frac{\mathrm{const}}{\sqrt{2\pi g t}} \mathrm{e}^{-2(1-g\lambda)t} + \mathcal{O}(n^{-c}), \qquad 1 \ll t \le n^c,$$

for some $c > 0$ with very high probability, where u_0 is uniformly distributed on the sphere.

Gaussian i.i.d. elliptic case with $t_{ij} = \mathrm{const}$ by [Mehlig, Chalker '00]

Translation to selfadjoint problem

Multi-resolvent problem

- Solve the ODE to get $u_t = \mathrm{e}^{gX-1} u_0$
- Take expectation with respect to initial data

$$\mathbb{E}_{u_0} \|u_t\|^2 = \mathbb{E}_{u_0} \langle u_0 \,, \mathrm{e}^{gX^*-1} \mathrm{e}^{gX-1} u_0 \rangle = \frac{1}{n} \operatorname{Tr} \mathrm{e}^{gX^*-1} \mathrm{e}^{gX-1}$$

- Represent in terms of resolvents

$$\frac{1}{n}\operatorname{Tr} f(X^*)g(X) = \oint \frac{\mathrm{d}\overline{w}}{2\pi\mathrm{i}} \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} K(z,\overline{w}) f(\overline{w}) g(z), \quad K(z,\overline{w}) := \frac{1}{n}\operatorname{Tr} \frac{1}{X-z} \frac{1}{X^*-\overline{w}}$$

Linearization technique

- Create resolvent product from self-adjoint model

$$\frac{1}{X-z} \frac{1}{X^*-\overline{w}} = \partial_\alpha G^\alpha_{31}(z,\overline{w})|_{\alpha=0}$$

- The resolvent here is $G^\alpha(z,\overline{w}) = \lim_{\eta\downarrow 0}(H^\alpha(z,\overline{w}) - \mathrm{i}\eta)^{-1}$ with

$$H^\alpha(z,\overline{w}) = \begin{pmatrix} 0 & 0 & 0 & X^* - \overline{w} \\ 0 & 0 & X - z & -\alpha \\ 0 & X^* - \overline{z} & 0 & 0 \\ X - w & -\alpha & 0 & 0 \end{pmatrix}$$

Thank you!