The memory of geometry in self-interacting random walks

Eccentricity and flip statistics

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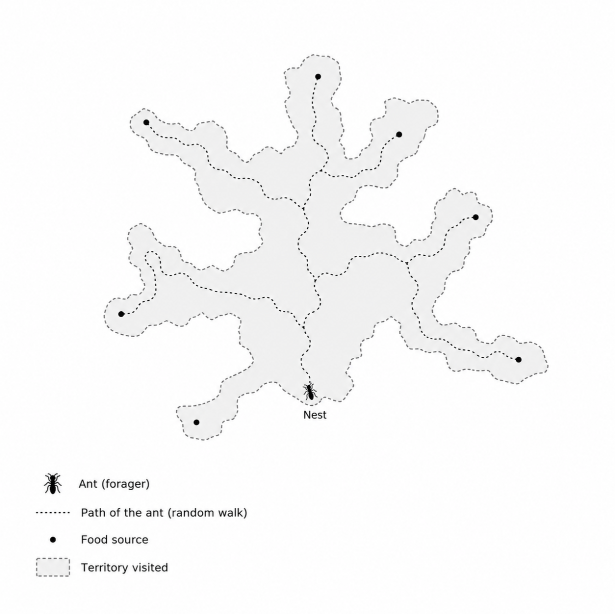
18/05/2026

Collège de France

The visited territory



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$\Rightarrow$ Given the agent's *microscopic* dynamics, modeled by a **random walk (RW)**, can we understand the *macroscopic* span dynamics ?

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However, this rough description cannot account for the **geometry** of the span.

Beyond span size : span geometry

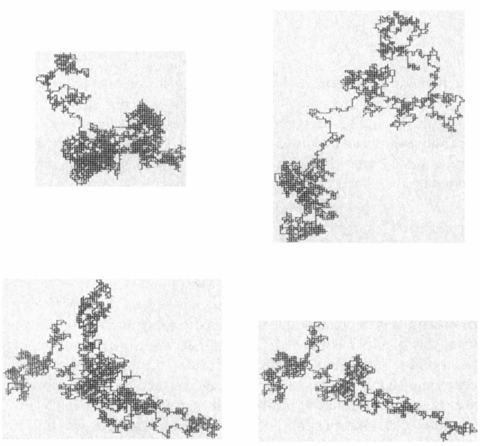

Fig. 1. Sample random walks in two dimensions of several thousand steps.

Source: The Shapes of Random Walks, Rudnick & Gaspari, Science 1987.

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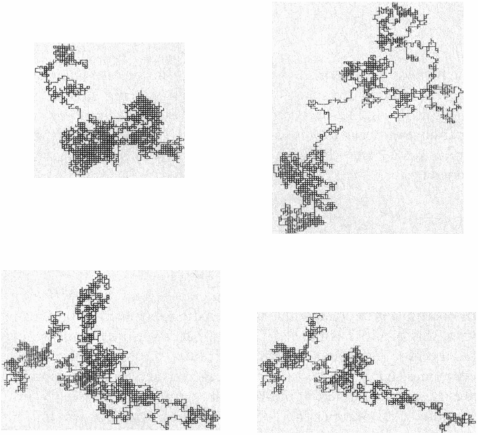

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⇒ Some results about non-sphericity of the span for a **simple, Markovian random walk in dimension** $d \geq 2$. Not a lot of results since !

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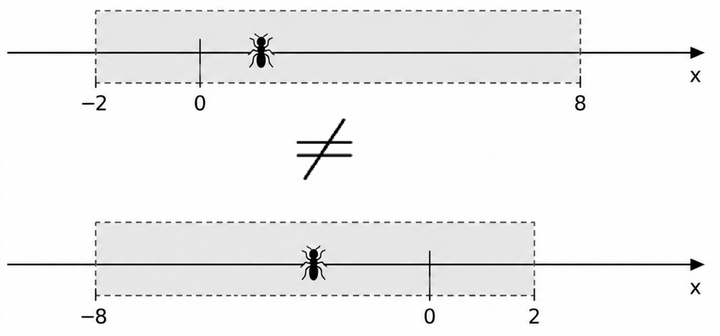

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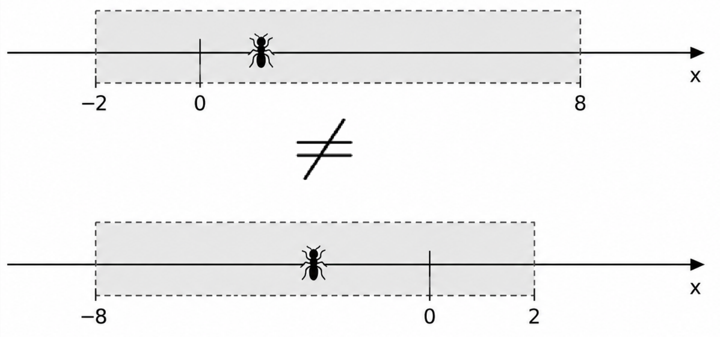

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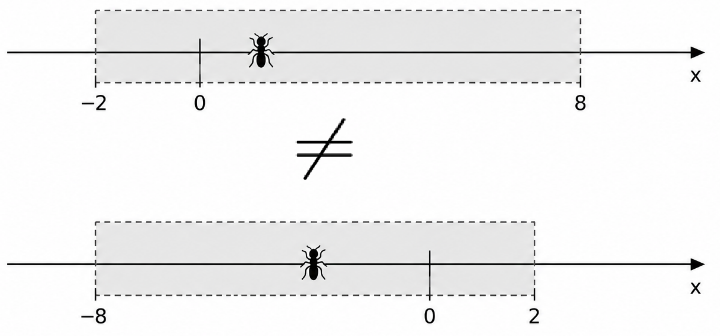

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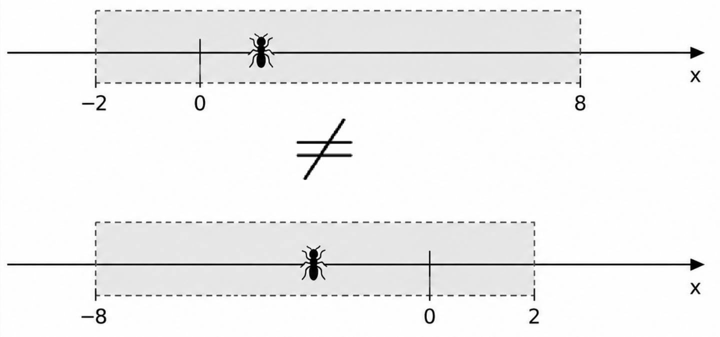

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We answer these questions for **self-interacting random walks (SIRWs)**.

$1D$ SIRW: Definition and general idea

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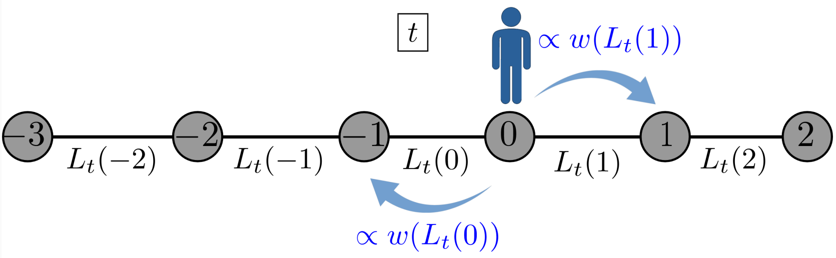
- If $w(n)$ decreases, the walk is **self-repelled** (exploratory); if $w(n)$ increases, it is **self-attracted** (timid).

SIRW: Illustration of the definition



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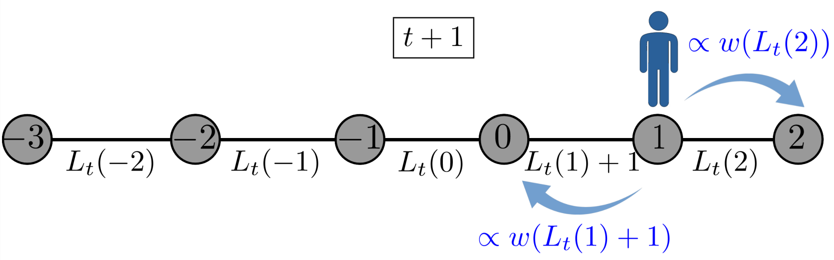

Three main classes of SIRWs

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Key difference: For saturating SIRWs, the span is remembered through its boundaries. For non-saturating SIRWs, the complete edge local time profile $(L_t(x))_x$ inside the span matters.

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Universal behavior

Belonging to a class depends on the **asymptotic behavior** of $w(n)$; in a single class, large-scale span properties are identical (as we proceed to show).

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Similarly, SIRWs whose self-interaction depends only on the **differences** of neighboring edge local times belong to non-saturating SIRWs. A prime example is TSAW defined by $w(n) = e^{-\beta n}$.

Key idea

SIRWs are difficult to deal with because they have an **infinite number of hidden degrees of freedom**: the edge local times $(L_t(x))_{x\in\mathbb{Z}}$.

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1. the current position y of the walker;
2. how many times the single edge $\{y-1, y\}$ has been visited.

Generalized Ray–Knight theorems (Tóth '96)

We can describe (a scaled version of) $(L_t(x))_x$ in terms of (powers of) squared Bessel (BESQ) processes, defined by their dimension δ and the SDE

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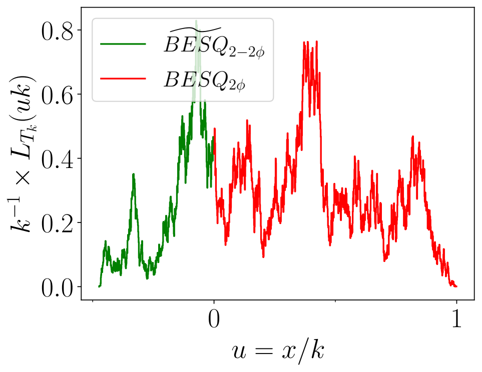
For e.g. TSAW, $(L_t(x))_x$ can be described using simple Brownian paths !

Illustration: Ray–Knight theory for saturating SIRWs



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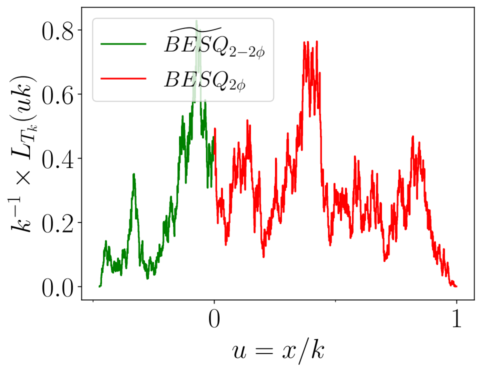

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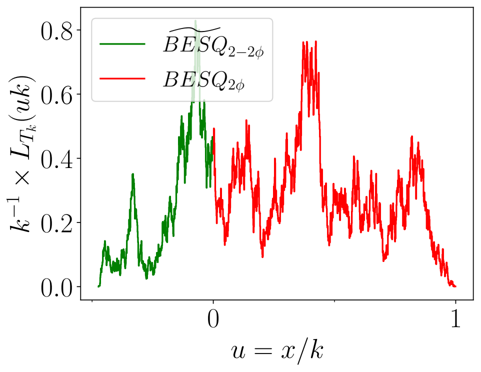

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$\Rightarrow$ First-passage time density to 0 of the edge local time process $\equiv$ **distribution** $q_+(k, m)$ **of the minimum** $-m$ **of the SIRW when it reaches its maximum** k**.**

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$\Rightarrow$ $q_+(k, m)$ gives access to the distribution of z_n !

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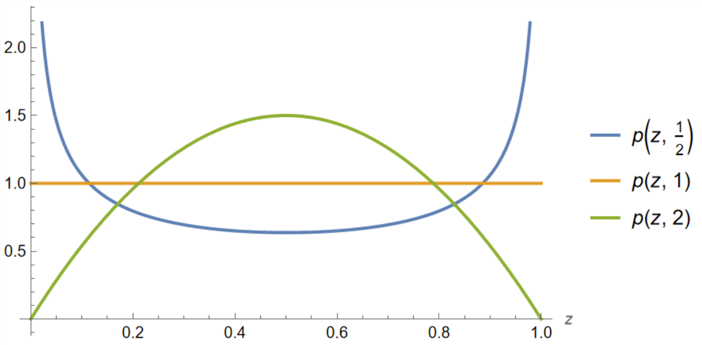
$\Rightarrow z_n \in \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ converges to a **Beta distributed** continuous

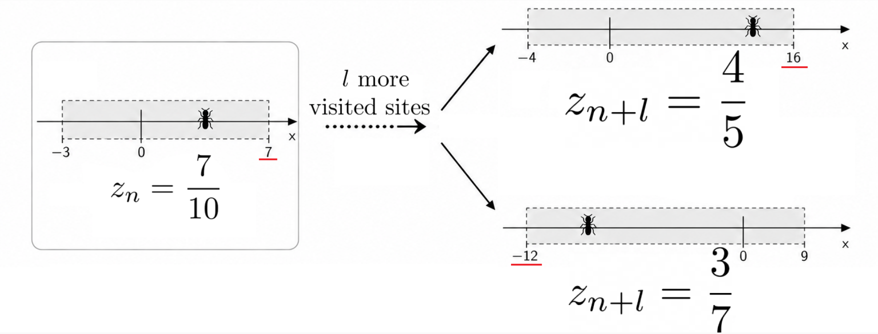
variable $z \in [0, 1]$! **Regardless of whether the SIRW saturates or not.**

Plots for $P(z, \phi = \delta/2)$



$\Rightarrow$ Among all $n+1$ possible spans of a given size n with eccentricities $z_n \in \{0, \frac{1}{n}, \ldots, 1\}$, **self-repulsion ($\phi < 1$) favors asymmetric spans, self-attraction ($\phi > 1$) favors symmetric spans, while Brownian-like walks ($\phi = 1$) are indifferent.**

What about aging ? Illustration



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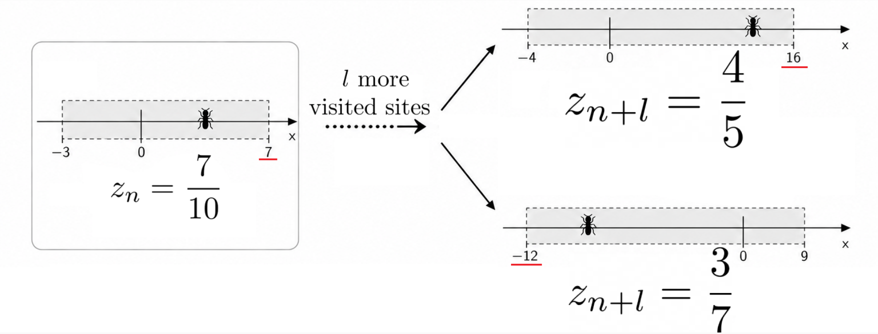

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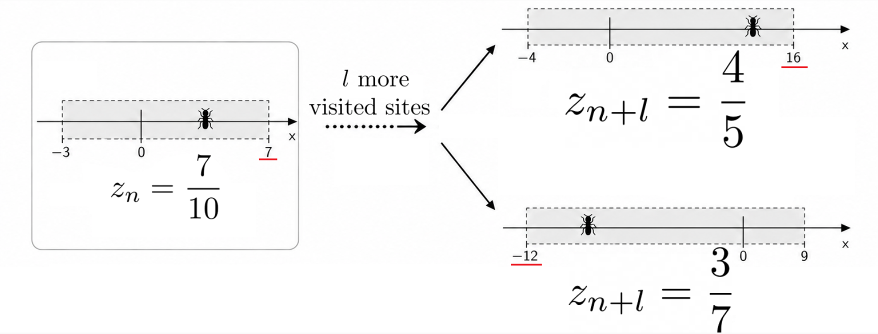

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$\Rightarrow$ How does the past eccentricity z_n, which conditions the walker's past trajectory, impact (by self-interaction) the future eccentricity z_{n+l} ?
This quantifies the "memory of past geometry".

The need for aged Ray–Knight-like results

To describe aging, we would like to have an *aged* Ray–Knight theorem: given a past span $[-m_1, \underline{k_1}]$, what does the future edge local time look like when reaching maximum $\underline{k_2}$?

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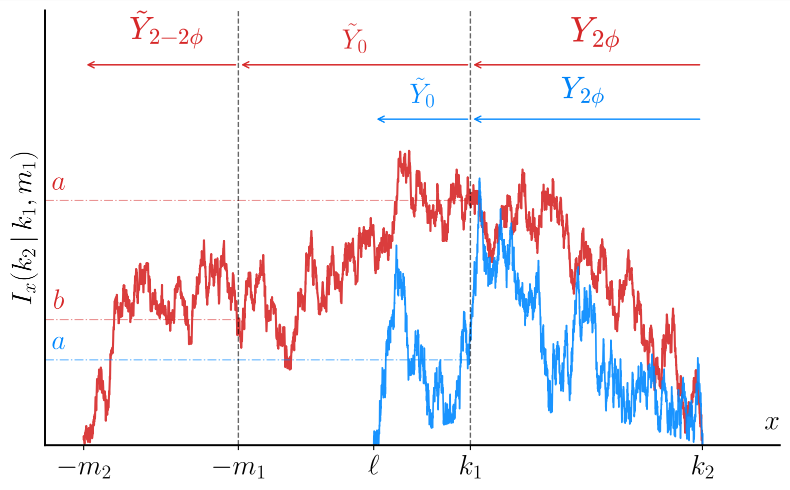
For **saturating** SIRWs, we can describe $(I_x(k_2 \mid k_1, m_1))_{x\in\mathbb{Z}}$ easily, thanks to Chaumont & Doney (2000).

Case of saturating SIRWs



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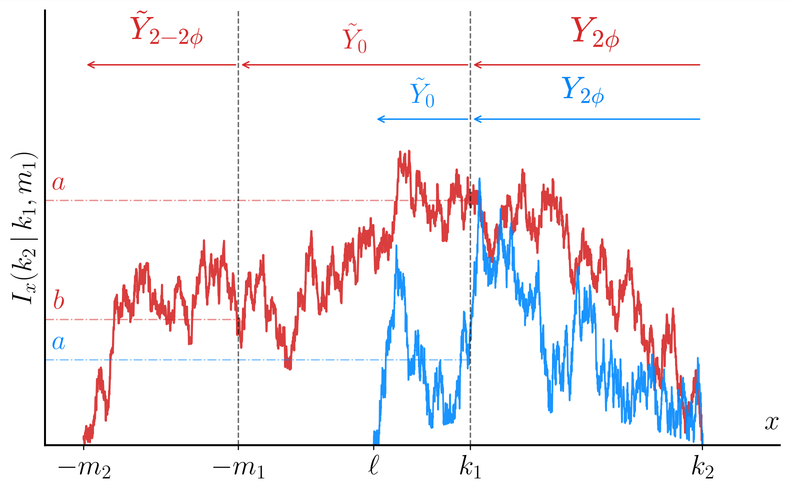

A related result was also used recently by Kosygina, Mountford and Peterson to establish convergence properties.

Result for saturating SIRWs

$$\mathbb{P}([-m_2, \underline{k_2}] \mid [-m_1, \underline{k_1}]) = \phi^2 \cdot \frac{(k_1 + m_1)^{\phi-1}}{(k_2 + m_2)^{\phi+1}} \cdot (k_2 - k_1) \times {}_2F_1\left(\phi + 1,\ 1 - \phi;\ 2;\ -\frac{(k_2 - k_1)(m_2 - m_1)}{(k_1 + m_1)(k_2 + m_2)}\right) + \delta(m_2 - m_1) \cdot \left(\frac{k_1 + m_1}{k_2 + m_1}\right)^{\phi}. \quad (2)$$

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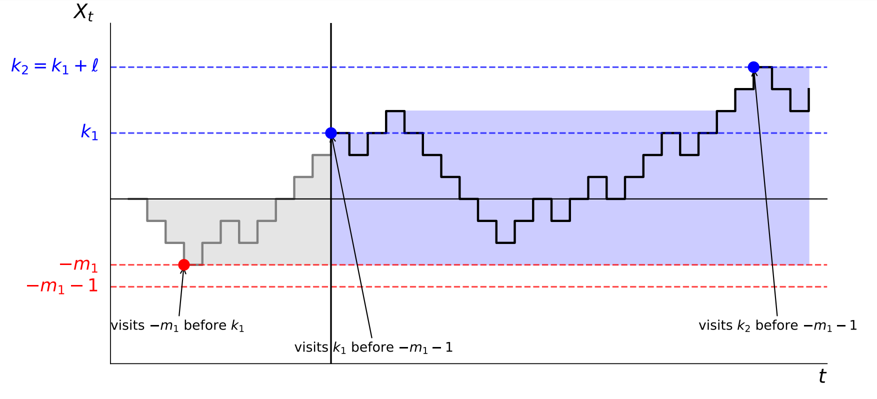
The counterpart where the path hits $-m_2$ before k_2 can also be computed.

The singular Dirac peak



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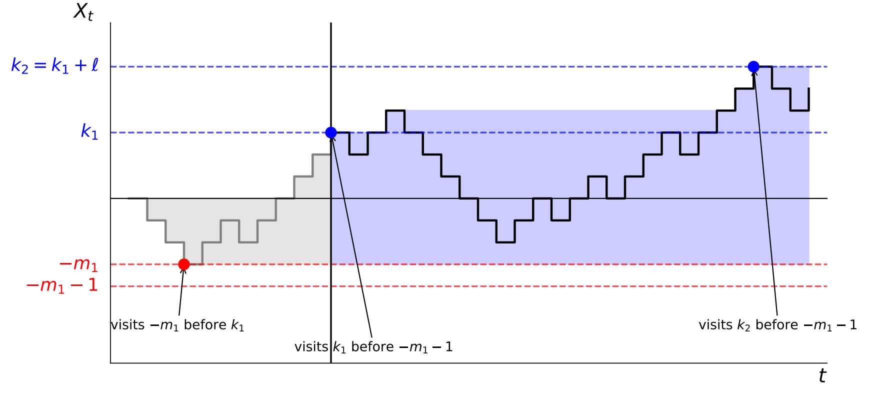
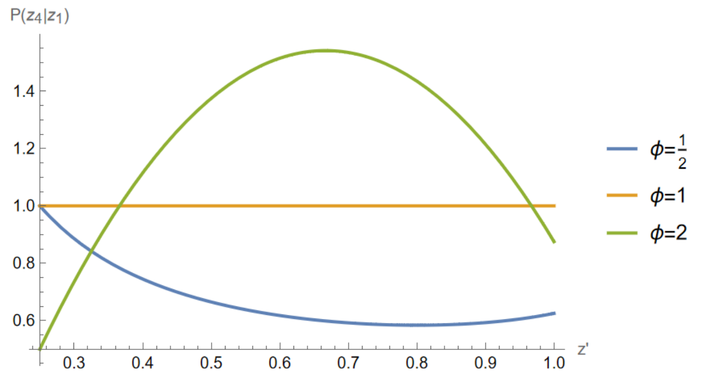


Weight of the Dirac peak

$\equiv \mathbb{P}(\text{next } \ell = k_2 - k_1 \text{ visited sites are on the same side as } k_1)$

$= \left(1 + \frac{\ell}{k_1 + m_1}\right)^{-\phi} \Rightarrow$ Measures **persistence in exploration.**

Nonsingular part



Influence of the past visited territory of size 1 on the eccentricity z_4 of the future visited territory of size 4.

⇒ quantitatively measures how memory **shifts** future eccentricity z_4.

The case of non-saturating SIRWs

We cannot describe easily the local time increment $(I_x(k_2 \mid k_1, m_1))_{x\in\mathbb{Z}}$ for e.g. TSAW or PSRW: memory is now **present everywhere in the visited interval** $[-m_1, \underline{k_1}]$**, not just at its boundaries**.

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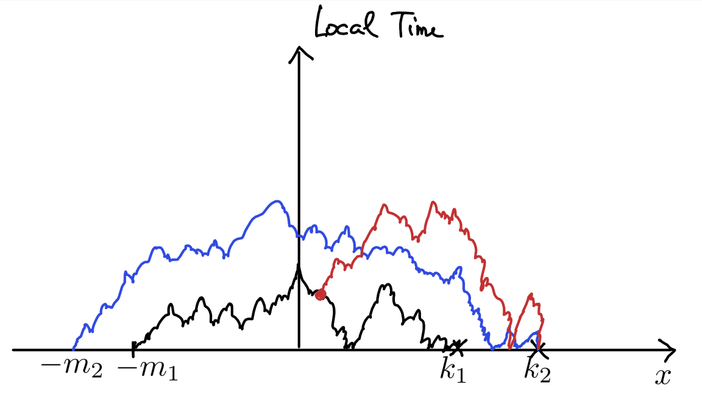
$\Rightarrow$ We will use them to compute aged properties such as $\mathbb{P}([-m_2, \underline{k_2}] \mid [-m_1, \underline{k_1}])$ for nonsaturating SIRWs.

Computation from the Brownian Web (1/3)



Black: initial edge local-time profile when visiting $[-m_1, \underline{k_1}]$.

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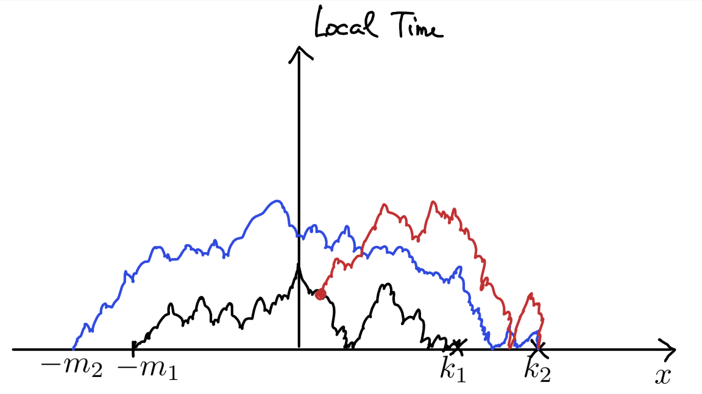

Black: initial edge local-time profile when visiting $[-m_1, \underline{k_1}]$. **Red**: future profile when visiting k_2, **while staying above** $-m_1$ (contributes to the singular Dirac peak).

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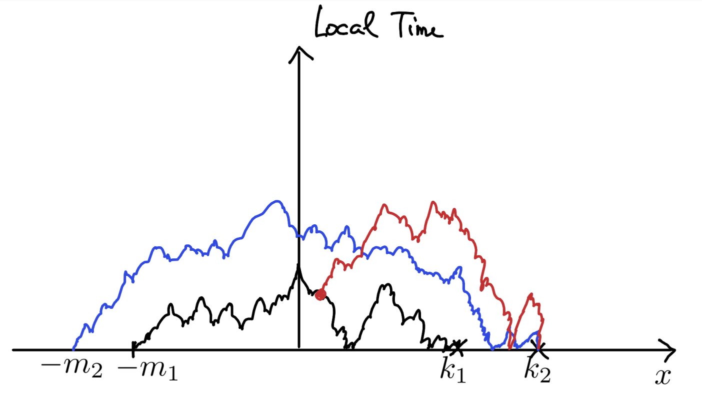
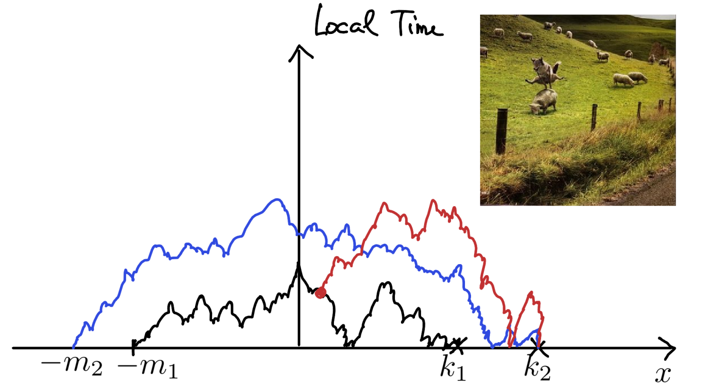


Black: initial edge local-time profile when visiting $[-m_1, \underline{k_1}]$. **Red**: future profile when visiting k_2, **while staying above** $-m_1$ (contributes to the singular Dirac peak). **Blue**: future profile when visiting k_2, **after going all the way to** $-m_2$ (contributes to the nonsingular part).

Computation from the Brownian Web (2/3)



In essence, the problem is reduced to the probability that **a (power of) Brownian motion "jumps" above another**.

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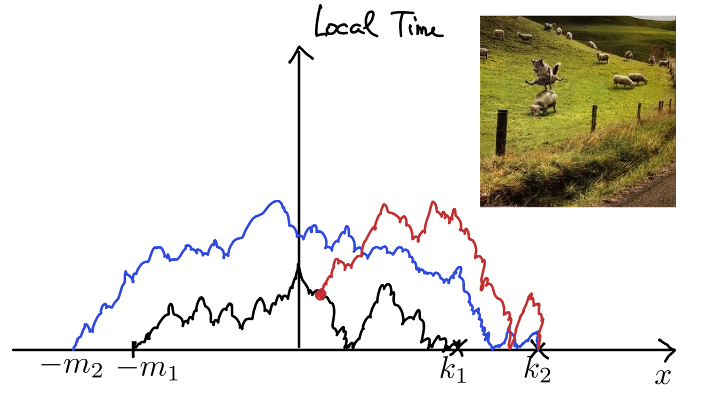
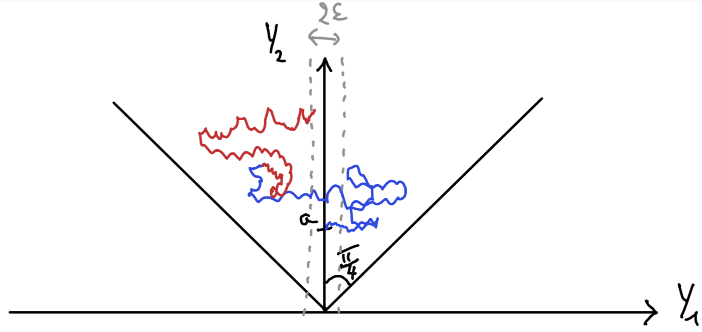


In essence, the problem is reduced to the probability that **a (power of) Brownian motion "jumps" above another**.

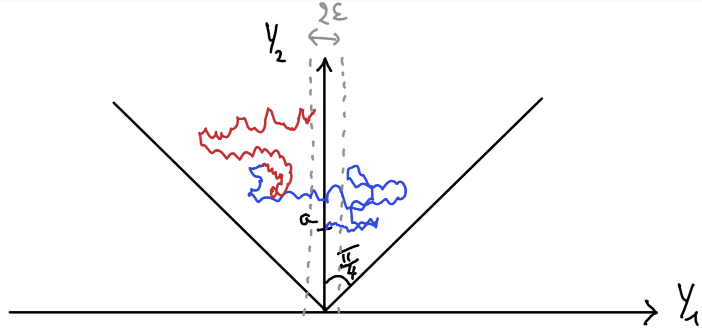
Difficulty: the initial profile (black) starts from 0 at "time" k_1, is reflected until time 0, and then stays above 0 before its absorption at time $-m_1$...

Computation from the Brownian Web (3/3)



We view the two one-dimensional Brownian motions as a **single two-dimensional Brownian motion** (Y_1, Y_2), where a is the value of the future local-time field at k_1.

Computation from the Brownian Web (3/3)



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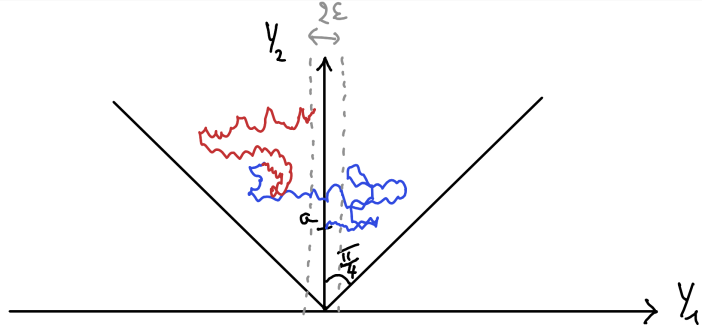

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- Duration k_1: motion in an **absorbing quadrant**.
- Duration m_1: motion in an **absorbing octant**.

Results for non-saturating SIRWs

$$\mathbb{P}([-m_2, \underline{k_2}] \,|\, [-m_1, \underline{k_1}])$$
$$= \frac{(k_2 - k_1)(m_2 - m_1)}{\pi k_2 m_2 (k_2 + m_2 - k_1 - m_1)} \sqrt{\frac{k_2}{m_2}}$$
$$+ \frac{2}{\pi} \delta(m_2 - m_1) \left[\frac{(k_1 + m_1)\sqrt{k_2/k_1}}{k_2 + m_1} \sin^{-1} \sqrt{\frac{k_1}{k_1 + m_1}} + \cos^{-1} \sqrt{\frac{k_2}{k_2 + m_1}} \right].$$

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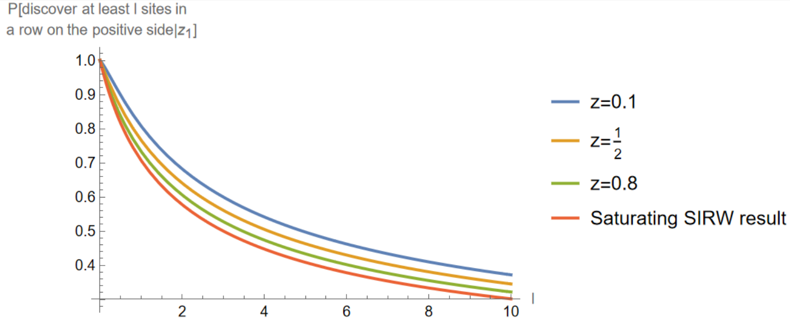
This result holds for **all non-saturating SIRWs (TSAW, PSRW, SESRW...)**

Singular part for nonsaturating SIRWs



⇒ **The less one side of the origin has been explored, the stronger the tendency to keep expanding this under-explored side.**

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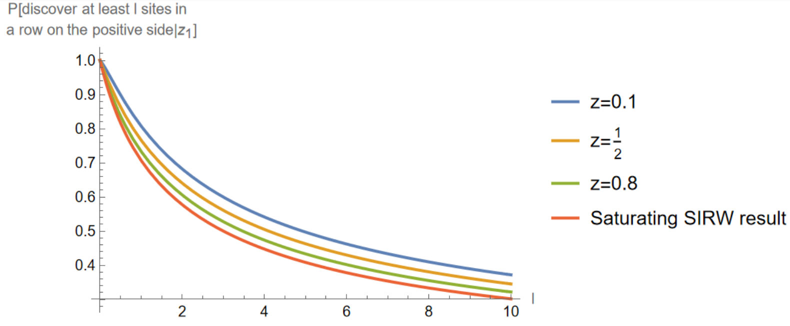

$\Rightarrow$ **The less one side of the origin has been explored, the stronger the tendency to keep expanding this under-explored side.**

This effect is **absent** for saturating SIRWs ! It is a consequence of the long-range influence of the **full** past edge local time.

Take-home message

Geometry has its own clock: span size n.

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Temporal dynamics can be very different

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Thank you for your attention !