Universality and delocalization of band matrix

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Joint work with
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Recall Wigner Random Matrix:

\[ H = H^\dagger, \quad H_{ij} \text{ i.i.d.} \quad \mathbb{E}(H_{ij}) = 0, \quad \mathbb{E}(|H_{ij}|^2) = 1/N \]

\[ 1 \leq i, j \leq N \]

In the last 10 years, we have seen many variations.

* Not identical distribution
* Not mean zero
* Not uniform variance
* Not quite independent
* With spikes (like E. R graph)

In a Wigner random band matrix, we have the properties 1, 3, 5, but the most important is that it is a Non-mean-field model.

\[ H_{xy} = 0, \quad \text{a.s. if } |x - y| \gg W \]

In a system with size length \( L \), there is no interaction between \( x \) and \( y \), if the distance between them is much greater than \( W \) (interaction scale).

\( W \) is called band width.
A more precise definition:

\[ H = H^\dagger, \quad H_{ij} \text{ i. d.} \quad \mathbb{E}(H_{xy}) = 0, \quad \mathbb{E}(|H_{xy}|^2) = S_{xy} \]

\[ 1 \leq x, y \leq (\mathbb{Z}_N)^d \]

\[ \sum_x S_{xy} = 1, \quad S_{xy} = O(1/W), \quad S_{xy} = 0 \text{ if } \|x - y\| >> W \]

The interesting case is

\[ W = L^a, \quad 0 \leq a \leq 1 \]
Global statistics: 2009 Erdos, Yau and Y:

Semi-Circle law holds up to scale $W^{-d}$.

Interest property:

But it could be still a band matrix, even if the power $K$ is very very large.
$d=1$

\[ K \to \text{infinity} \]

\[ \begin{array}{c c c c}
    & & & 0 \\
    L & w & & \\
    & & 0 & \\
\end{array} \quad \to \quad
\begin{array}{c c c c}
    & & w^2 & \sim 0 \\
    & & & \\
\end{array} \quad \sim 0

The reason is that in this case most eigenvectors of $H$ are localized in the scale of $W^2$:

\[ \sim 0 \quad (////) \quad \sim 0 \]

$W^2$

Recall:

\[ H^k = \sum_{\alpha} \lambda_{\alpha}^k v_\alpha^\dagger v_\alpha \]

On the other hand, physicists want to know why sometimes it is \textit{delocalized}
30 years ago, numerical calculations (P.R.L) and non-rigorous super symmetry (P.R.L) arguments showed that there is a phase transition between localization and delocalization.

**Random Band Matrix Conjecture (bulk)**

\[ d = 1 \quad W \sim L^{1/2} \]

\[ d = 2 \quad W \sim (\log L)^{1/2} \]

\[ d \geq 3 \quad W \gg 1 \quad (W \geq 3) \]

It is also the threshold for local statistics:

- Small \( W \): Poisson distribution
- Large \( W \): RM distribution (Sine-kernel)

The main new work for this talk is: 2020, Yau, Yang and Y

**Delocalization of eigenvectors** holds if

\[ d \geq 10, \quad W = L^a, \quad a > 0 \]
A quick review on previous results

Universality part:

\[ W \sim L \quad \text{Bourgade, Erdos, Yau and Y 2015} \]

\[ W \gg L^{(3/4)} \quad \text{Bourgade, Yau, Yang and Y 2017-18} \]

Localization part: for \( d=1 \) Gaussian band matrices

\[ W \ll L^{(1/8)} \quad \text{J. Schenker} \]

\[ W \ll L^{(1/7)} \quad \text{R. Peled, J. Schenker, M. Shamis and S. Sodin} \]
For delocalization part:  \( d=1 \)

\[ W \gg L^{(6/7)} \quad \text{Erdos and Knowles} \]

\[ W \gg L^{(4/5)} \quad \text{Erdos, Knowles, Yau and Y.} \]

\[ W \gg L^{(7/9)} \quad \text{Y. He and M. Marcozzi} \]

\[ W \gg L^{(3/4)} \quad \text{Bourgade, Yau, Yang and Y} \]

For Edge eigenvector:

\[ W \sim L^{(5/6)} \quad \text{S. Sodin (moment method)} \]
For supersymmetry:

When the entries are Gaussian with some specific covariance profile, one can apply supersymmetry techniques.

- $W \sim L$  
  T. Shcherbina 2014

- $W \sim L^{(6/7)}$  
  Bao and Erdos 2017

- $W \sim L^{(1/2)}$  
  T. Shcherbina and M. Shcherbina 2017, 2019

- $W \sim L^{(1/2)}$  
  M. Disertori, M. Lohmann, S. Sodin 2018
For delocalization, \( d > 1 \), \( W = L^a \)

\[
a > \frac{6}{d + 6} \quad \text{L. Erdos and A. Knowles}
\]

\[
a > \frac{d + 2}{2d + 2} \quad \text{L. Erdos, A. Knowles, Yau and Y}
\]

\[
a > \frac{d + 1}{2d + 1} \quad \text{Y. He and M. Marcozzi}
\]

\[
a > \frac{2}{2d + 2} \quad \text{Yang and Y. 2019-2020}
\]

Main new result of this talk

\[
a > 0, \ d \geq 10 \quad \text{Yau, Yang and Y 2020}
\]

Hope to finish the proof for \( d \geq 6 \) in the next year.
Resolvent Tool

\[ G = (H - z)^{-1}, \quad z \in \mathbb{C} \]

Delocalization of eigenvectors is implied by the following sense of delocalization of resolvent:

\[ \exists \eta > 0, \text{ s.t. } \forall \ E \in (-2, 2), \ \forall \ \ell \ll L \]

\[ \max_x \frac{\sum \limits_{y-x \leq \ell} |G_{xy}(z)|^2}{\sum \limits_{y} |G_{xy}(z)|^2} \ll 1, \quad z = E_0 + i\eta \]

Usually people choose \( \eta \sim W^2/L^2 \)

When \( \eta \) is larger, the above inequality does not hold.
But actually resolvent polynomially decays (slower than power of $d$)

$$|G_{xy}|^2 \sim \frac{L}{W^2}, \quad d = 1$$

$$|G_{xy}|^2 \sim \frac{1}{W^2}, \quad d = 2$$

$$|G_{xy}|^2 \sim \frac{1}{W^d + W^2|x-y|^{d-2}}, \quad d \geq 3$$

The main contribution comes from $|x-y| \sim L$ (min part)

Therefore, really we need to prove this profile for delocalization.
Previous tool / work

\[ L_{\text{max}} : \max_{x \neq y} |G_{xy}| \sim W^{-\frac{d}{2}} \] (highly non-trivial)

\[ L_2 : \sum_y |G_{xy}|^2 \sim 1/\eta \]

In this work, we need to

\[ G_{xy} \sim (W^{-d/2} - \text{term}) + (W^{-d} - \text{term}) + (W^{-3d/2} - \text{term}) \]

\[ + (\ldots) + (W^{-k\cdot d/2} - \text{term}) + (\ldots) \]

\[ + (W^{-2|x-y|^{-d+2}} - \text{term}) \]

Actually, we do (term by term) prove these terms are zeros, except the last one.
Universal - Expansion

We expand the resolvent with

- some resolvent identities
- some conditional expectation tricks
- and a well designed rule on expansion order,

(which will be the main component of the first paper in this series)

Why Universal:

There is a large universal expansion, for $W = L^a$, for fixed $a > 0$, we will only use a subpart of the whole expansion.

$$\mathcal{P}_a \subset \mathcal{P}'_a, \quad a \geq a$$

The limit expansion has a very clear fractal structure

$$\lim_{a \to 0} \mathcal{P}_a = \mathcal{P}_0$$

Different $W$ cases share the same expansion is a very important property for our proof. (You will see soon)
\[ T_{xy} := \sum_{y': |y' - y| \sim W} |G_{xy'}|^2 \]

\[ T = B + B \cdot A \cdot T, \quad B = S(I - S)^{-1}, \quad S_{xy} = \mathbb{E}|H_{xy}|^2 \]

(B has the profile as above)

\[ \|B\|_{L_1 \to L_\infty} \sim \eta^{-1} \]

Here \( A^{(k)} \) is the sub graphs with \( \sum_{z'} A_{zz'}^{(k)} \sim (W^{-d/2})^k \)

With some non-trivial cancellation (i.e., Z lemma idea), one can show \( A^{1} \) term is zero.

Therefore, it would be an effective expansion if

\[ \eta^{-1} W^{-d} \ll 1, \quad \text{i.e.,} \quad L \ll W^{\frac{d+2}{2}} \]
Complexity of the expansion

Even for such simple case:

Each free index: \( K^d \)
Each G edge: \( K^{-d/2+1} \) if \(|x-y| \sim K\)
Each B, T edge: \( K^{-d+2} \)

Expansion seems not a good idea, since one extra edge cannot cancel the factor contributed by one free index.

The expansion does not produce enough edges.

The more expansion (with no correct plan),
the messier (out of control) the graphs will be.
The number of graphs

$k$-th order: $A^k$ has about $(2k)^{3k}$ graphs.

$k=2$, with some simplification,

there are about hundreds of graphs

$k=3$, with help of computer (Matlab),

it took iMac 1 hr on computation and me 1 month in coding

$k=4$, computer basically never stop
Due to shortage of time, I will only focus on one of main novelty in the proof. The whole proof is composed of 3 papers (or more).

\[ \sum \text{ ZERO} \]

\[ L^d \eta^{-1} + L^d \eta^{-1} + \eta^{-2} \sum_{zz'} A_{zz'}^{(2)} \]

It turns out if we choose

\[ \frac{L^2}{W^2} \cdot W^{-3d/2} \ll 1, \quad \frac{L^2}{W^2} \cdot W^{-d} \gg 1 \]

Then

\[ \frac{L^2}{W^2} \sum_{z'} \left( A_{z}^{(2)} \right)^{z'} \ll 1 \]
\[ L^2/W^2 \sum_{z'} A^{(2)} z' = 0 \]

**SUM ZERO**

Example:

\[ \sum_{z''} - \sum_{z''} \]

\[ S - \text{edge} \]

**Why is Universal so important:**

For certain L, we obtain the **sum zero property** of some \( A^{(k)} \)

From sum zero property, we know the **cancellation** between the graphs in \( A^{(k)} \)

Since for other (larger) L’s, we use **same** graphs, so in those cases, \( A^{(k)} \) also has **sum zero** property for larger L case.
Why is it so important?

\[ \sum_{z'} B_{xz'} A_{z'z} = \sum_{z'} (B_{xz'} - B_{xz} - \partial_z B_{xz}(z' - z)) A_{z'z} \]

Thanks