A CLT for characteristic polynomial of GβE

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with Fanny Augeri and Raphael Butez

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Asymptotically Gaussian fields in random matrix theory

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Central limit theorem

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Central limit theorem $f : \mathbb{R} \to \mathbb{R}$ compactly supported, smooth. Consider

$$W_{f,N} = \sum_{i=1}^{N} f(\lambda_i) - N \int f d\sigma.$$

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Theorem (Johansson '98; β ensembles)

$W_{f,N}$ satisfies CLT, mean $(2/\beta - 1)\int f d\nu$, variance

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CLT's of this type go back at least to CLT of Jonsson for moments ('82), Pastur and co-workers, Bai-Silverstein, Shcherbina, ...

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Our basic object of interest: $\log|P_N(z)| = \log|\det(zI - X_N)|$.

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For $z \in (-2, 2) \setminus \{0\}$, define $\omega_k = z\sqrt{n/k}$, $k_0 = z^2 n/4$, and $\alpha(\omega_k) = \omega_k/2 + \sqrt{\omega_k^2/4 - 1}$.

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The α rescaling is natural as it relates to eigenvalues of certain transfer matrices. At exponential scale, the product of $\alpha's$ relates to the logarithmic potential of the semicircle.

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I will describe the proof, after a short digression toward circular ensembles.

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If it is log-correlated, what about the extrema?

CUE char poly

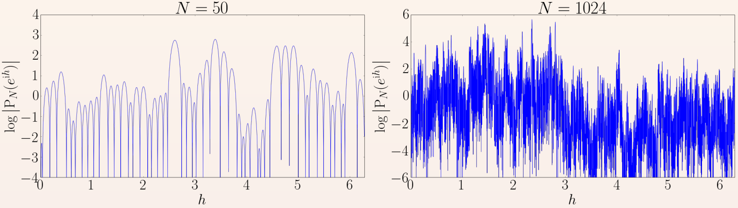

Figure 1: Realizations of $\log |\mathrm{P}_N(e^{\mathrm{i}h})|$, $0 \le h < 2\pi$, for $N = 50$ and $N = 1024$. At microscopic scales, the field is smooth away from the eigenvalues, in contrast with the rugged landscape at mesoscopic and macroscopic scales.

(From Arguin, Belius, Bourgade '17)

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Both use in essential way CUE (aka $\beta = 2$), where joint distribution of eigenvalues is

$$\prod_{i<j} |\lambda_i - \lambda_j|^2$$

for which Gaussianity of traces follows from Diaconis-Shashahani and moments of determinant (=exponential moments of $M_N(z)$) are Toeplitz determinants.

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$\alpha_k = B_k e^{2\pi i \theta_k}$, $EB_k^2 \sim 2/\beta k$, beta variable. $\alpha_k \sim g_k + i g_k'$, Gaussian.

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$\alpha_k = B_k e^{2\pi i \theta_k}$, $EB_k^2 \sim 2/\beta k$, beta variable. $\alpha_k \sim g_k + i g_k'$, Gaussian.
In addition, $\sup_{|z|=1} |\log|M_N(z)| - \log|\Phi_k^*(z)||$ is tight.

Recursions - Circular ensembles

$$\log \Phi_k^*(e^{i\theta}) - \log \Phi_{k-1}^*(e^{i\theta}) = \log(1 - \alpha_j e^{i\Psi_{k-1}(\theta)}) \sim -\alpha_j e^{i\Psi_{k-1}(\theta)}$$

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Chhaibi-Najnudel '19 $P_N(\cdot)$ converges to the GMC with parameter $\sqrt{2/\beta}$.

$\beta = 2$: Nikula, Saksman, Webb '18, Webb '15

Recursions - Circular ensembles

$$\log \Phi_k^*(e^{i\theta}) - \log \Phi_{k-1}^*(e^{i\theta}) = \log(1 - \alpha_j e^{i\Psi_{k-1}(\theta)}) \sim -\alpha_j e^{i\Psi_{k-1}(\theta)}$$

$$\Psi_k(\theta) = \Psi_k(\theta) + \theta - 2\Im \log(1 - \alpha_j e^{i\Psi_{k-1}(\theta)}).$$

Thus, marginal of $\log |\Phi_N^*(e^{i\theta})|$ is essentially Gaussian, of variance $(2/\beta) \log N$.
Log correlated, but joint law is not Gaussian.
Use a branching structure.
Chhaibi-Najnudel '19 $P_N(\cdot)$ converges to the GMC with parameter $\sqrt{2/\beta}$.

$\beta = 2$: Nikula, Saksman, Webb '18, Webb '15
Work in progress: Paquette-Z ('20?) Convergence in law of $\max \log |\Phi_N^*(e^{i\theta})|$ to Gumbel shifted by (unknown) r.v..
Some new phenomena for log-determinant of random permutations: Cook-Z. '20

Back to GβE

We take $X_N \sim$ GβE, ie joint distribution of eigenvalues on $\mathbb{R}^N$:

$$\prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\beta \frac{N}{4} \sum \lambda_i^2}.$$

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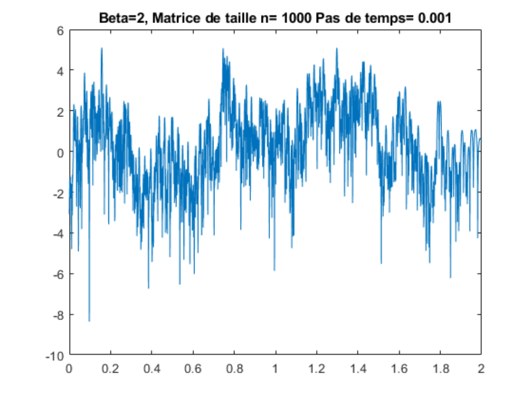
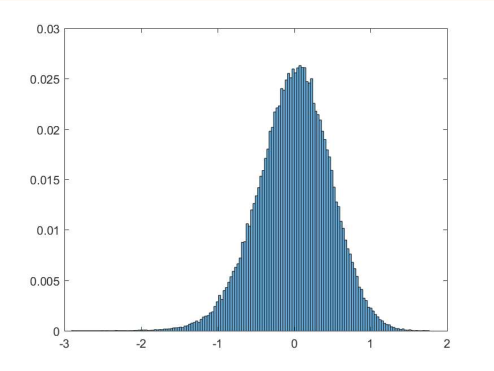
Recent result of Claeys, Fahs, Lambert, Webb: sharp CLT's for counting functions, GMC convergence.

log-det trajectory



Empirical facts



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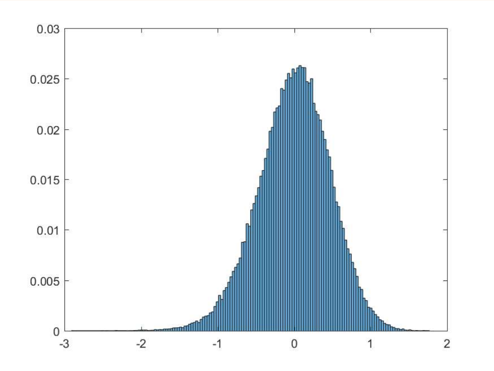
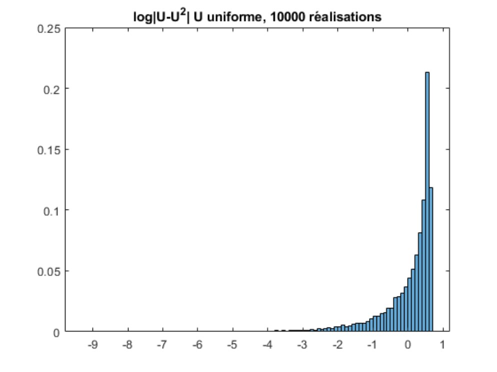


Skewed?

Reason for skewness in simulations



CLT for log determinant GβE

The case $z = 0$ is special.

Theorem (Tao-Vu '11)

$(M_N(0) - aN - b\log N)/\sqrt{\beta \log N}$ converges (for Wigner matrices, 4 matching moments) to standard Gaussian.

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Bourgade-Mody '19: extends w/out matching 4 moments.
By replacement principle, the key step in the TV proof is the result for GβE, $\beta = 1, 2$. Their proof extends to general $\beta > 0$, and is based on recursions.

The Dumitriu-Edelman representation

Theorem (Dumitriu-Edelman '05)

X_N from $G\beta E$ is unitarily equivalent to the following 3-diagonal Jacobi matrix

$$\frac{1}{\sqrt{N}} X_N = \frac{1}{\sqrt{N}} \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \mathbf{0} & a_{N-1} & b_N \end{pmatrix}$$

where $b_i \sim N(0, \sqrt{2/\beta})$, $a_i \sim \chi_{i\beta}/\sqrt{\beta}$.

Here $a_i \sim \chi_{i\beta}/\sqrt{\beta}$; here $\chi^2_{i\beta}$ has chi-square distribution with $i\beta$ degrees of freedom, ie $\chi_{i\beta}/\sqrt{\beta} \sim \sqrt{i\beta} + G/\sqrt{2\beta} + O(1/i)$.

Recursions

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Recall: $\omega_k = z\sqrt{n/k}$, $k_0 = z^2 n/4$, and $\alpha(\omega_k) = \omega_k/2 + \sqrt{\omega_k^2/4 - 1}$ if $k < k_0$, $\alpha(\omega_k) = 1$ if $k \geq k_0$.

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We set

$$\Psi_k(z) = \phi_k(z\sqrt{N})\frac{1}{\sqrt{k!}\prod_{i=1}^{k}\alpha(\omega_i)}$$

and then

$$\Psi_k(z) = \frac{z\sqrt{N} - b_k}{\sqrt{k}\alpha(\omega_k)}\Psi_{k-1}(z) - \frac{a_{k-1}^2}{\sqrt{k(k-1)}\alpha(\omega_k)\alpha(\omega_{k-1})}\Psi_{k-2}(z).$$

Recursions

Recall that k_0 satisfies $\omega_{k_0} = 2$ (if $z = 0$ then $k_0 = 1$). In matrix form, for $k \geq k_0$,

$$\begin{pmatrix} \Psi_{k+1}(z) \\ \Psi_k(z) \end{pmatrix} \sim \begin{pmatrix} \omega_k & -1 + 1/2k \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_k(z) \\ \Psi_{k-1}(z) \end{pmatrix} + \begin{pmatrix} b_k/\sqrt{k} & g_k/\sqrt{k} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Psi_k(z) \\ \Psi_{k-1}(z) \end{pmatrix}$$

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In the Tao-Vu $z = 0$ case, $\omega_k = 0$, and except for perturbation, we have a pure rotation.
Tao-Vu show that $\Psi_{k-1}(z)^2 + \Psi_{k-1}(z)^2$ (essentially) forms a martingale with quadratic variation process of increment $\sim 1/k$. This gives the CLT.

$$\begin{pmatrix} \Psi_{k+1}(z) \\ \Psi_k(z) \end{pmatrix} = A_k \begin{pmatrix} \Psi_k(z) \\ \Psi_{k-1}(z) \end{pmatrix} + E_k \begin{pmatrix} \Psi_k(z) \\ \Psi_{k-1}(z) \end{pmatrix}$$

where

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and E_k is a small noise matrix.

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The eigenvalues of A_k are roughly $\frac{1}{2}\omega_k \pm \frac{1}{2}\sqrt{\omega_k^2 - 4}$.
For $k < k_0$, eigenvalues real and smaller that 1.
For $k > k_0$, eigenvalues imaginary, of modulus roughly 1.

Recursions - general z

There are several regimes to consider. Fix $\epsilon > 0$, recall that $k_0 = z^2 N/4$.

- $k < (1-\epsilon)k_0$: one checks that $\Psi_k(z) \sim 1$.
- $k \in [(1-\epsilon)k_0, k_0]$: write

$$X_k = \Psi_k/\Psi_{k-1} = 1 + \delta_k, \quad X_k = A_k + B_k/X_{k-1}$$

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for appropriate A_k, B_k. In this regime, $\delta_k \sim 0$ and one obtains a recursion

$$\delta_k \sim u_k + v_k \delta_{k-1}$$

where, with $\alpha_k = \alpha(\omega_k)$,

$$u_k \sim \frac{b_k}{\sqrt{k\alpha_k^2}} + \frac{1}{2k\alpha_k^2} - \frac{g_k}{\sqrt{k\alpha_k^4}}, \quad v_k = \frac{1 - \frac{1}{2k} + \frac{g_k}{\sqrt{k}}}{\alpha_k^2},$$

which one solves.

- $k > k_0$: Oscillatory regime, most interesting.

Recursions - general z - the scalar regime

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$\bar{\delta}_k$ is a martingale, and small. We can compute $\sum \bar{\delta}_k$, and $\bar{\delta}_k$ are correlated!

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Using a-priori bounds on $\bar{\delta}_k$, control size of Δ_k, and of the error terms.

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At the end, Δ_k contributes only to the mean.We conclude that

$$\log \Psi_{k_0 - k_0^{1/3}} = \sigma_n G - \frac{1}{2}\sigma_n^2, \quad \delta_{k_0 - Ck_0^{1/3}} = O\left(\frac{1}{k_0^{1/3}}\right)$$

where $\sigma_n^2 = \frac{2}{3\beta} \log n$ and G is a standard Gaussian.

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$$\bar{\delta}_k = u_k + v_k \bar{\delta}_{k-1}, \quad \bar{\delta}_k = \sum_{j=2}^{k} u_j \prod_{\ell=j+1}^{k} v_\ell .$$

Turns out contribution occurs only for $k < k_0 - k_0^{1/3}$, and then get a CLT with blocks of length $(k_0/i)^{1/3}$ to the left of k_0 contributing order $1/i$ to the variance. Also, correlation between different z's computable.
We need to control $\Delta_k = \delta_k - \bar{\delta}_k$. Using a-priori bounds on $\bar{\delta}_k$, control size of Δ_k, and of the error terms.
At the end, Δ_k contributes only to the mean.We conclude that

$$\log \Psi_{k_0 - k_0^{1/3}} = \sigma_n G - \frac{1}{2}\sigma_n^2, \quad \delta_{k_0 - Ck_0^{1/3}} = O\left(\frac{1}{k_0^{1/3}}\right)$$

where $\sigma_n^2 = \frac{2}{3\beta} \log n$ and G is a standard Gaussian.

A much finer analysis (up to $k_0 - k_0^{1/3}(\log k_0)^{2/3}$) by Lambert-Paquette (hyperbolic regime) - arXiv:2001.09042

Recursions - general z - the oscilatory regime

$$X_k = \begin{pmatrix} \Psi_{k+1} \\ \Psi_k \end{pmatrix}, k > k_0.$$

We have

$$X_{k+1} = (A_k + W_k) X_k,$$

where

$$A_k = \begin{pmatrix} \omega_k & -1 + \frac{1}{2k} \\ 1 & 0 \end{pmatrix}, \ W_k = \begin{pmatrix} \frac{-b_k}{\sqrt{k}} & \frac{g_k}{\sqrt{k}} \\ 0 & 0 \end{pmatrix},$$

$z_k = z\sqrt{\frac{n}{k}} = 2 - \frac{l}{k_0}$ and $b_k \sim \mathcal{N}(0, 2/\beta)$ and $g_k \sim \mathcal{N}(0, 2/\beta)$.

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Eigenvalues of A_k for $k > k_0$ are complex of (essentially) unit norm. Change basis to eigenvector basis, get

$$\hat{X}_k = Q_k \prod_{i=k_0}^{k-1} Q_{i+1}^{-1} Q_i \left(R_i + \hat{W}_i\right) Q_{k_0}^{-1} \hat{X}_{k_0},$$

where R_i are rotation matrices of angle $\theta_k \sim \sqrt{k/k_0 - 1}$.

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For lower bound on norm, use anti-concentration.

Recursions - general z - the oscilatory regime

Problem 2: Noncommutative product - control

$$\hat{X}_k = Q_k \prod_{i=k_0}^{k-1} Q_{i+1}^{-1} Q_i (R_i + \hat{W}_i) \hat{Y}_{k_0 + C k_0^{1/3}}$$

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Solution: along block we have $\prod R_i = I$, but the vector $(1, 0)^T$ is not mapped to $\rho_i(1, 0)^T$ due to the noise. So instead, stop (at random time) where

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Of course, cannot achieve exactly $(1,0)^T$, but can control error by choosing when to stop.

Recursions - general z - the oscilatory regime

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Within a block, linearization is a good approximation:

$$\prod_{j=\ell_i+1}^{\ell_{i+1}} Q_{j+1}^{-1} Q_j \big(R_i + \hat{W}_i\big) = \prod_{j=\ell_i+1}^{\ell_{i+1}} (I + \Delta_j)\big(R_i + \hat{W}_i\big)$$

$$= \sum_{k=\ell_i+1}^{\ell_{i+1}} \mathbf{R_k}(I + \Delta_k + \hat{W}_k)\mathbf{R_k}^{-1} + \text{error terms} = I + \sum_{k=\ell_i+1}^{\ell_{i+1}} \mathbf{R_k}(\Delta_k + \hat{W}_k)\mathbf{R_k}^{-1} + \text{error term}$$

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where $\mathbf{R_k}$ is a rotation by an angle between 0 and 2π.
Easy to compute effect of linearization, get that $\rho_i \sim 1 + g_i + c'/i$ where g_i has variance c/i.

Recursions - general z - the oscilatory regime

Caveat: Complication when blocks get too small - cannot ensure the approximation, e.g. if block is of length 1; But variance is small there, so can combine blocks!

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Working on it!