Collective behavior of self-interacting random walkers/particles

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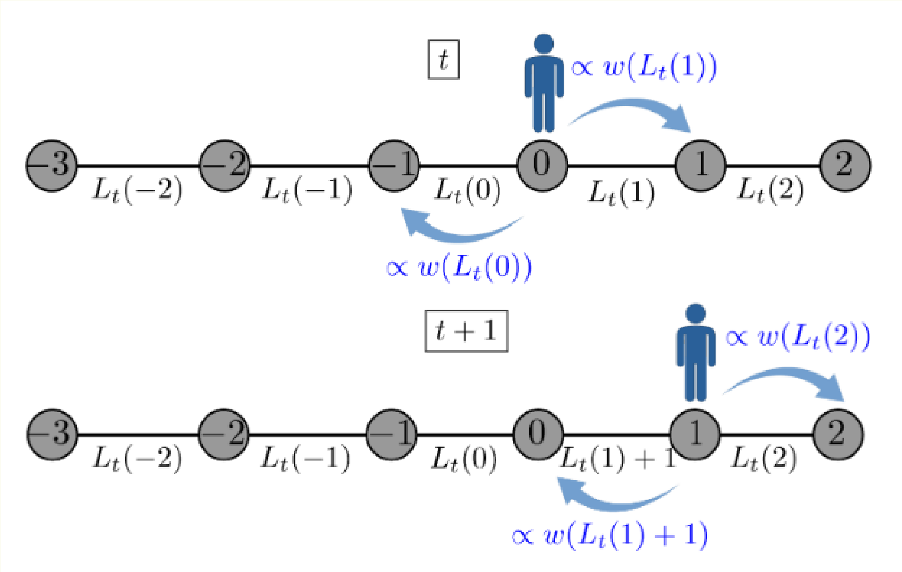
Self-interacting random walk (SIRW)

- Local time

  $L_t(x) = \{time\ spent\ at\ x\ until\ time\ t\}$
- Probability of jumping depends locally on L_t



[Brémont, PRL 2025]

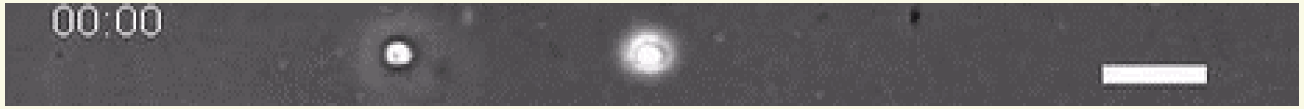
- Transition probabilities

$$P(x+1,t+1|x,t) = \frac{w\big(L_t(x+1)\big)}{w\big(L_t(x)\big) + w\big(L_t(x+1)\big)}$$

Several classes of SIRW:

- Asymptoticaly free, or “saturating” case:
  - Saturation: $w(n) \to 1$
  - Simplest model is the $SATW_\phi$:

$$w(n) = \begin{cases} 1/\phi & if\ n = 0 \\ 1 & else \end{cases}$$



[Barbier—Chebbah, Benichou, Voituriez, d’Alessandro, Ladoux (IJM), Nat. Comm. 21]

- TSAW:
  - L_t acts as a potential on the random walker
  - $w(n) = e^{-\beta n}$ with $\beta > 0$
- PSRW, SESRW…

Collective behaviors

[Experiments from K.Alim group]

- Particles interact **repulsively** with
  → their own trail
  → others' trail

- Collective behavior ≠ lonely behavior

- New mathematical difficulties

Outlines

- Part 1: Splitting probabilities and first passage times of N SATW

(with Julien Brémont)

  - Extension of the Generalized Ray-Knight theorems of B. Tóth for N SATWs

- Part 2: Hydrodynamic limit for several "TSAW-like" particles

in d-dimension (with Ram Adar)

  - Spreading in free space
  - Effective interaction in bounded space

Part 1:
Splitting probabilities and first passage times of N SATW

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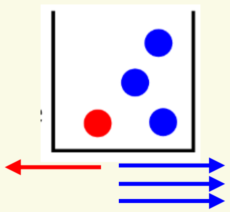
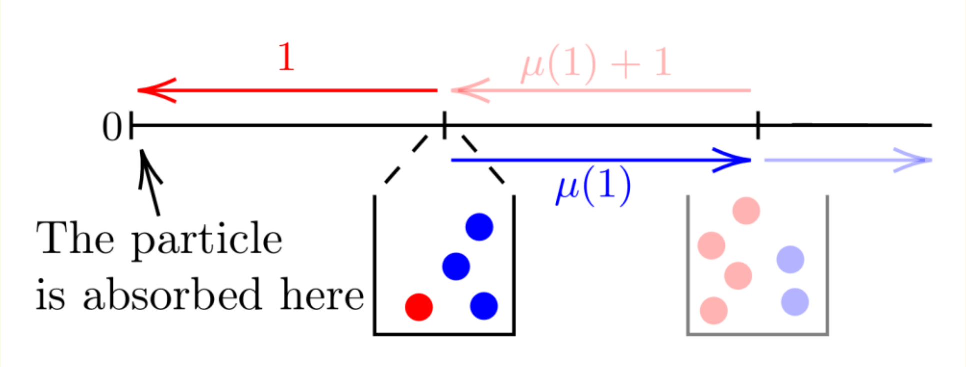
Pólya urn representation of SIRW

- Attach every site with a Pólya urn:
  - Draw red ball → jump left
  - Draw a blue ball → jump right
  - Add a ball of the same color





- Local time ≃ number of balls

- $N = 1$, if 0 is absorbent

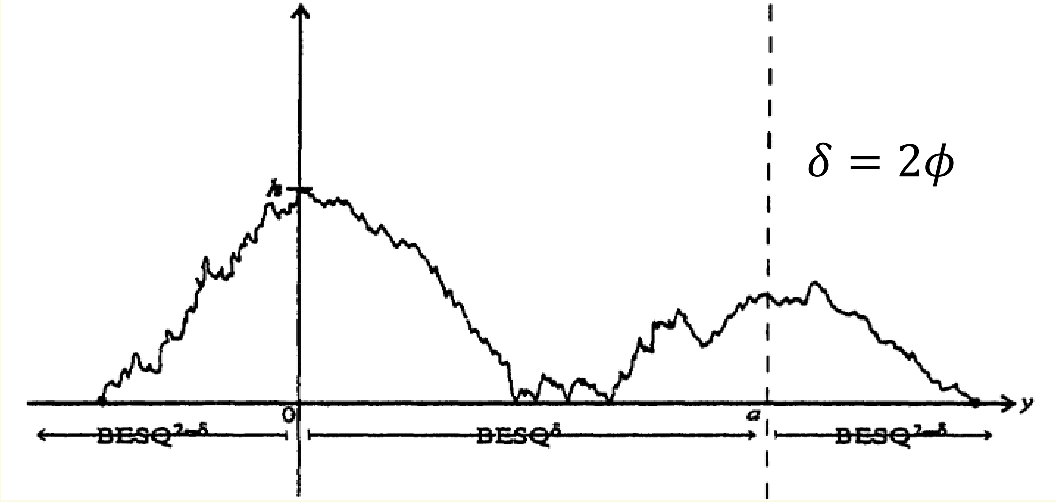
  → Propagation of information by introducing :

$\mu(m) =$ Number of blue balls knowing that there are m red balls

- The local time description is Markovian

- [B.Tóth 94,95,96]: The local time can be described with squared Bessel processes

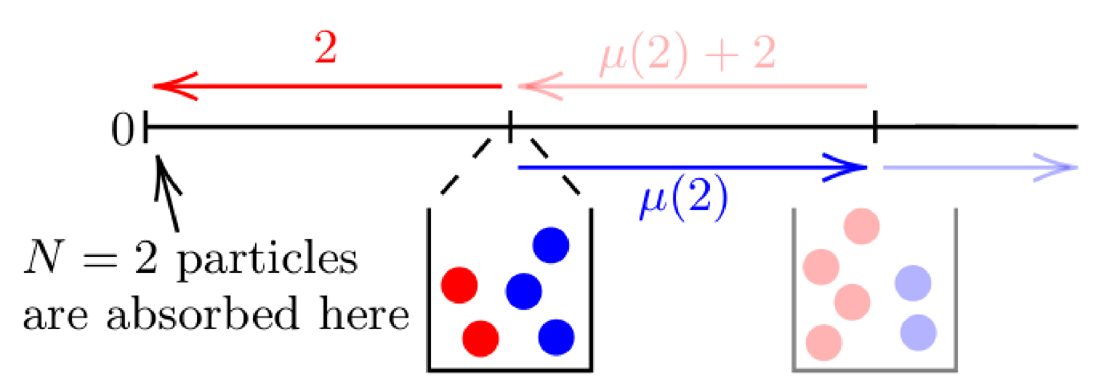
Example for the $SATW_\phi$:



- **Can we still do this kind of description with $N \geq 2$ particles ?**

Similar description for $N = 2$?

2 particles starting at $x_0 > 0$.



What are the conditions allowing this description ?

→ Important feature: Makovianity

- the urn must be blind to the way balls are drawn

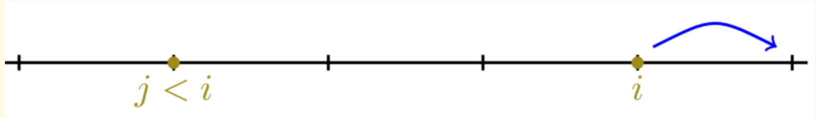
→ Consider the event

A =“first a red then a blue ball are drawn at site i”

- First situation: A single particle draws both balls

$$P(A_1) = \frac{w(0)}{w(0) + w(1)} \cdot \frac{w(1)}{w(2) + w(1)}$$

- Second situation: A first particle draws the red ball, and a second draws the blue ball



$$P(A_2) = \frac{w(0)}{w(0) + w(1)} \cdot \frac{w(2)}{w(2) + w(1)}$$

To have $P(A_1) = P(A_2)$, one needs

$$w(1) = w(2)$$

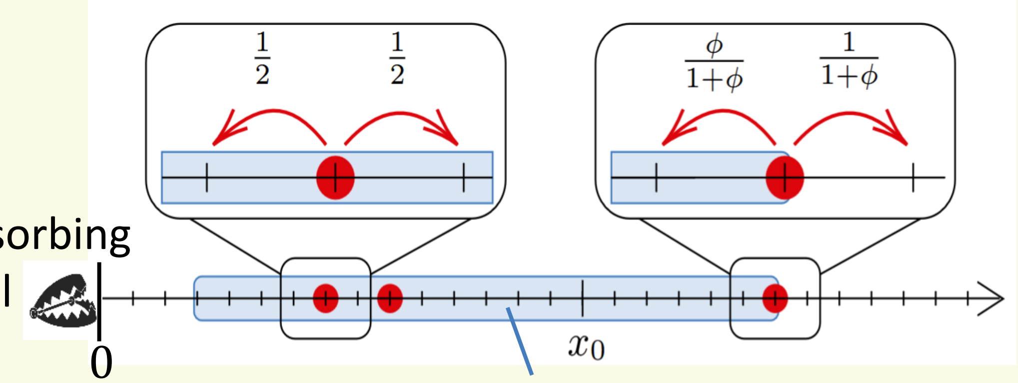
- To keep Markovianity, one needs

$$w(n) = w(1) \quad \forall n \geq 1$$

→ This description still works **only** for **SATW**

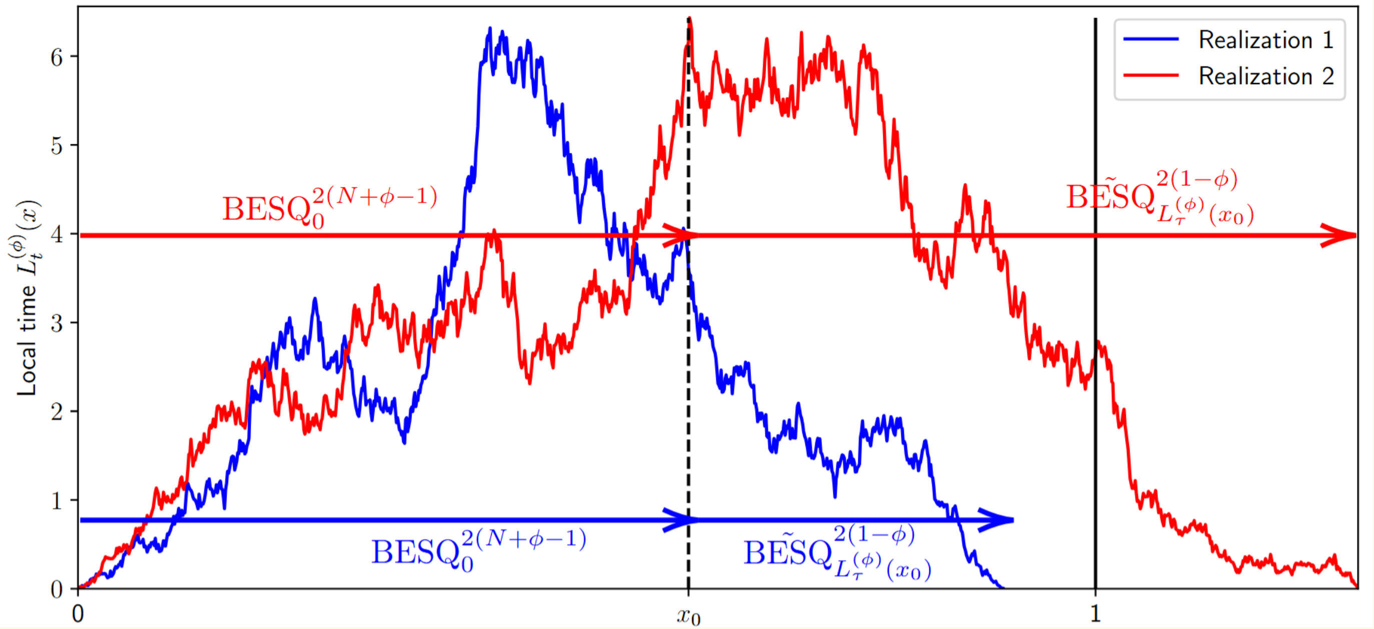
(Generalized)^2 Ray-Knight Theory and Limit Theorem for several SATWs on $\mathbb{Z}$

- Consider N $SATW_\phi$ starting at $x_0 > 0$

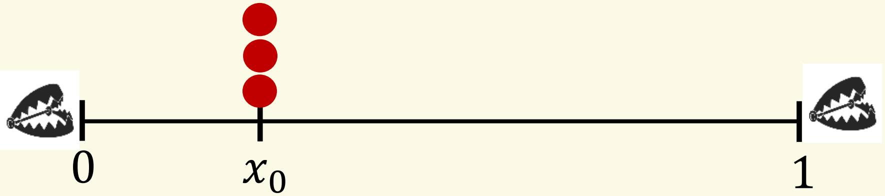


- After the N particles are absorbed at 0, the local time process follow:

$$L(x) = \begin{cases} \mathrm{BESQ}_0^{2(N+\phi-1)}(x) \text{ if } x \in [0, x_0] \\ \tilde{\mathrm{BESQ}}_{L(x_0)}^{2(1-\phi)}(x - x_0) \text{ if } x > x_0 \end{cases}$$



- Splitting probabilities:

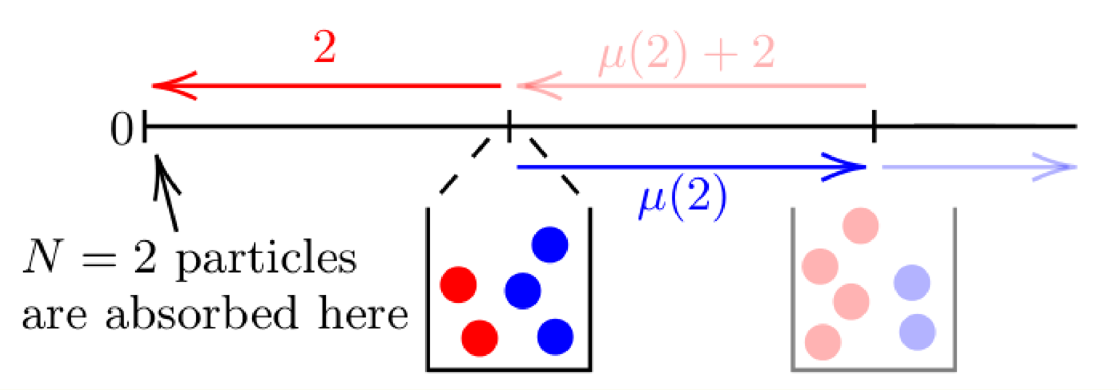


$P(+^0|x_0)$ probability that 0 walkers are trapped at 1, *ie* N walkers are trapped at 0

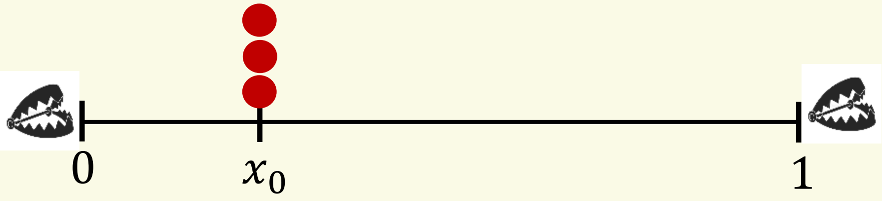
$$P(+^0|x_0) = I_{1-x_0}(N + \phi - 1, \phi)$$

where I is the regularized incomplete Beta function

Irrelevance of time in the description

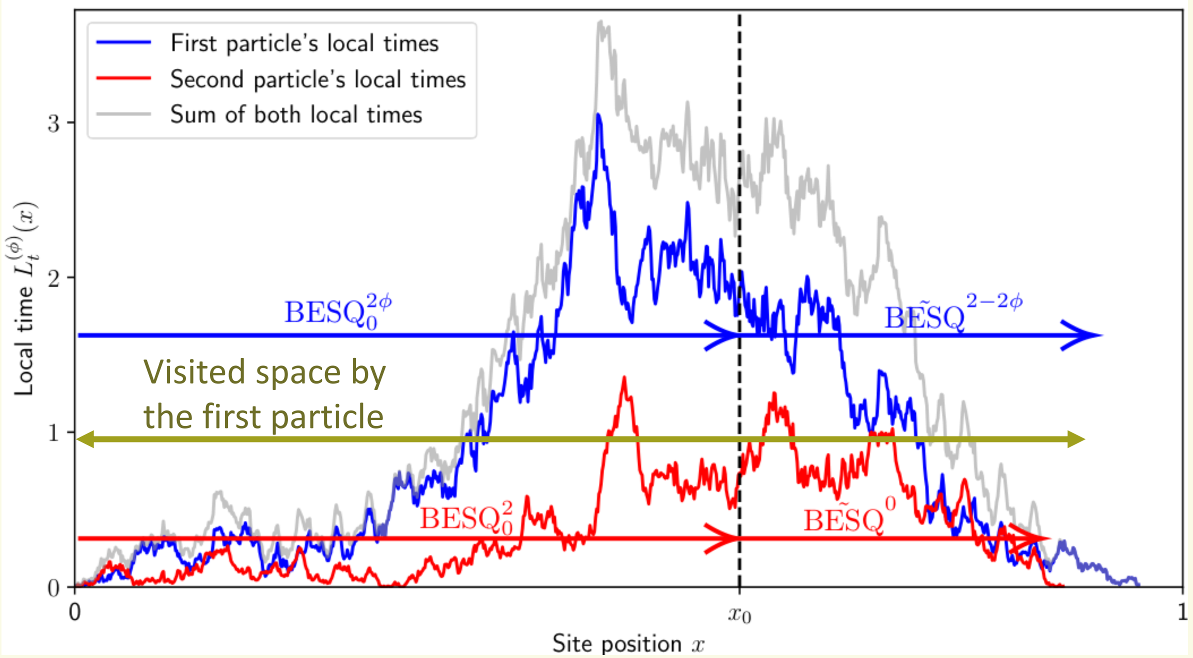


- The description get rid of the time
  - What if the walkers do not walk at the same speed ?
  - If they are launched sequentially ?



→ Local time description should not change

- Example with $N = 2$:

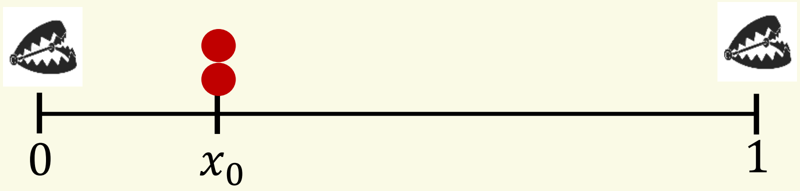


Using additivity property, [A. Göing-Jaeschke et M. Yor, 03]:

$$BESQ_a^\delta + BESQ_b^\gamma = BESQ_{a+b}^{\delta+\gamma} \text{ (in law)}$$

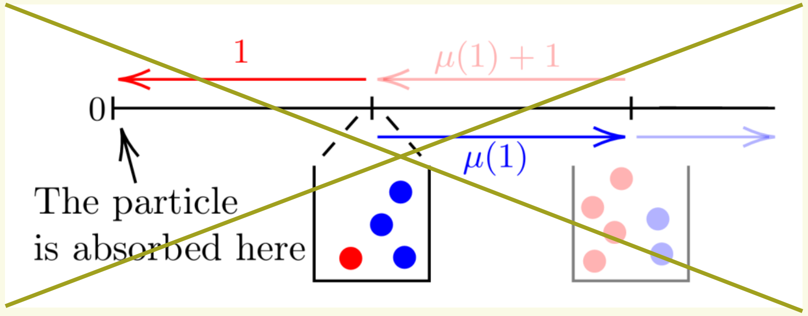
We recover the same local time formula as before.

Splitting probability $P(+^1|x_0)$



$\rightarrow N = 2, k = 1$

Can we propagate as before ?



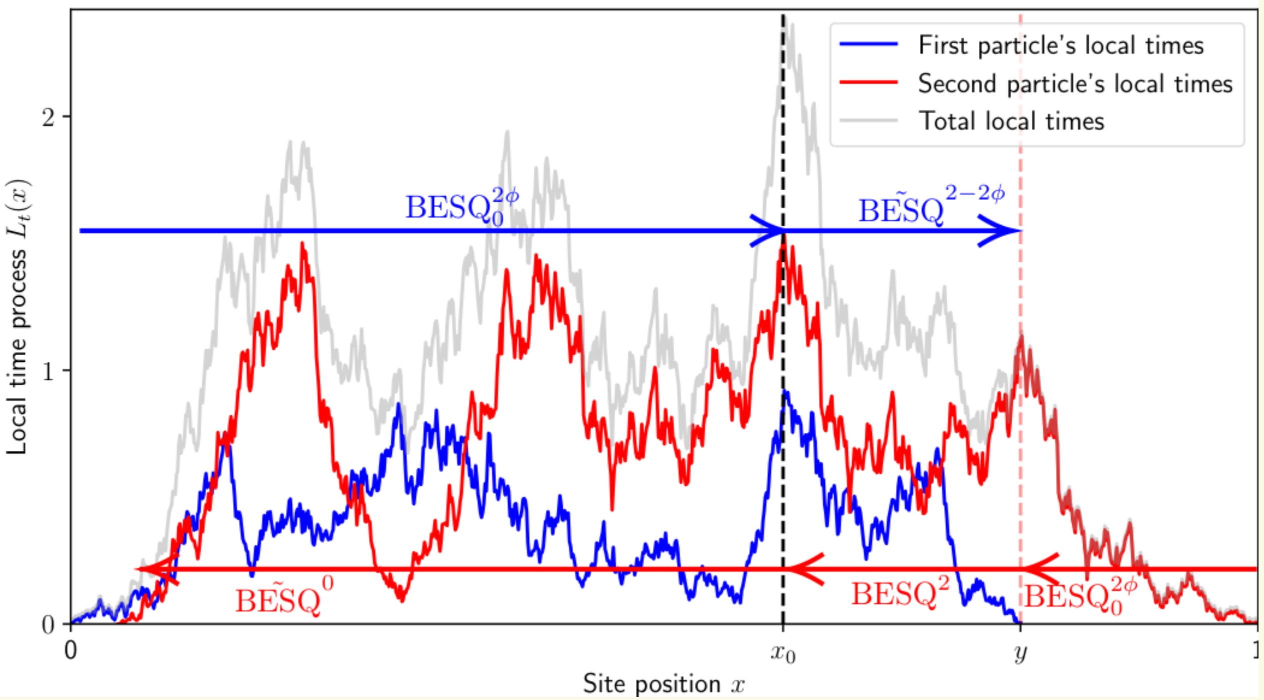
No

→ Boundary condition on the right wall makes the description inhomogeneous in space !

→ Build a space-inhomogeneous description with urns

→ The description is still independent of time

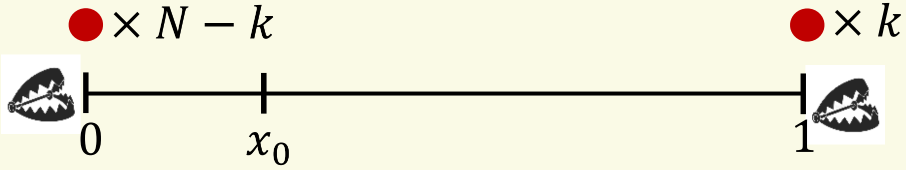
- Launch sequentially 2 walkers from x_0, and describe the local time when:
  - → the first particle is absorbed at 0
  - → the second particles is absorbed at 1



After calculation:

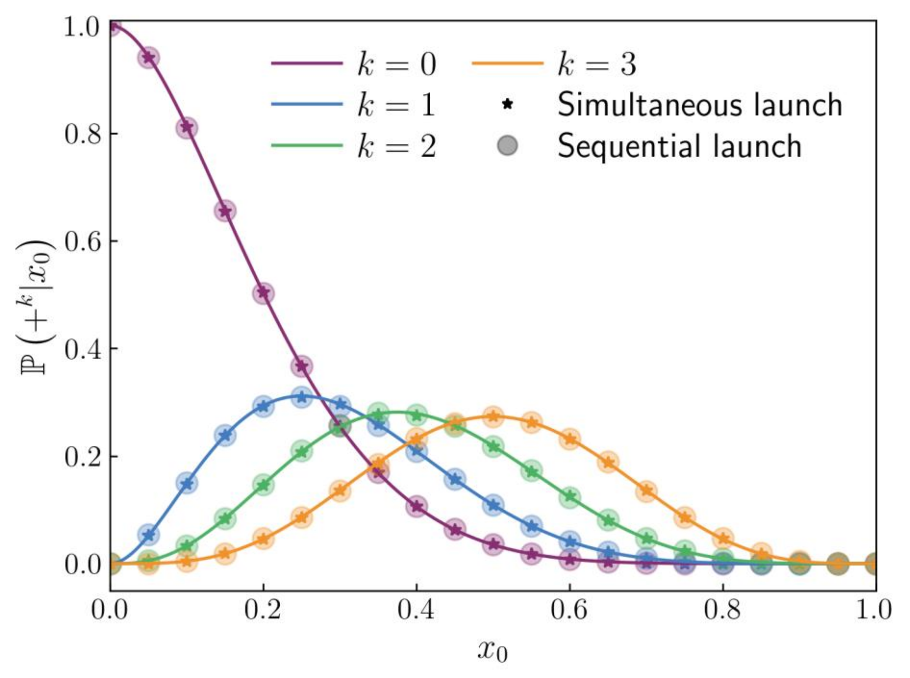
$$P(+^1|x_0) = \binom{2\phi}{\phi}\left((1-x_0)x_0\right)^{\phi}$$

Splitting probabilities $P(+^k|x_0)$



- For $1 \le k < N$:

$$P(+^k|x_0) = \binom{N + 2(\phi - 1)}{k + \phi - 1} (1 - x_0)^{N-k+\phi-1} x_0^{k+\phi-1}$$



Comparison between simulations and predictions for $\phi = 0.5$

Generalization for any saturating weight function

- As long as $w(n) \xrightarrow{n\to\infty} 1$, one can define

$$\phi = w(0)^{-1} + \sum_{j=1}^{\infty} \left(w(2j)^{-1} - w(2j-1)^{-1}\right)$$

→ the same results apply

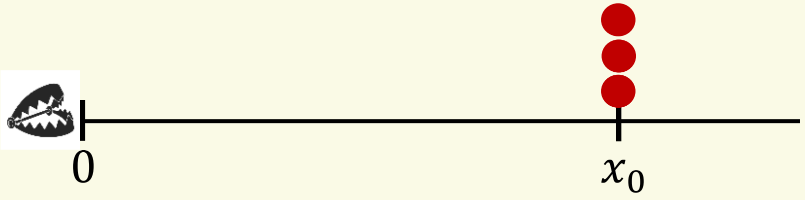
- Example:

$$w(n) = \begin{cases} 1 + \dfrac{1}{(n+4)^2} & if\ n\ even \\ 1 - \dfrac{1}{(n+4)^2} & if\ n\ odd \end{cases}$$

gives $\phi \simeq 0.7186$

→ Still works

First-passage statistics: persistent exponents



- Define the probability $S_{k,N}(t)$ that at least k walkers out of N are not trapped. Then

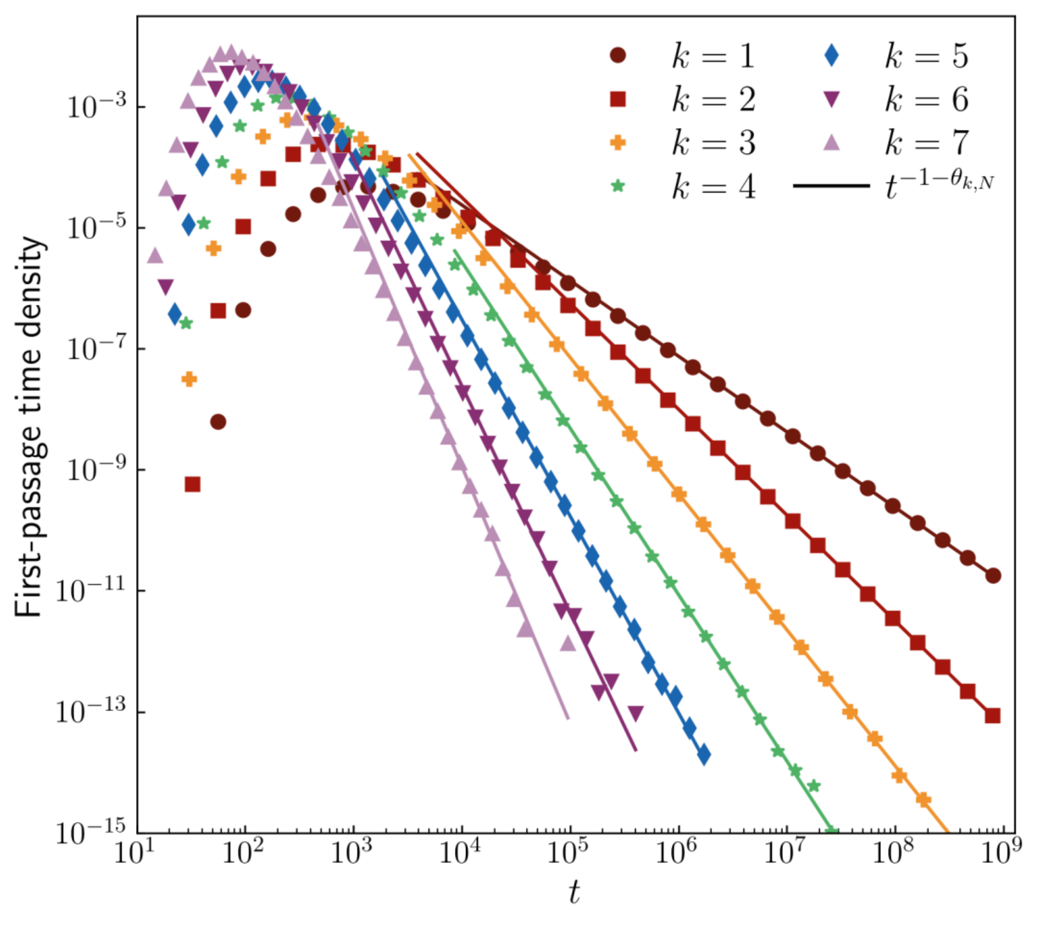
$$S_{k,N}(t) \sim t^{-\theta_{k,N}}$$

with $\theta_{k,N}$ the "generalized persistence" exponent.

- For a $SATW_\phi$:

$$\theta_{k,N} = \frac{k+\phi-1}{2}$$

- The absorbing times have tails in $\sim t^{-1-\theta_{k,N}}$



Simulations for $N = 10$ SATWs with $\phi = 0.5$

Conclusion of part 1

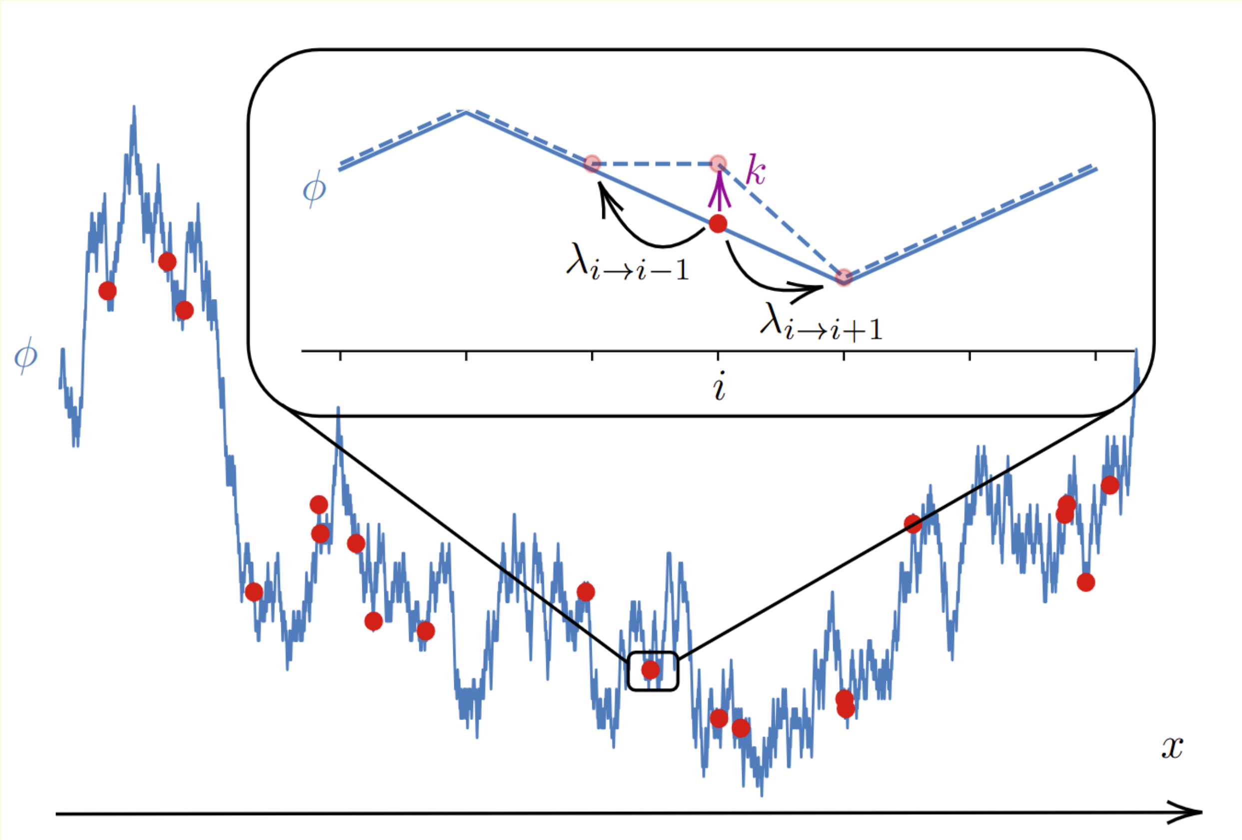
- Pólya urn description for several particles only works for SATW
- Limit process for N SATWs absorbed at 0
- Simultaneous and Sequential launch give the same results
- Splitting probabilities
- First-passage time statistics
- Generalization for any saturating weight functions (numerical)
- On arXiv very soon…

Part 2: Hydrodynamic limit for several "TSAW-like" particles in d-dimensions

Paul Pineau
Raphaël Voituriez
Ram Adar

A microscopic model

- particles **deposit** a cue field ϕ
- particles are **repelled** by this cue



ρ: density of particles

- Jump rates: $\lambda_{i\to i\pm1} \propto e^{-h\phi_{i\pm1}}$
  → ϕ **act as a potential** on the particles
  → h is the interaction parameter

- Deposition rate: k
  → Increase locally ϕ by 1 (--------- event)

Hydrodynamic equations in d dimensions:

- Field-theory methods give

Diffusion equation — Interaction term

$$\partial_t \rho = \Delta\rho + h\nabla[\rho\nabla\phi] + \nabla\left(\sqrt{2\rho}\,\eta\right)$$

$$\partial_t \phi = k\rho + \sqrt{k\rho}\,\xi$$

Gaussian white noises
$\eta(t,x)dtdx = d^2W(x,t)$

→ Non-linear coupled equation
→ Stochastic equations (η,ξ)

Scaling solutions in infinite space

- Our d-dimensional equations:

$$\partial_t \rho = \Delta\rho + h\nabla[\rho\nabla\phi] + \nabla(\sqrt{2\rho}\,\eta)$$
$$\partial_t \phi = k\rho + \sqrt{k\rho}\,\xi$$

- Scaling Ansatz:

$$\rho(t,r) = \frac{t^{-\alpha}}{r^{d-1}} f\left(\frac{r}{t^{\alpha}}\right)$$
$$\phi(t,r) = \frac{t^{1-\alpha}}{r^{d-1}} F\left(\frac{r}{t^{\alpha}}\right)$$

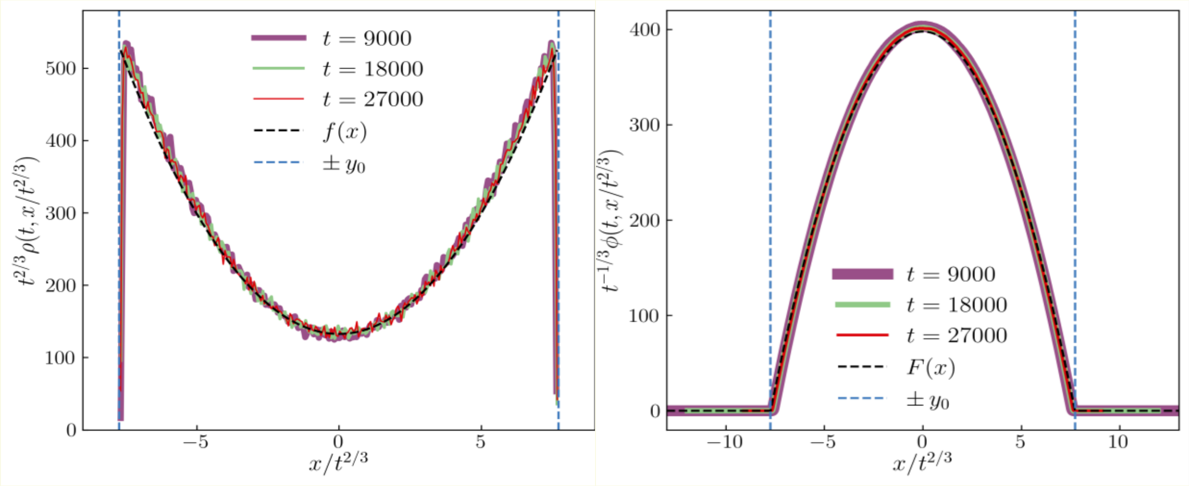
- In one dimension:

→ Superdiffusive scaling $\alpha = 2/3$

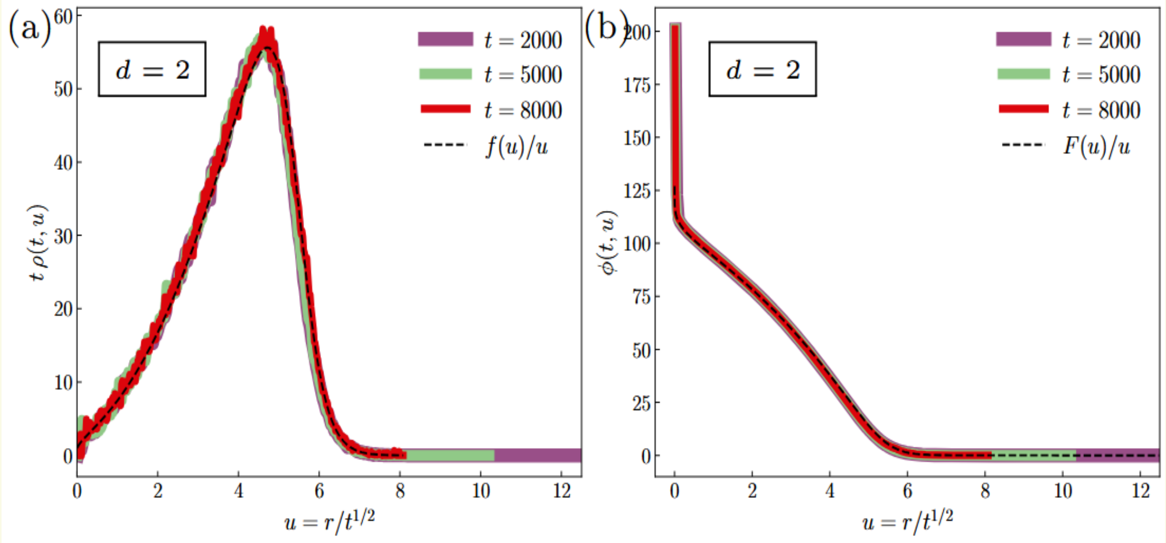
→ [Tóth, Werner '02]'s 1D equations :

$$\partial_t \rho = h\nabla[\rho\nabla\phi]$$
$$\partial_{xt}\phi = k\partial_x \rho$$

→ Solve it in similar way of [Maggs,'26]

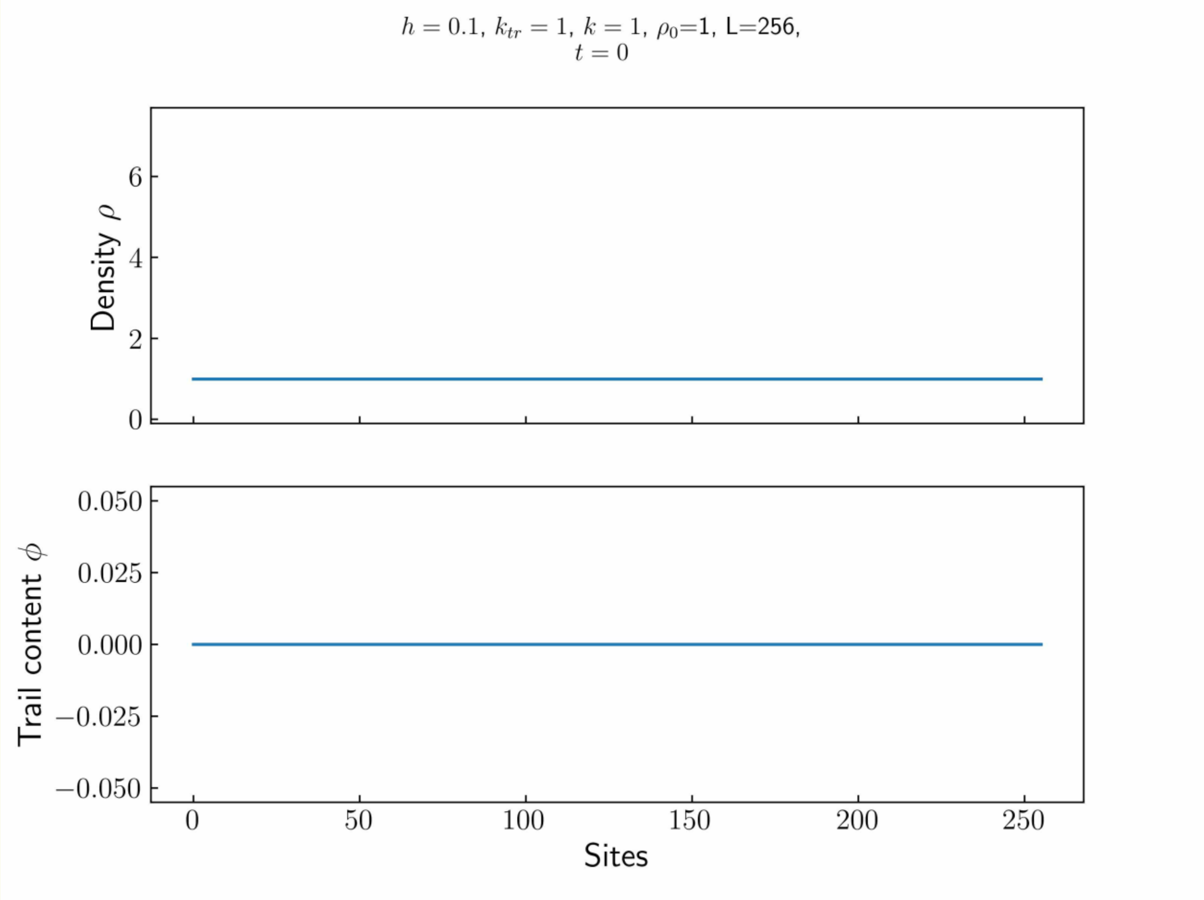


- In two dimensions:
  - $\alpha = 1/2$: All the terms are important
  - Solve the equations:



- No log corrections ?

In periodic boundary conditions:



- Growing fluctuating interface ϕ
- Linearize the equations: $\delta\phi = \phi - \langle\phi\rangle$

$$[\partial_{tt} + q^2\partial_t + khq^2]\,\delta\phi = \frac{\Lambda}{\sqrt{N}}$$

with $\Lambda = \sqrt{k}(\partial_t\xi + q^2\xi - i\sqrt{2k}q\eta)$

- Calculate the Green function G

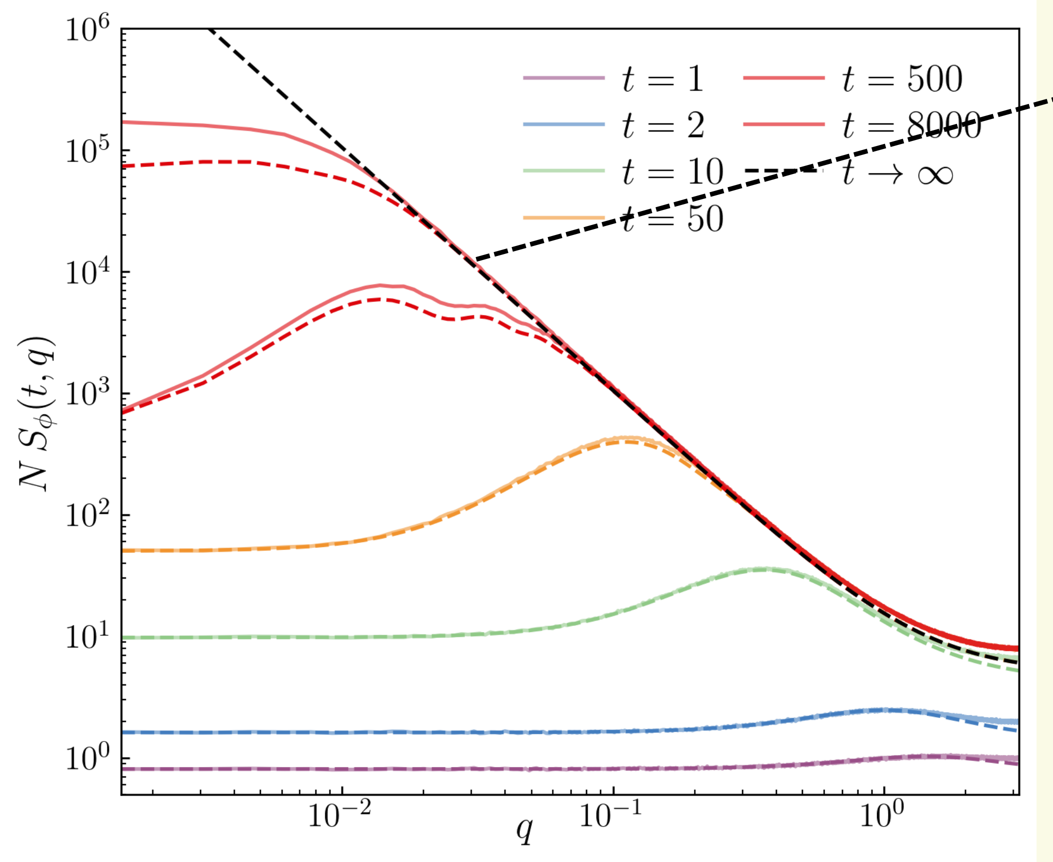
  → System reaches a **steady-state**

  → $\delta\phi(\mathrm{t,q}) = \frac{1}{\sqrt{N}}\int_0^t \Lambda(\mathrm{s,q})G(\mathrm{t-s,q})\mathrm{ds}$

→ Static structure factors of ϕ:

$$S_\phi(t,q) = \frac{1}{L}\langle|\delta\phi(t,q)|^2\rangle = \int_0^L \langle\delta\phi(t,x)\delta\phi(t,x+z)\rangle e^{iqz}dz$$

Structure factor of ϕ



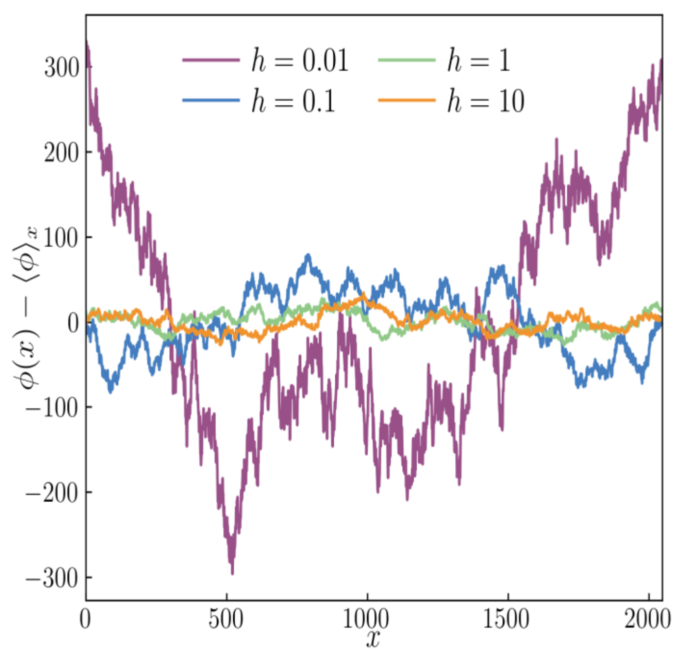
Comparison between theoretical predictions (plain lines) with theoretical predictions (dotted lines)

(works for $d \in \{1,2\}$)

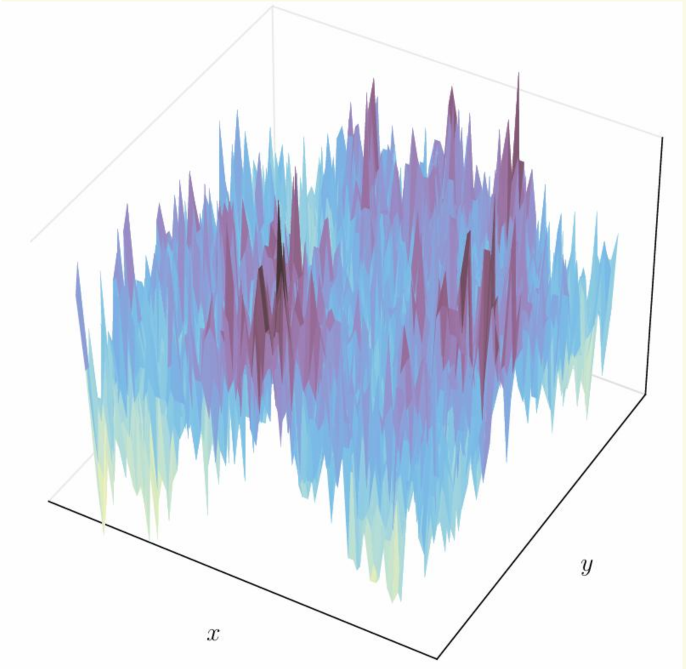
- $S_\phi(t,q) \xrightarrow[t\to\infty]{} \frac{k(2+h)+q^2}{2hq^2} \sim_{q \ll \sqrt{k(2+h)}} \frac{k(2+h)}{2h} \cdot \frac{1}{q^2}$

→ Edward Wilkinson interface

$d = 1$: Brownian motion



$d = 2$: Gaussian free field



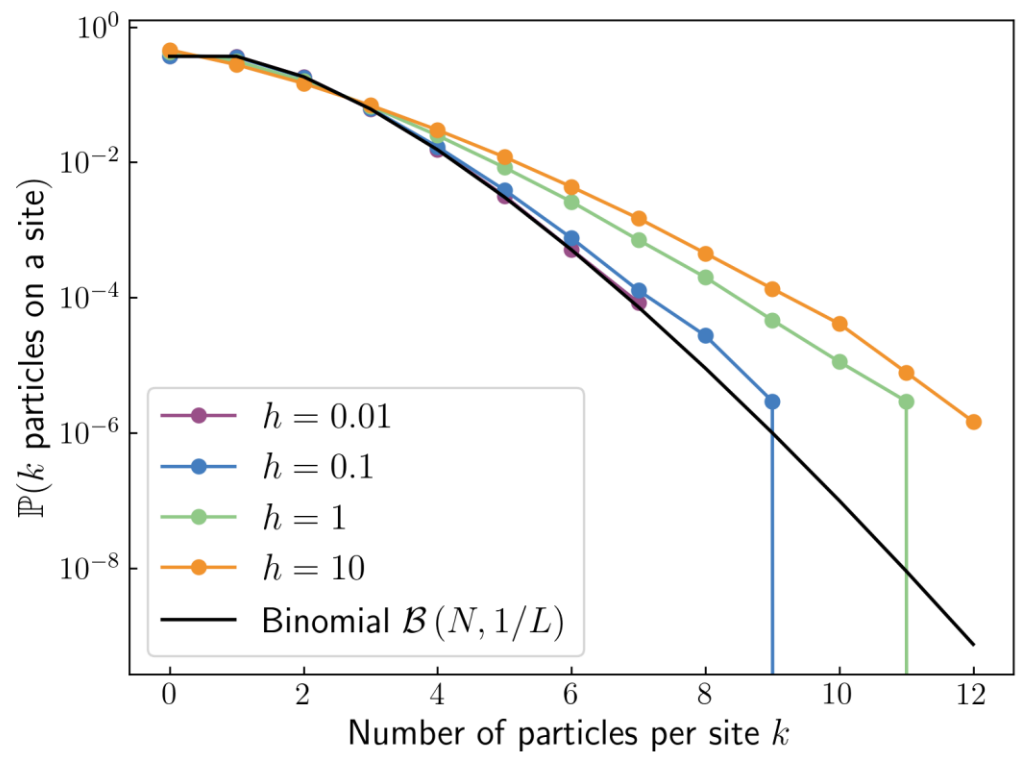
- Open questions (future directions)
  - → Is ϕ a KPZ interface in 1d
  - → How to characterize it ?

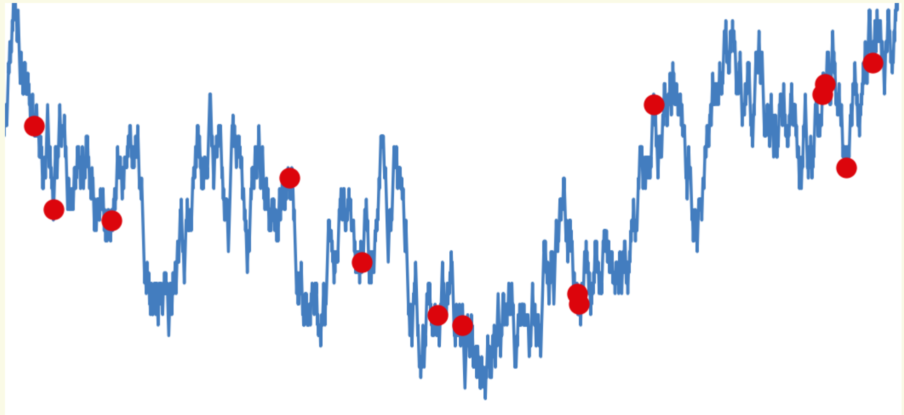
Structure factor of ρ

- $S_\rho(t,q) = \frac{1}{L}\langle|\delta\rho(t,q)|^2\rangle$
- $N\, S_\rho(t,q) \xrightarrow[t\to\infty]{} 1 + h/2$
- Particles exhibits surprising effective interactions:

  → **attractive interaction** if on the same site

  → independent if not on the same site



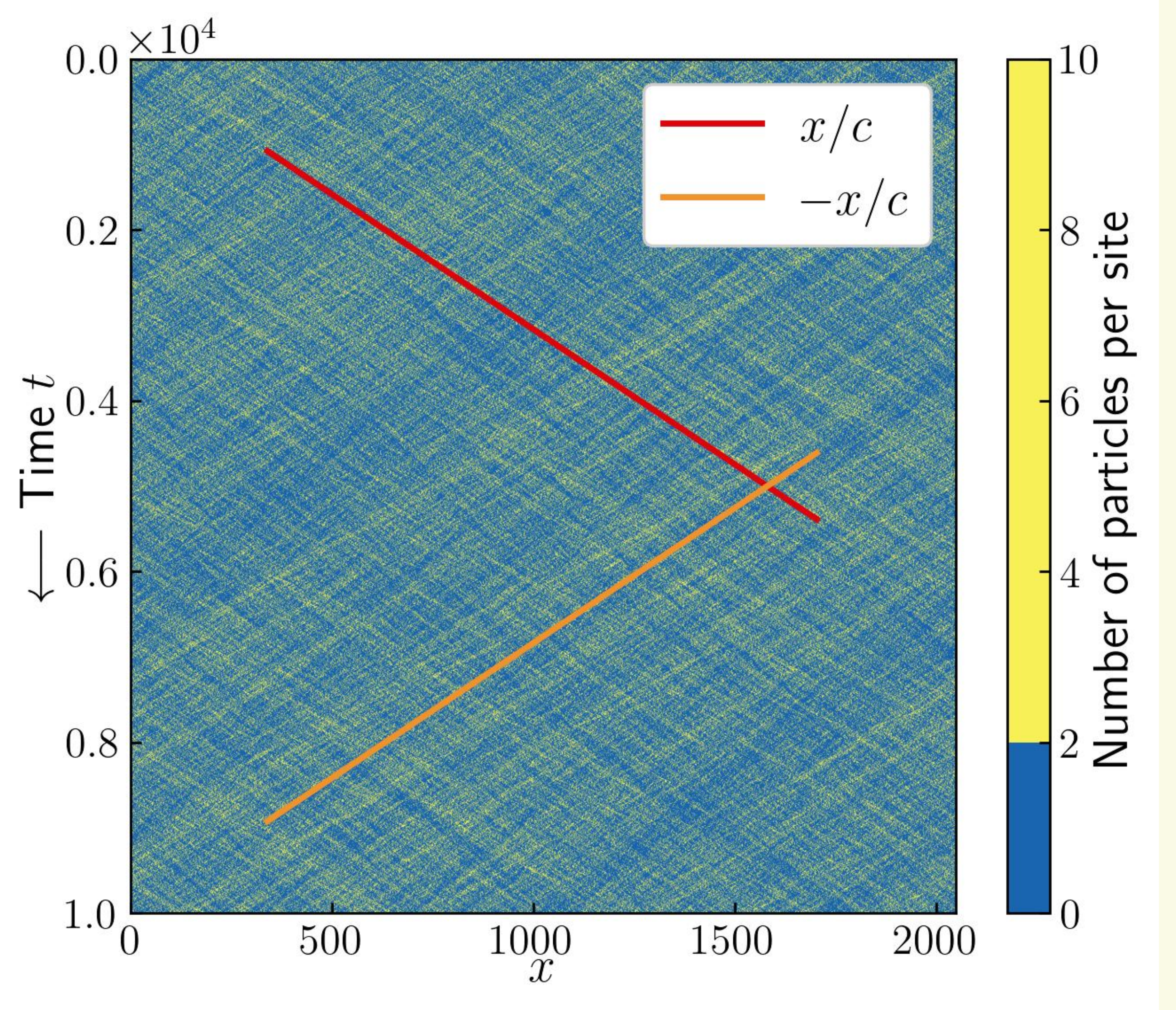


Dynamic structure factor

- $S_\rho(\omega,q) = \frac{1}{L}\langle|\delta\phi(\omega,q)|^2\rangle$

$$= \frac{1}{Nq^2} \frac{2 + \frac{hc^2q^2}{\omega^2}}{1 + \frac{\omega^2}{q^4}\left(1 - \frac{c^2q^2}{\omega^2}\right)^2}$$

Wave propagation mode at speed $c = \sqrt{hk}$!

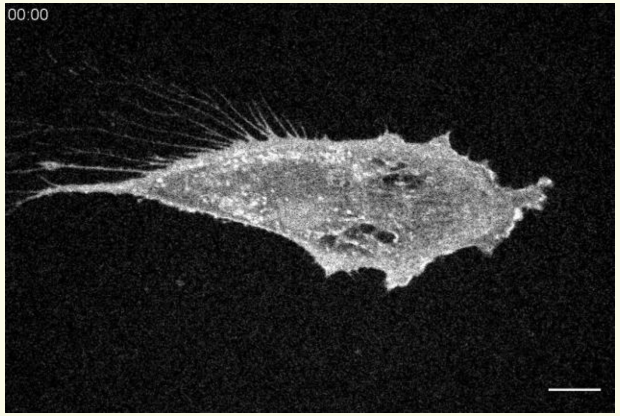


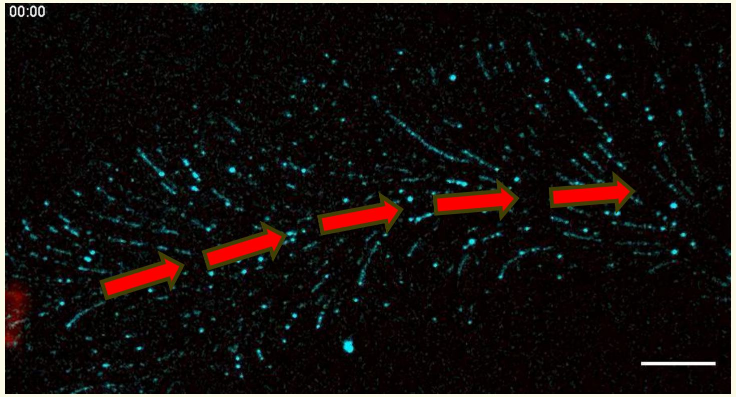
Conclusion of part 2

- Hydrodynamic equations from microscopic description
- Infinite space: mean-field working for all d
  - $d = 2$: No log corrections for $N \to \infty$?
- Periodic boundary conditions:
  - Growing interface converging to Edward-Wilkinson (KPZ for $d = 1$?)
  - Effective *attractive* zero-range interaction despite *repulsive* deposition
  - Traveling waves
- First version on arXiv:2512.03950, second version very soon…

Future work: polar deposition

[Montagnac group,Sc Adv 23]





Depositions define a polar field

Benoit Ladoux's team
at Institut Jacques Monot