Limit theorems for self-interacting random walks: a Ray-Knight approach

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Joint work with

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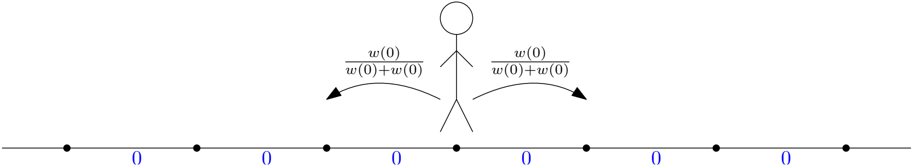
A class of self-interacting random walks

Weight function $w : \mathbb{Z}_+ \to (0, \infty)$

- Transition probabilities proportional to the weight function of the number of edge crossings.



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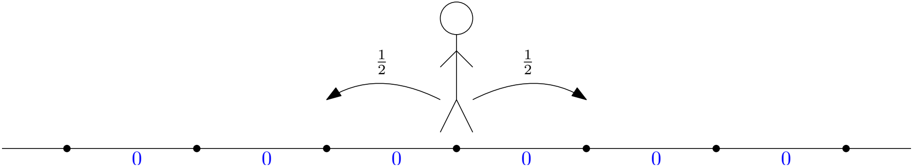

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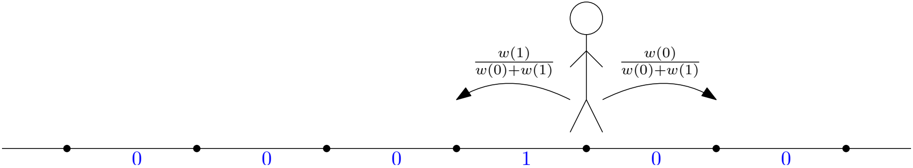

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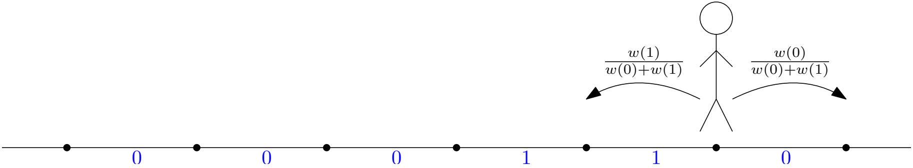

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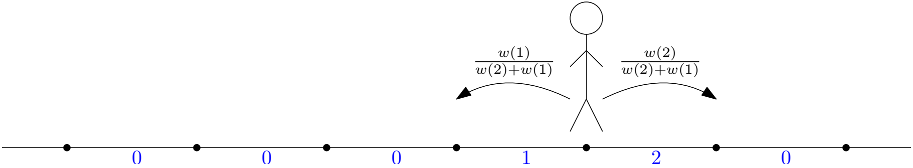

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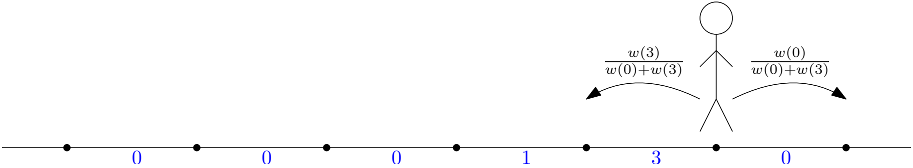

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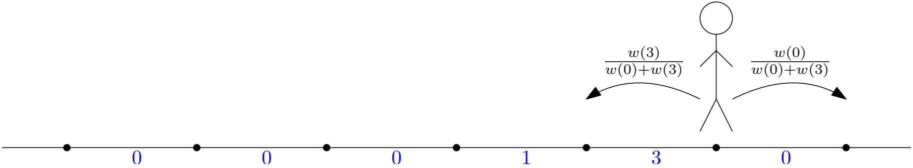

Assume $w(\cdot)$ is monotone.

- Decreasing $w(\cdot)$: self-repelling random walk
- Increasing $w(\cdot)$: self-attracting random walk

Examples

- **Linear ERRW:** $w(n) = a + n$
  - X_n converges in distribution without scaling (Pemantle '88).
- **"True" self-avoiding walk:** $w(n) = e^{-\beta n}$, for some $\beta > 0$.
- **Polynomially self-repelling:** $w(n) = n^{-\alpha}\left(1 + \frac{C}{n} + \mathcal{O}(n^{-2})\right)$ for some $\alpha > 0$.
- **Asymptotically free random walk:** $w(n) = 1 + \frac{C}{n} + \mathcal{O}(n^{-2})$

Markovian structure of local times

Directed edge local times:

$$\mathcal{E}(n,x) = \sum_{i=1}^{n} \mathbf{1}\{X_{i-1} = x, X_i = x+1\}$$

Stopping times: $\tau_{x,m}$ is the time of the $(m+1)$-st visit to x.

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$\boxed{\{\mathcal{E}(\tau_{x,m}, y)\}_{y\in\mathbb{Z}} \text{ is a gluing together of Markov chains.}}$

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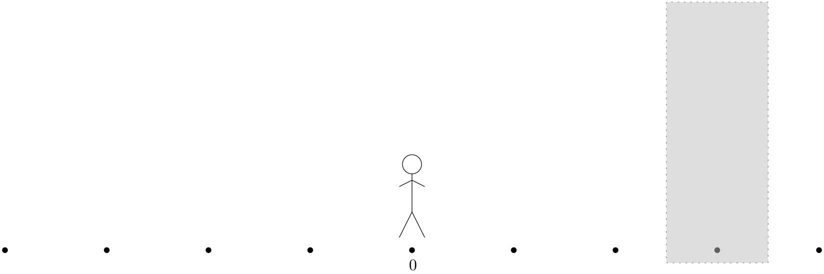

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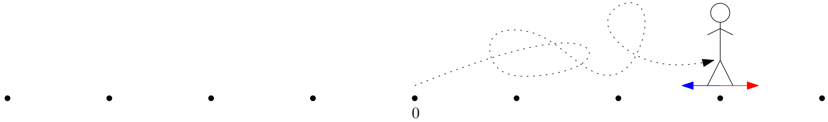

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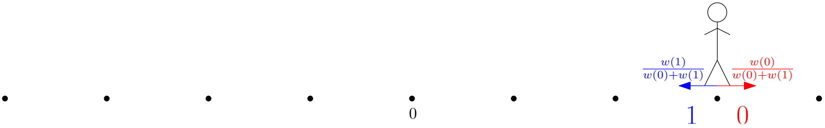

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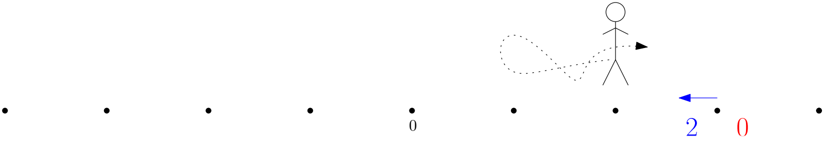

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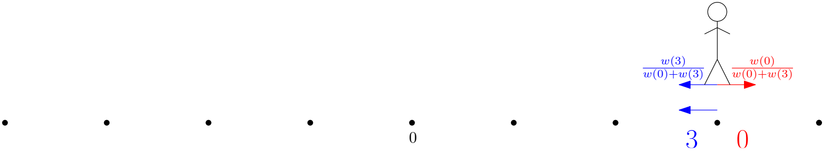

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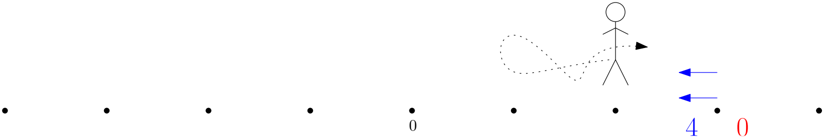

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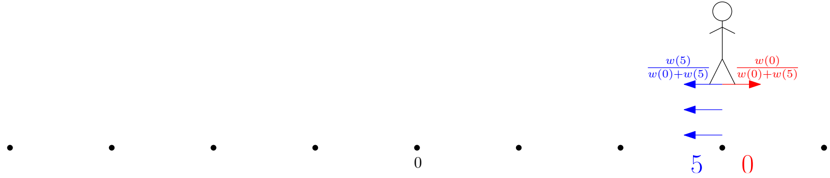

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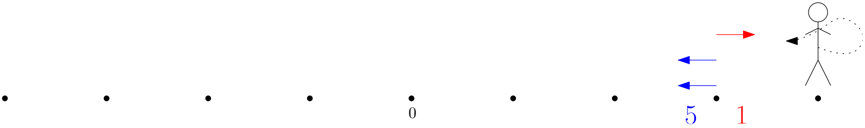

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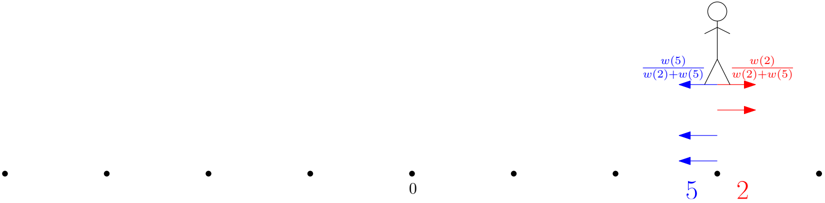

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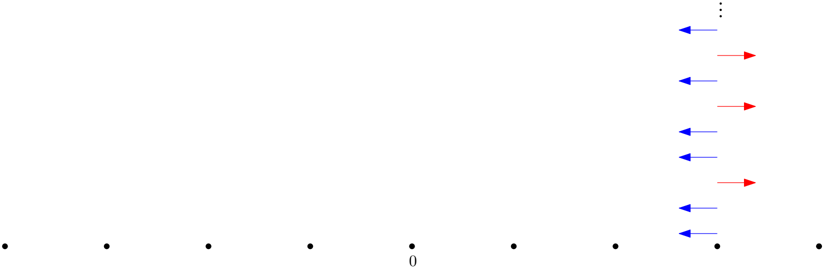

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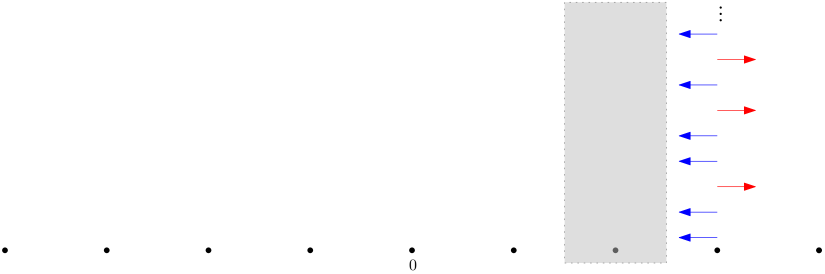

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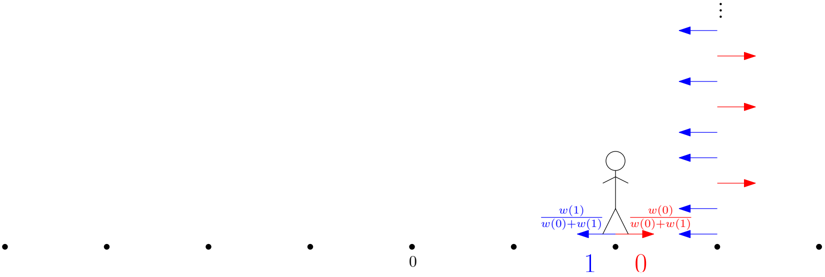

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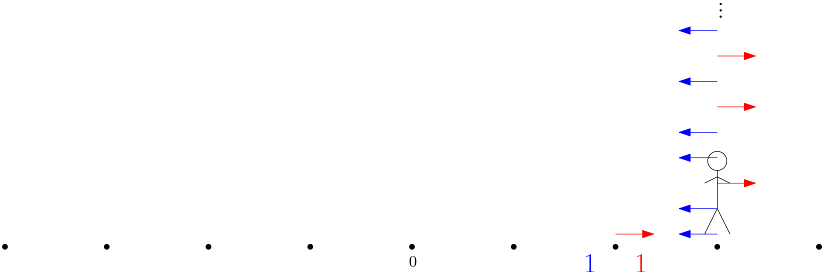

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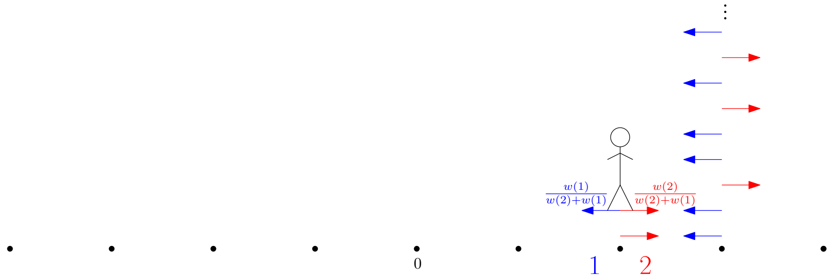

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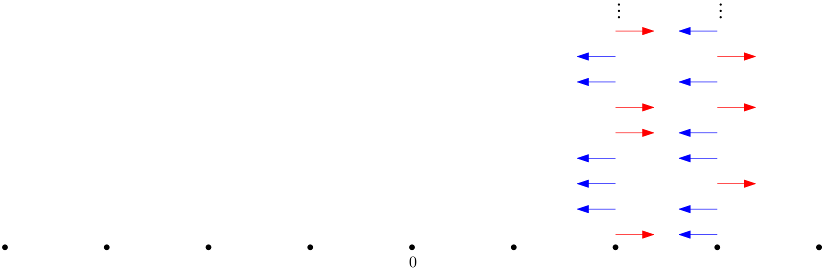

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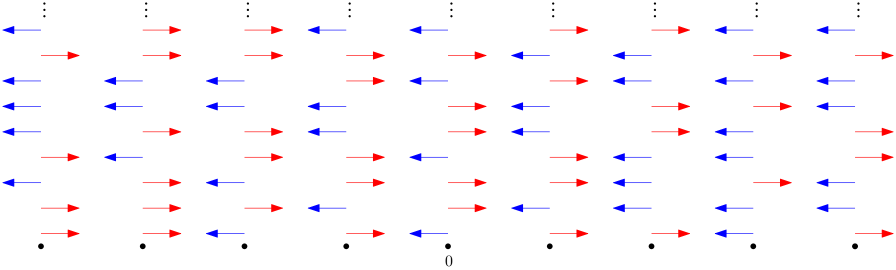

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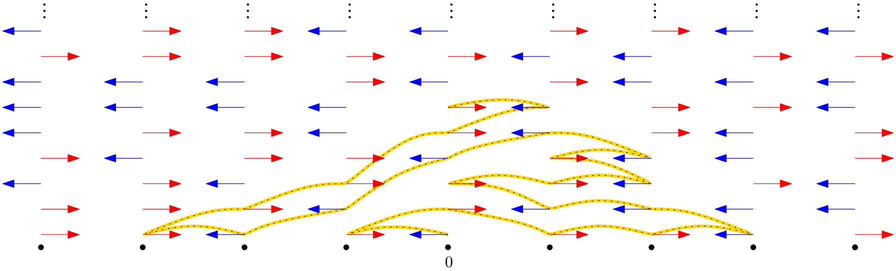

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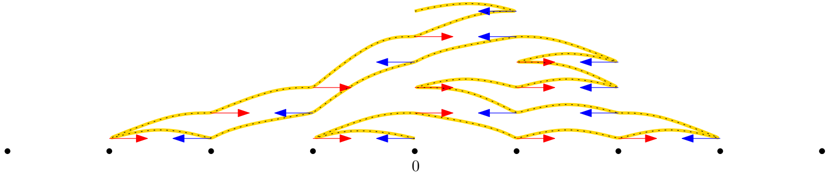

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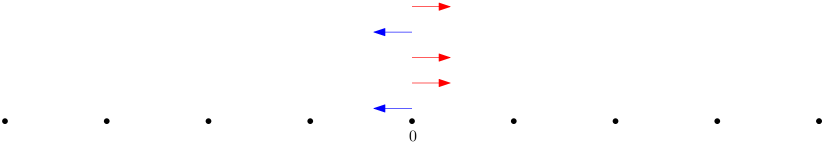

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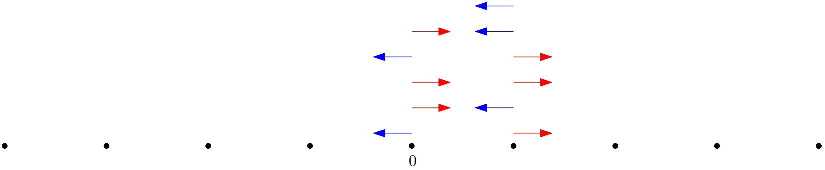

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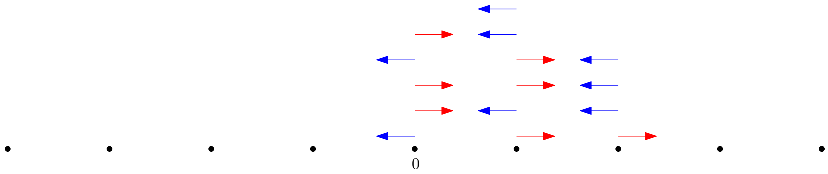

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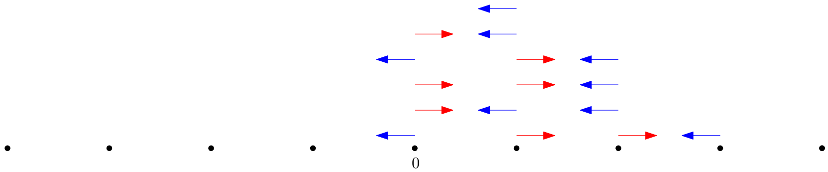

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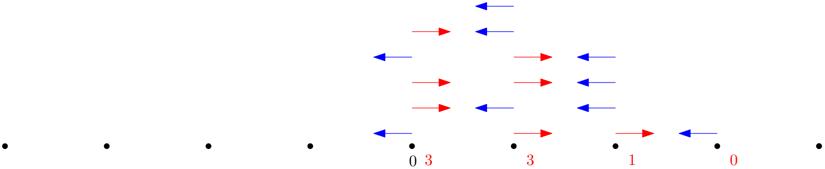

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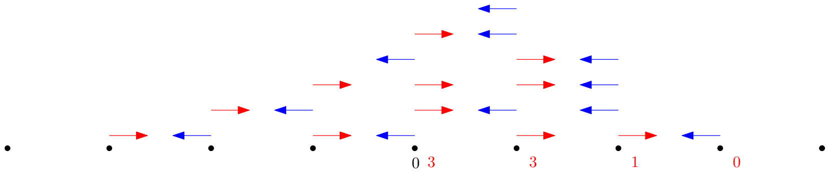

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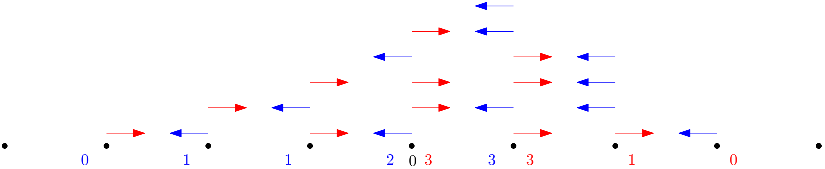

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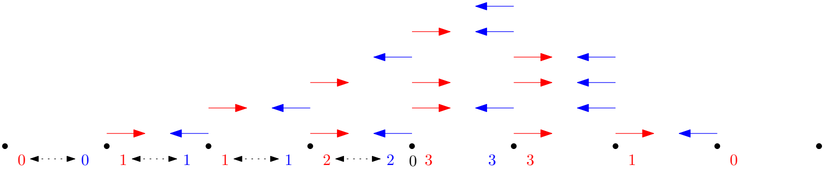

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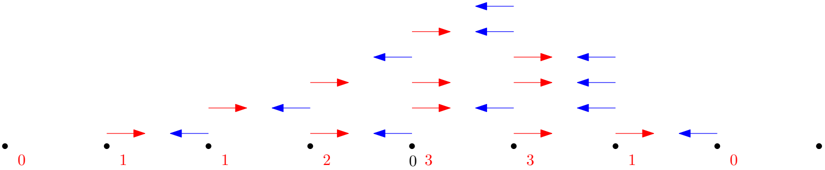

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Ray-Knight Theorems for the TSAW

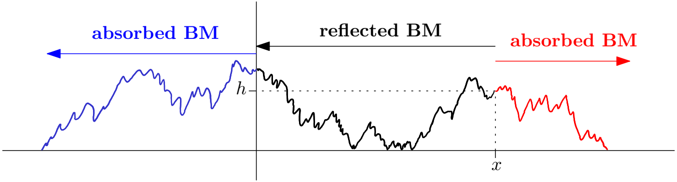
TSAW: $w(n) = e^{-\beta n}$, and let $\sigma^2 = \frac{\sum_{x\in\mathbb{Z}} x^2 e^{-\beta x^2}}{\sum_{x\in\mathbb{Z}} e^{-\beta x^2}}$.

Theorem (Tóth '95)

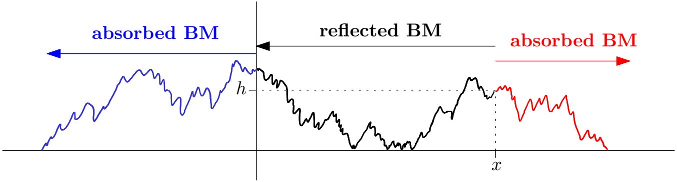
For any $x \in \mathbb{R}$ and $h \geq 0$, let

$$\bar{\Lambda}^n_{x,h}(y) = \frac{\mathcal{E}(\tau_{\lfloor xn\rfloor,\lfloor 2\sigma h\sqrt{n}\rfloor}, \lfloor yn\rfloor)}{\sigma\sqrt{n}}, \qquad y \in \mathbb{R}.$$

Then $\bar{\Lambda}^n_{x,h}(\cdot) \Longrightarrow \Lambda_{x,h}(\cdot)$, where $\Lambda_{x,h}(\cdot)$ is a two-sided, reflected/absorbed Brownian motion started at $\Lambda_{x,h}(x) = h$.



TSAW: Consequences of the Ray-Knight Theorem



- $\frac{\tau_{\lfloor xn \rfloor, \lfloor 2\sigma h\sqrt{n} \rfloor}}{2\sigma n^{3/2}} \Longrightarrow \int_{\mathbb{R}} \Lambda_{x,h}(y)\, dy.$

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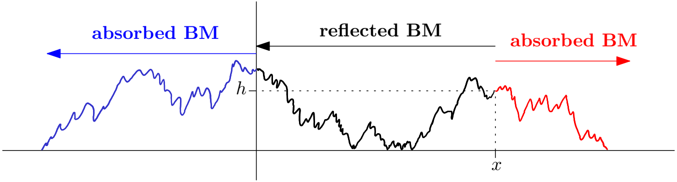

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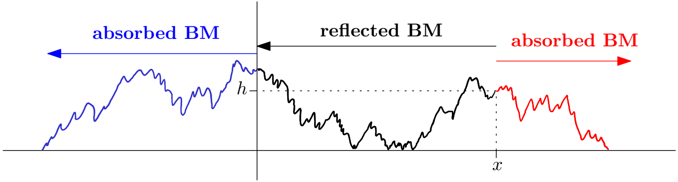

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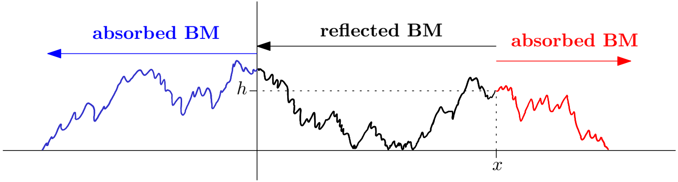

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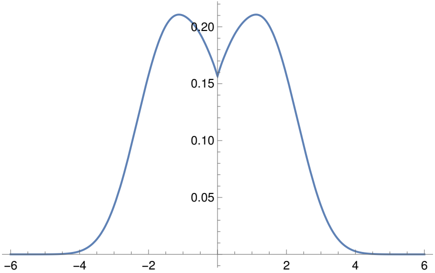
  This is enough to identify the limiting distribution of $\frac{X_n}{n^{2/3}}$ ***if it exists***.

TSAW limiting density

Semi-explicit formula for limiting distribution of $\frac{X_n}{n^{2/3}}$ given by Dumaz and Tóth ('13).



Limiting Distributions from Ray-Knight Theorems

Remaining questions:

- Can you use the Ray-Knight Theorems to prove a limiting distribution for the walk?
- What about functional limit laws for $(\frac{X_{\lfloor nt \rfloor}}{n^{2/3}})_{t \geq 0}$?

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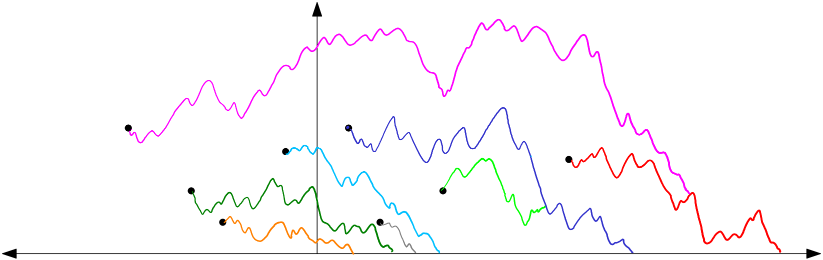
- **"True" self-repelling motion (TSRM)** - candidate limiting process (Tóth and Werner '98)

“True” self-repelling motion

Constructed by Tóth and Werner (’98)
Brownian web: $\{\Lambda_{x,h}(\cdot)\}_{x\in\mathbb{R},\, h\geq 0}$



Forward paths are independent, reflected/absorbed BM.
Backward paths are dual in the web.

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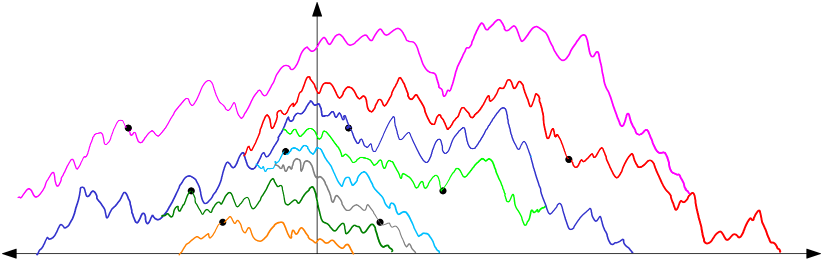

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Inverse of $\mathfrak{T}$ gives a random space filling curve

$$t \mapsto (\mathfrak{X}_t, \mathfrak{H}_t).$$

$\mathfrak{X}_t$ is the **"true" self-repelling motion.**

"True" self-repelling motion

Properties of the "true" self-repelling motion

- Continuous, recurrent.
- Not an SDE nor a semi-martingale (finite variation of order $3/2$).
- Scaling: $\{\mathfrak{X}_{at}\}_{t\geq 0} \overset{\text{Law}}{=} \{a^{2/3}\mathfrak{X}_t\}_{t\geq 0}$.

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- Local times $\mathfrak{L}(t,x)$ exist.

Inverse local times $\mathfrak{T}_{x,h} = \inf\{t \geq 0 : \mathfrak{L}(t,x) > h\}$

Joint Ray-Knight Theorems $\Lambda_{x,h}(y) = \mathfrak{L}(\mathfrak{T}_{x,h}, y)$

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- Formal dynamics

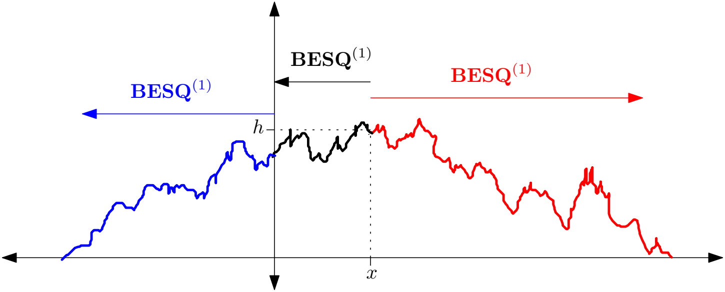
$$d\mathfrak{X}_t \text{ ``$=$'' } -\frac{\partial}{\partial x}\mathfrak{L}(t,x) + \text{(boundary effects)}$$

Ray-Knight theorems for PSR random walk

Polynomially self-repelling RW: $w(n) = n^{-\alpha}\left(1 + \frac{C}{n} + O(n^{-2})\right)$

Generalized Ray-Knight Theorem (Tóth '96)

$$\boxed{\Lambda^n_{x,h}(y) = \frac{2\mathcal{E}\left(\tau_{\lfloor xn\rfloor,\lfloor h\beta n\rfloor}, \lfloor yn\rfloor\right)}{\beta n} \underset{n\to\infty}{\Longrightarrow} \Lambda_{x,h}(y), \qquad \beta = \frac{1}{1+2\alpha}}$$



Brownian motion perturbed at its extrema

(α, β)**-BMPE:**

$\alpha, \beta < 1$

$$Y^{\alpha,\beta}(t) = B(t) + \alpha(\sup_{s \leq t} Y^{\alpha,\beta}(s)) + \beta(\inf_{s \leq t} Y^{\alpha,\beta}(s))$$

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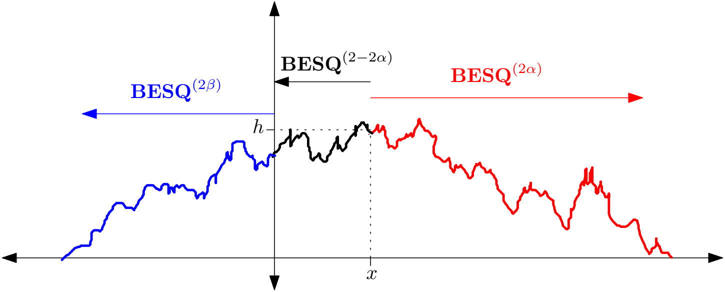

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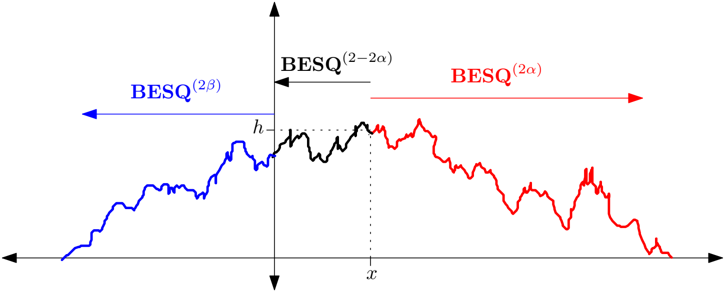

Suggests PSRW converge to $(\frac{1}{2},\frac{1}{2})$-BMPE.

Scaling limit for PSR random walk

$$\mathcal{X}_n(t) = \frac{X_{\lfloor nt \rfloor}}{\sqrt{n}}, \quad t \geq 0.$$

Theorem (Kosygina, Mountford, P. '23)

For a PSR random walk with $w(n) \approx n^{-\alpha}$,
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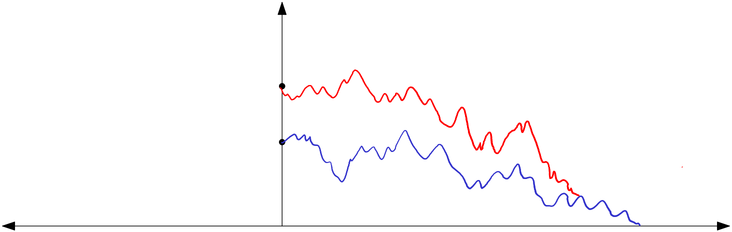
- Does $\mathcal{X}_n(t)$ converge in distribution for $t > 0$ fixed?
- Does $\mathcal{X}_n(\cdot)$ converge in distribution?

PSR walks don't converge to BMPE

Idea of argument in (Kosygina, Mountford, P. '23)
Fix $0 < h_1 < h_2$.

$$(\Lambda^n_{0,h_1}(t), \Lambda^n_{0,h_2}(t)) \Longrightarrow (X(t), Y(t)), \qquad t \geq 0.$$

$$\begin{cases} dX(t) = dt + 2\sqrt{X(t)}\, dB_1(t) \\ dY(t) = dt + 2\sqrt{X(t)}\, dB_1(t) + 2\sqrt{Y(t)}\sqrt{1 - \left(\frac{X(t)}{Y(t)}\right)^{2\alpha+1}}\, dB_2(t) \end{cases}$$



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BUT limit for $(1/2, 1/2)$-BMPE should be

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Joint Ray-Knight Theorems

$$\Lambda^n_{x,h}(y) = \frac{L(\tau_{\lfloor xn\rfloor, \lfloor han^\gamma\rfloor}, \lfloor yn\rfloor)}{an^\gamma}$$

Joint Ray-Knight: For any $(x_1, h_1), \ldots, (x_N, h_N) \in \mathbb{R} \times [0, \infty)$, joint convergence of the paths

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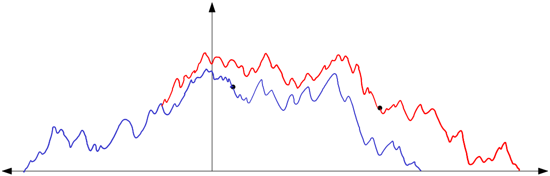

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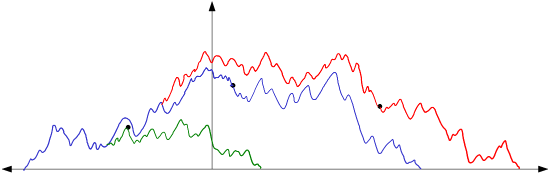

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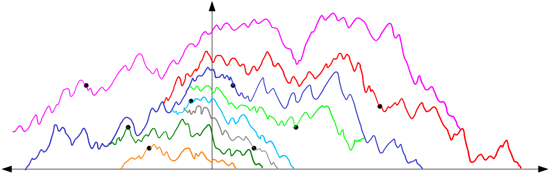

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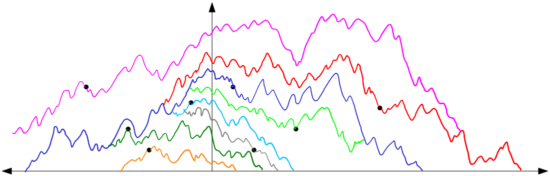

Augmented Joint Ray-Knight: Joint convergence of paths together with merge points and endpoints.

Scaling limits from Joint Ray-Knight

Random walk X_n. Local time $L(n,x) = \sum_{k=0}^n \mathbf{1}_{\{X_k = x\}}$.

Rescaled path: $\mathcal{X}_n(t) = \dfrac{X_{\lfloor tn^{1+\gamma} \rfloor}}{n}, \quad t > 0$

Rescaled local time: $\mathcal{L}_n(t,x) = \dfrac{L(\lfloor tn^{1+\gamma} \rfloor, \lfloor xn \rfloor)}{n^\gamma}, \quad t > 0, x \in \mathbb{R}$

Theorem (Kosygina, Marêché, Mountford, P. '26+)

If the walk satisfies an augmented Joint Ray-Knight theorem (+ certain technical conditions), then there is a continuous process $\mathfrak{X}(\cdot)$ with local time $\mathfrak{L}(\cdot,\cdot)$ such that

- $(\mathcal{X}_n(\cdot), \mathcal{L}_n(\cdot,\cdot)) \Longrightarrow (\mathfrak{X}(\cdot), \mathfrak{L}(\cdot,\cdot))$
- $\mathfrak{X}(\cdot)$ *is the unique continuous process with the same joint Ray-Knight theorems as the walk.*

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    - Joint RK proved in (Kosygina, P. ’26)
2. PSR walk converges to a new process $\mathfrak{X}_\alpha(\cdot)$
    - New proces: α-polynomially self-repelling motion
    - Joint RK proved in (Yu ’26+)
    - Augmented joint RK proved in (Kosygina, Marêché, Mountford, P. ’26+)

The α-polynomially self-repelling motion

Similarities with $(\frac{1}{2}, \frac{1}{2})$-BMPE
(Tóth '96, Carmona, Petit, Yor '98, Brémont, Bénichou, Voituriez '24)

1. $\mathfrak{X}_\alpha(\cdot)$ has the same Ray-Knight theorems as $Y^{\frac{1}{2},\frac{1}{2}}(\cdot)$

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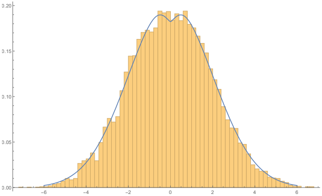

Simulations of PSR walk with $\alpha = 4$ for 10,000 steps

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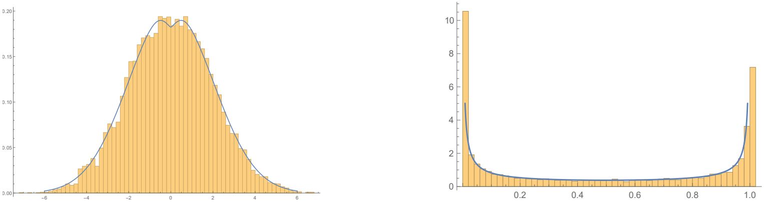
3. For any $\alpha > 0$ and $t > 0$,

$$\frac{1}{t}\int_0^t \mathbf{1}_{\{\mathfrak{X}_\alpha(s)\geq 0\}}\,ds \overset{\text{Law}}{=} \frac{1}{t}\int_0^t \mathbf{1}_{\{Y^{\frac{1}{2},\frac{1}{2}}(s)\geq 0\}}\,ds \sim \text{Beta}\left(\frac{1}{4}, \frac{1}{4}\right)$$



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Differences with $(\frac{1}{2},\frac{1}{2})$-BMPE

1. $\mathfrak{X}_\alpha(\cdot)$ and $Y^{\frac{1}{2},\frac{1}{2}}(\cdot)$ have different joint Ray-Knight theorems.
2. For any $0 < s < t$

$$\mathfrak{X}_\alpha(t)-\mathfrak{X}_\alpha(s) \overset{\text{Law}}{\neq} Y^{\frac{1}{2},\frac{1}{2}}(t)-Y^{\frac{1}{2},\frac{1}{2}}(s)$$

Simulations for $n = 10{,}000$ steps with $s = 0.9$, $t = 1$.
PSR walk with $\alpha = 4$ (left) and AF walk with $\gamma = 1/2$ (right).

The α-polynomially self-repelling motion

Heuristic driving mechanism

$$\mathfrak{X}_\alpha(t) \text{ ``$=$'' } B(t) - \frac{\alpha}{2}\int_0^t \frac{\partial}{\partial x}\log\left(\mathfrak{L}_\alpha(s,x)\right)\Big|_{x=\mathfrak{X}_\alpha(s)} ds$$
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Compare with "true" self-repelling motion (Tóth, Werner '98)

$$\mathcal{X}(t) \text{ ``$=$'' } - \int_0^t \frac{\partial}{\partial x} \mathcal{L}_\alpha(s,x)\Big|_{x=\mathcal{X}(s)} ds + \text{(perturbations at boundary)}$$

Technical Conditions for Main Theorem

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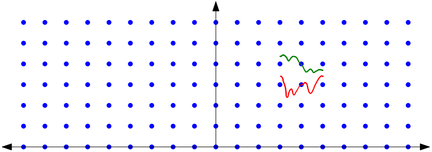
- Non-degeneracy condition

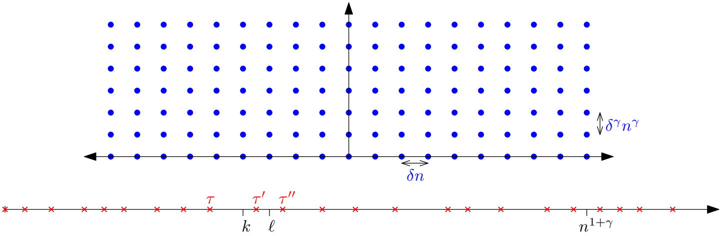
$$\max_{k\in\mathbb{Z},\,\ell\in\mathbb{Z}_+} P\left(\Lambda_{k,\ell}(k\pm 1) < \Lambda_{k,\ell+1}(k\pm 1)\right) \leq 1 - c_0$$



Tightness from joint Ray-Knight theorems

Need: for any $\varepsilon, \eta > 0$ there exists $\delta' > 0$ such that

$$\limsup_{n\to\infty} P\left(\max_{\substack{k,\ell \leq n^{1+\gamma} \\ |k-\ell| \leq \delta' n^{1+\gamma}}} |X_k - X_\ell| > \varepsilon n \right) < \eta.$$



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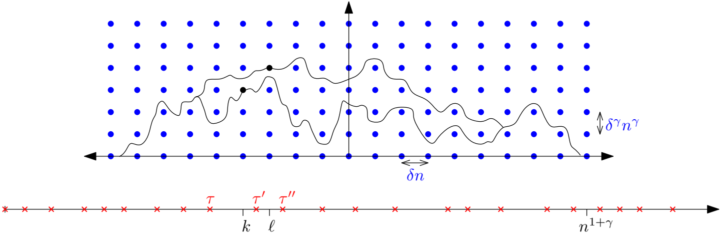

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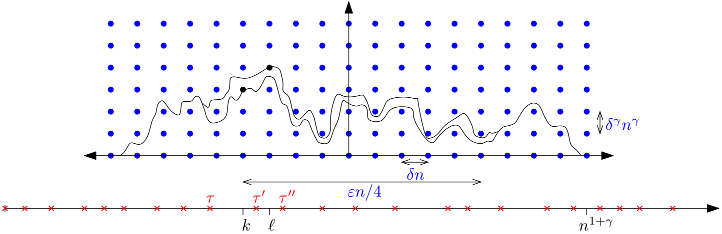

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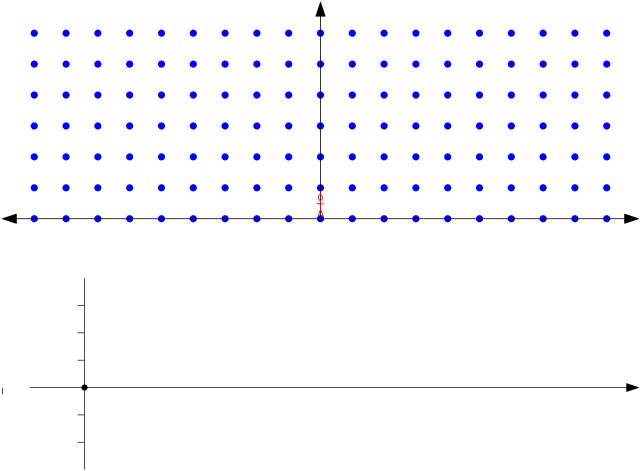
Identification of finite dimensional distributions

Use stopping times $\tau_{\lfloor xn \rfloor, \lfloor han^\gamma \rfloor}$ for (x, h) on δ-grid $(\delta\mathbb{Z}) \times (\delta^\gamma \mathbb{Z}_+)$.



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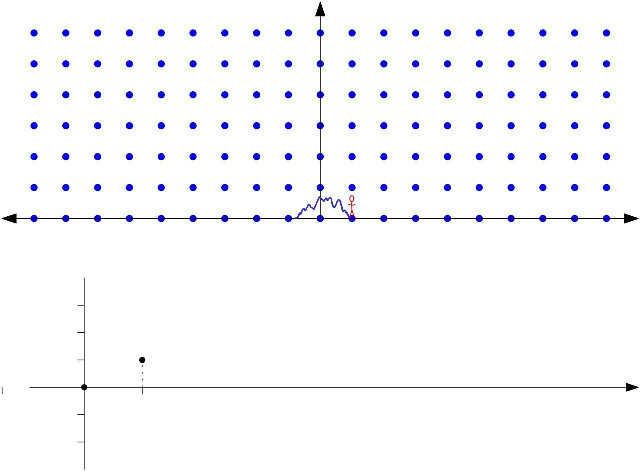

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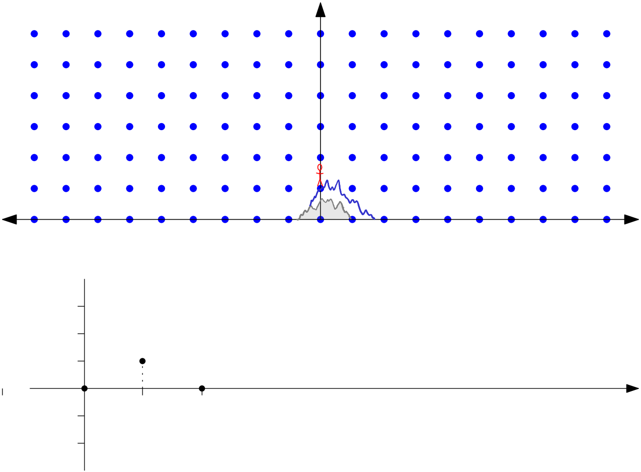

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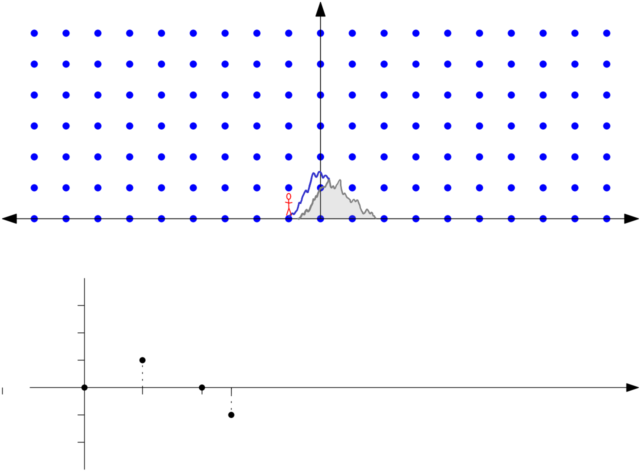

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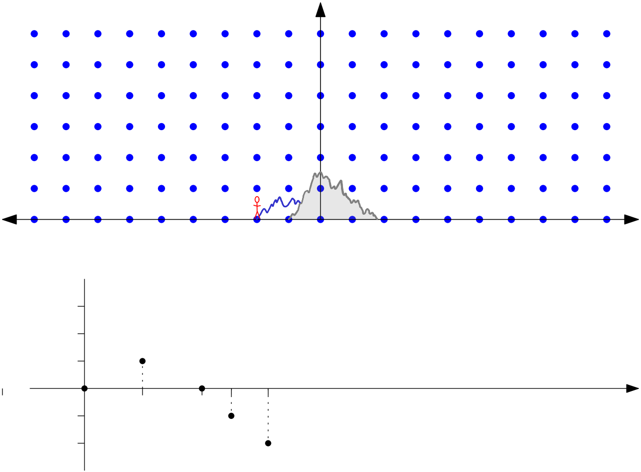

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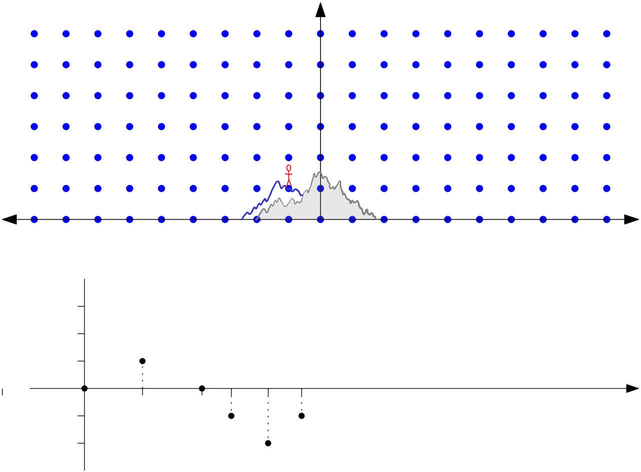

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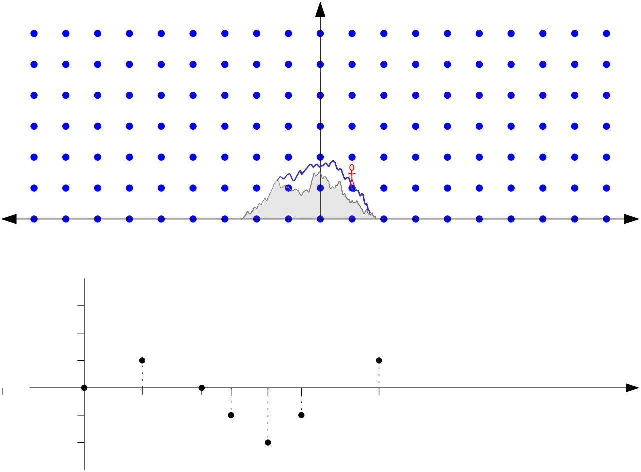

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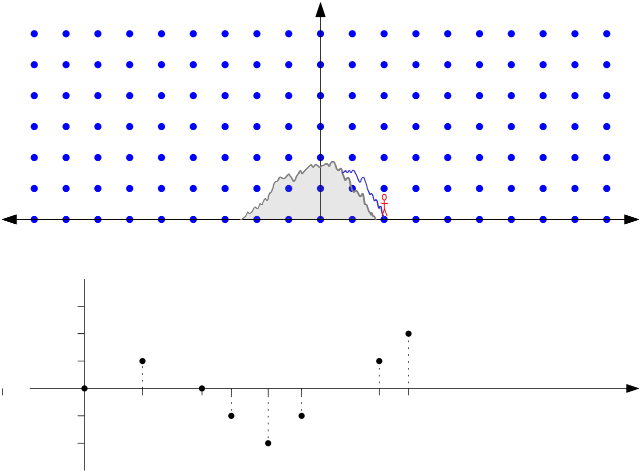

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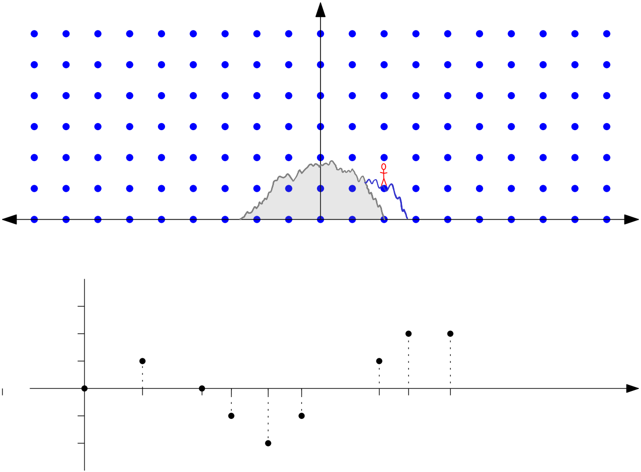

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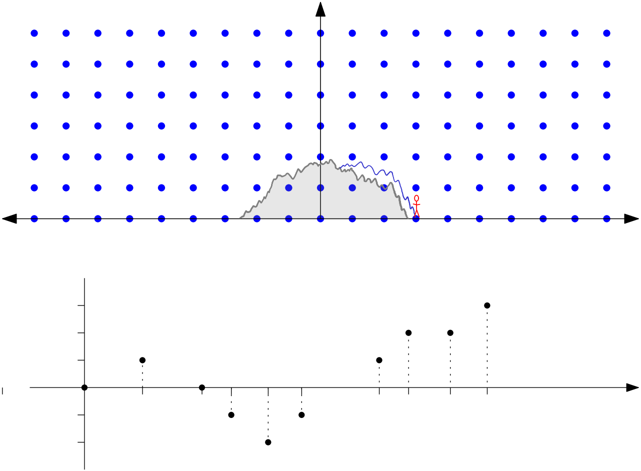

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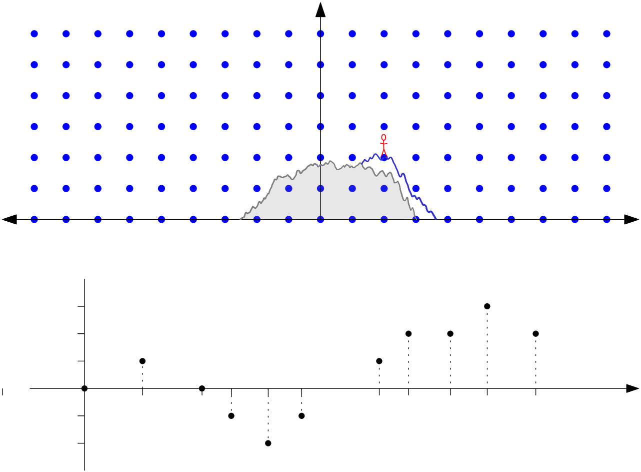

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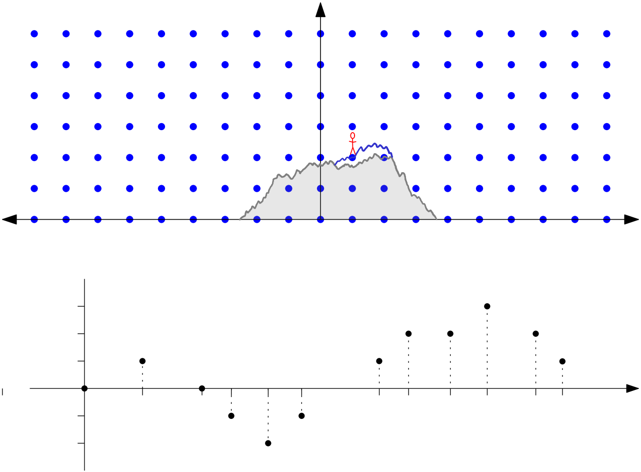

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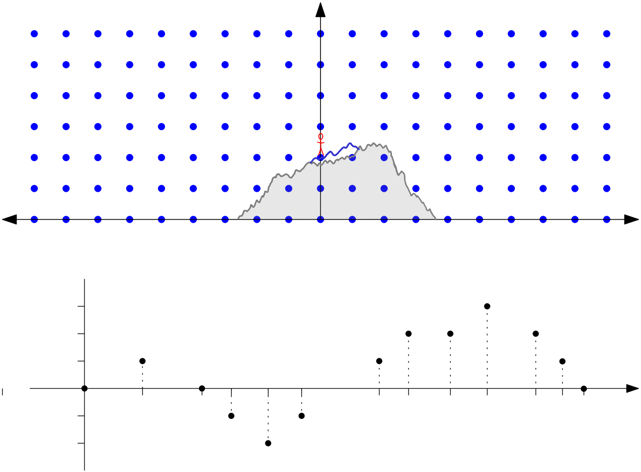

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    - $\mu_{\delta,n}$ is distribution of $\mathcal{X}_{\delta,n}(\cdot)$.
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- Sequence $\{\mu_n\}_{n\geq 1}$ is Cauchy.