The Brownian web distance

Bálint Vető

Budapest University of Technology and Economics
HUN–REN Rényi Institute

Recent Progress on Self-Interacting Processes
and Non-Reversible Monte Carlo

20th May 2026

Outline

- Introduction
- Random walk web distance
- Brownian web and Brownian web distance
- Bernoulli-Exponential last passage percolation: a weighted version
- Proof ideas

Introduction

joint work with Martin Hairer, Julian Ransford and Bálint Virág

Kardar–Parisi–Zhang (KPZ) universality conjecture, 1:2:3 scaling: for a wide class of surface growth models with height function $h(t,x)$,

$$\frac{h(n^{3/3}t, n^{2/3}x) - \mathrm{E}(h(nt, n^{2/3}x))}{n^{1/3}}$$

converges as $n \to \infty$

Directed landscape $\mathcal{L}(x,t;y,s)$: coupling of the scaling limit of the height at (s,y) started from narrow wedge initial condition at (t,x) (Dauvergne, Ortmann, Virág, 2018)

MŰEGYETEM 1782

Directed landscape and the 1:2:3 scaling

Last passage percolation: let ξ_{ij} for $(i,j) \in \mathbb{Z}^2$ be i.i.d. random variables with geometric or exponential distribution. The last passage value

$$L((i,j),(k,l)) = \max_{\pi:(i,j)\nearrow(k,l)} \sum_{(a,b)\in\pi} \xi_{ab}$$

Theorem (Dauvergne, Virág, 2022)

Let $u = (1,1)$, $v = (1,-1)$, then

$$\frac{L(tn^3u + xn^2v, sn^3u + yn^2v) - \alpha n^3(t-s)}{\chi n} \stackrel{\mathrm{d}}{\Longrightarrow} \mathcal{L}(x,t;y,s)$$

as $n \to \infty$ for some explicit constants α, χ.

The directed landscape is invariant under the 1:2:3 scaling:

Proposition

For $\alpha > 0$,

$$\alpha^{-1}\mathcal{L}(\alpha^2 x, \alpha^3 t; \alpha^2 y, \alpha^3 s) \stackrel{\mathrm{d}}{=} \mathcal{L}(x,t;y,s)$$

Other universality classes

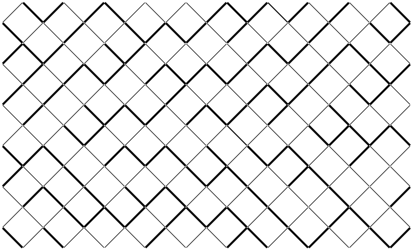
Examples:

- 1:2:4 scaling:
  Edwards–Wilkinson class: additive stochastic heat equation
- 1:1:2 scaling:
  Brownian castle (Hairer–Cannizzaro, 2022): Brownian motion on the Brownian web

In this talk, we show a natural example for another universality class built on a different environment.

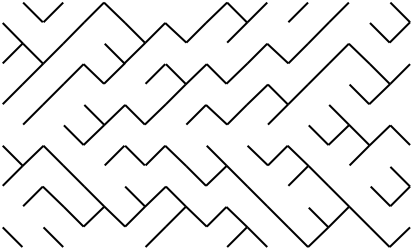
Random walk web

- Lattice: $\{(i,n) \in \mathbb{Z}^2 : i+n \text{ is even}\}$ with directed lattice edges from (i,n) to $(i+1, n\pm 1)$
- Graph of free edges: exactly one of the outgoing lattice edges with equal probabilities independently, i.e. coalescing random walks to the right



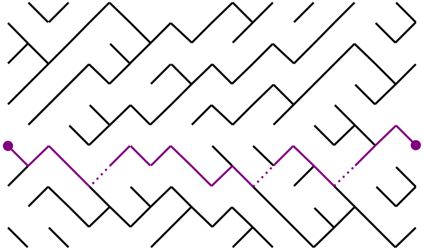
Random walk web

- Lattice:
  $\{(i,n) \in \mathbb{Z}^2 : i+n \text{ is even}\}$
  with directed lattice edges from (i,n) to $(i+1, n\pm 1)$
- Graph of free edges: exactly one of the outgoing lattice edges with equal probabilities independently, i.e. coalescing random walks to the right
- Edge weights: edges of the graph with weight 0, other lattice edges with weight 1
- Distance $D^{\mathrm{RW}}(i,n;j,m)$: weight of the directed path between (i,n) and (j,m) with minimal total weight



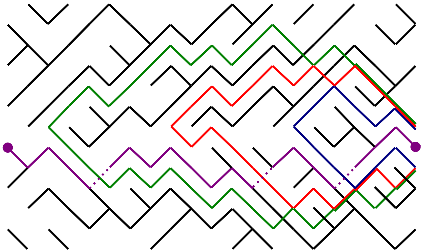
Random walk web distance

- Distance $D^{\mathrm{RW}}(i,n;j,m)$: weight of the directed path between (i,n) and (j,m) with minimal total weight
- In other words: minimal number of jumps to get from (i,n) to (j,m)



Random walk web distance

- Distance $D^{\mathrm{RW}}(i,n;j,m)$: weight of the directed path between (i,n) and (j,m) with minimal total weight
- In other words: minimal number of jumps to get from (i,n) to (j,m)
- Blue, red, green regions: set of starting points with 0, 1 and 2 jumps to the purple target point
- Aim: distance function between remote points, scaling, continuum limit



Brownian web and its dual

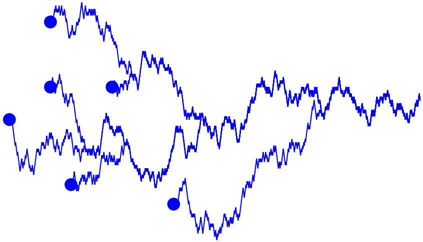
Brownian web: coalescing Brownian motions starting at all $(t, x) \in \mathbb{R}^2$

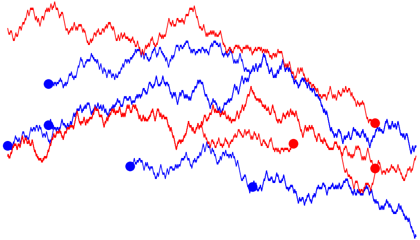
History:

- Arratia, 1979, unpublished
- Tóth, Werner, 1998, construction, special points, local time of true self-repelling motion
- Fontes, Isopi, Newman, Ravishankar, 2004, topology, „Brownian web”

Dual: coalescing backward Brownian motions

Forward and backward paths do intersect but they do not cross





Special points of the Brownian web

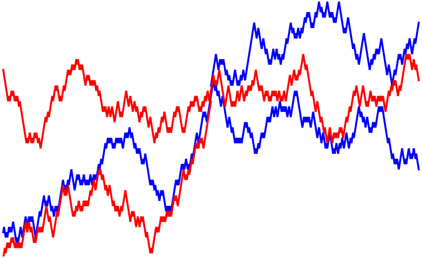
Special points: point of type $(m_{\text{in}}, m_{\text{out}})$ has m_{in} incoming and m_{out} outgoing paths

Possible types: $(0,1)$, $(0,2)$, $(0,3)$, $(1,1)$, $(1,2)$, $(2,1)$

Almost all points of $\mathbb{R}^2$ are of type $(0,1)$

Characterization of $(1,2)$ points (see figure): those hit by a forward and a backward path

Brownian web: unique path starting from almost every point of $\mathbb{R}^2$, an *additional path* at each $(1,2)$ point



Brownian web distance

Brownian web distance $D^{\mathrm{Br}}(t,x;s,y)$: minimal number of jumps from (t,x) to (s,y) using Brownian web paths and with jumps at $(1,2)$ points from the incoming path to the additional path

Basic properties:

- D^{Br} is integer valued (∞ included)
- D^{Br} is non-symmetric
- $D^{\mathrm{Br}}(t,x;s,y)=\infty$ for a typical (s,y) which is not hit by a Brownian web path
- $D^{\mathrm{Br}}(t,x;t,x)=0$
- Triangle inequality:

$$D^{\mathrm{Br}}(t,x;s,y)\le D^{\mathrm{Br}}(t,x;u,z)+D^{\mathrm{Br}}(u,z;s,y)$$

Lower semicontinuous version $D^{\mathrm{Br,LSC}}(t,x;s,y)$ differs from $D^{\mathrm{Br}}(t,x;s,y)$ by at most 1

Main results

0:1:2 scale invariance (c.f. 1:2:3 scaling in the KPZ class):

Proposition

For all $\alpha > 0$, *it holds that*

$$D^{\mathrm{Br}}(\alpha^2 t, \alpha x; \alpha^2 s, \alpha y) \overset{\mathrm{d}}{=} D^{\mathrm{Br}}(t, x; s, y).$$

Convergence:

Theorem (V., Virág, 2023)

- *The Brownian web distance as a function* $D^{\mathrm{Br,LSC}} : \mathbb{R}^4 \to \mathbb{R} \cup \{\infty\}$ *is almost surely lower semicontinuous.*
- *There is a coupling of the underlying random walk webs and Brownian web such that*

$$D^{\mathrm{RW}}(nt, n^{1/2}x; ns, n^{1/2}y) \to D^{\mathrm{Br,LSC}}(t, x; s, y)$$

as $n \to \infty$ *almost surely in the epigraph sense.*

KPZ limit for Brownian web distance after shear mapping

Theorem (V., Virág, 2023)

As $m \to \infty$, we have that

$$\frac{tm + 2zm^{2/3} - D^{\mathrm{Br}}(-tm, 2tm + 2zm^{2/3}; 0, \mathbb{R}_-)}{m^{1/3}} \xrightarrow{\mathrm{d}} \mathcal{L}(0,0;z,t)$$

where $\mathcal{L}$ is the directed landscape.

Conjecture (resolved by Dauvergne, Zhang, 2024)

As $m \to \infty$, we have that

$$\frac{m + 2(z-y)m^{2/3} - D^{\mathrm{Br}}(-m, 2m + 2zm^{2/3}; 0, (-\infty, 2ym^{2/3}])}{m^{1/3}} \xrightarrow{\mathrm{d}} \mathcal{L}(y,0;z,1)$$

For the interval I and $* \in \{\mathrm{Br}, \mathrm{RW}\}$, $D^*(t,x;s,I) = \inf_{y \in I} D^*(t,x;s,y)$.

KPZ limit for random walk web distance

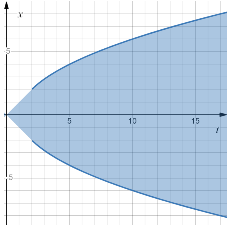
Theorem (V., Virág, 2023)

For any $\eta \in (0,1)$, we have that

$$\frac{c_1(\eta)n - c_2(\eta)zn^{2/3} - D^{\mathrm{RW}}(-n, \eta n + c_3(\eta)zn^{2/3}; 0, \mathbb{Z}_-)}{c_4(\eta)n^{1/3}} \overset{\mathrm{d}}{\Longrightarrow} \mathcal{L}(0,0;z,1)$$

as $n \to \infty$ where $\mathcal{L}(0,0;z,1) = \mathcal{A}(z) - z^2$ is the parabolic Airy process.

The function $c_1(\eta) = \frac{1-\sqrt{1-\eta^2}}{2}$ for $\eta \in (0,1)$ determines the limit shape of discs in D^{RW} in directions different from horizontal: $t - \sqrt{t^2 - x^2} \leq 2$



Horizontal scale of random walk web distance

Theorem (Hairer, V., Virág, 2026)

There is $\mu \in \mathbb{R}$ *and* $\sigma \in (0, \infty)$ *such that as* $n \to \infty$

- $$\frac{D^{\mathrm{RW}}(-2n, 0; 0, 0)}{\log(2n)} \longrightarrow \mu$$

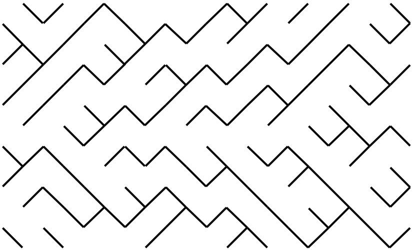
  almost surely and

- $$\frac{D^{\mathrm{RW}}(-2n, 0; 0, 0) - \mu \log(2n)}{\sigma \sqrt{\log n}} \xrightarrow{\mathrm{d}} \chi$$

  where χ *has standard normal distribution.*

Bernoulli-Exponential first passage percolation

- Lattice: $\{(i,n) \in \mathbb{Z}^2 : i+n \text{ is even}\}$ with directed lattice edges from (i,n) to $(i+1, n\pm 1)$
- Free edges: coalescing random walks
- Edge weights: 0 for free edges, independent EXP(1) weights for all other edges
- Distance $T(i,n;j,m)$: weight of the directed path between (i,n) and (j,m) with minimal total weight



$$T(i,n;j,[m,\infty)) = \min_{l\in[m,\infty)} T(i,m;j,l)$$

Bernoulli-Exponential first passage percolation history

- Barraquand–Corwin, 2017: explicit Fredholm determinant formulas for the distribution of $T(0,0;n,[m,\infty))$
- Barraquand–Corwin, 2017: for $\kappa > 0$, as $n \to \infty$

$$\frac{T(0,0;n,[\kappa n,\infty)) - c_1(\kappa)n}{c_2(\kappa)n^{1/3}} \stackrel{\mathrm{d}}{\Longrightarrow} \mathrm{TW}$$

- Barraquand–Rychnovsky, 2019: for all $d > 0$, as $n \to \infty$

$$\frac{T(0,0;n,[dn^{2/3},\infty)) - dn^{2/3}}{c(d)n^{4/9}} \stackrel{\mathrm{d}}{\Longrightarrow} \mathrm{TW}$$

- Question: what are the fluctuations on the scale $\sqrt{n}$ where the free path starting at $(0,0)$ is visible?

MŰEGYETEM 1782

Results on Bernoulli-Exponential first passage percolation

Theorem (V., 2024)

For any $h \in \mathbb{R}$, *as* $n \to \infty$,

$$\sqrt{n}\, T(0,0;n,[h\sqrt{n},\infty)) \xrightarrow{\mathrm{d}} T_h$$

where the distribution of the random variable T_h *is given by*

$$\mathrm{P}(T_h = 0) = \int_h^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,\mathrm{d}y$$

$$\mathrm{P}(T_h > s) = \det(\mathbb{1} - K_s)_{L^2((h,\infty))}$$

and for all $s > 0$ *with*

$$K_s(x,y) = \frac{1}{(2\pi i)^2} \int_{1+i\mathbb{R}} \mathrm{d}u \int_{\mathcal{C}_0} \mathrm{d}v\, \frac{e^{u^2/2 - yu - s/u}}{e^{v^2/2 - xv - s/v}} \frac{u}{v} \frac{1}{v-u}$$

where the integration contour $\mathcal{C}_0$ *is a small circle around* 0 *with positive orientation such that it does not intersect* $1 + i\mathbb{R}$.

Hard edge Bessel kernel recovered as the limit of biorthogonal ensembles Claeys, Zhang, 2026

Height function

In terms of the height function $H(n,r) = \max\{k \in \mathbb{Z} : T(0,0;n,k) \le r\}$:

Theorem

For any $s > 0$, as $n \to \infty$,

$$\frac{1}{\sqrt{n}} H\left(n, \frac{s}{\sqrt{n}}\right) \overset{\mathrm{d}}{\Longrightarrow} H_s$$

where the distribution of the random variable H_s is given by

$$\mathrm{P}(H_s < h) = \det(\mathbb{1} - K_s)_{L^2((h,\infty))}$$

with the same kernel.

- For $s = 0$, clearly $\frac{1}{\sqrt{n}} H(n,0)$ converges to a standard normal which is also obtained by the formal substitution $s = 0$ to K_s
- As $s \to \infty$,

$$2^{4/9} 3^{-1/3} s^{1/9} \left(H_s - 2^{-2/3} 3 s^{1/3}\right) \overset{\mathrm{d}}{\Longrightarrow} \mathrm{TW}$$

Relation to the Brownian net

$T(i,n;j,m)$ is a distance between points of $\mathbb{Z}^2$ which uses coalescing random walk path and exponential weights at jumps to neighbouring paths.

Similar to the marking construction of the Brownian net.

One expects the existence of a distance on the Bronwian net $D^{\mathrm{BN}}(t,x;s,y)$ which is the continuum limit.

Conjecture

$$n^{1/2}T(nt,n^{1/2}x;ns,n^{1/2}y)\to D^{\mathrm{BN}}(t,x;s,y)$$

as $n\to\infty$ where

$$D^{\mathrm{BN}}(0,0;1,[h,\infty))\stackrel{\mathrm{d}}{=}T_h$$

with distribution given by explicit formulas.

Then $D^{\mathrm{BN}}(t,x;s,y)$ should be invariant under the $-1:1:2$ scaling:

$$n^{1/2}D^{\mathrm{BN}}(nt,n^{1/2}x;ns,n^{1/2}y)\stackrel{\mathrm{d}}{=}D^{\mathrm{BN}}(t,x;s,y).$$

Existence and uniqueness of geodesics

Proposition

Assume that $(t,x,s,y)\in\mathbb{R}^4$ with $t<s$ and so that (s,y) has an incoming Brownian web path. Then $D^{\mathrm{Br}}(t,x;s,y)$ and $D^{\mathrm{BN}}(t,x;s,y)$ are finite.

The length of a path $\gamma:[a,b]\to\mathbb{R}$ is

$$|\gamma| = \sup_{a=t_0<\ldots<t_n=b} \sum_{i=1}^{n} D(t_{i-1},\gamma(t_{i-1});t_i,\gamma(t_i))$$

A path γ is a geodesic for D if $|\gamma| = D(a,\gamma(a);b,\gamma(b))$.

Proposition (Non-uniqueness of geodesics for D^{Br})

Assume that $D^{\mathrm{Br}}(t,x;s,y)$ is a non-zero finite integer. Then there are infinitely many geodesics for D^{Br} between (t,x) and (s,y) almost surely.

Proposition (Uniqueness of geodesics for D^{BN})

Assume that $D^{\mathrm{BN}}(t,x;s,y)$ is finite. Then there is a unique geodesic for D^{BN} between (t,x) and (s,y) almost surely.

Discrete-time TASEP with right to left sequential update

$x_j(i)$: position of the jth particle at time i for $i = 0, 1, 2, \dots$ and $j = 0, 1, 2, \dots$

Ordered particles on different locations: $\dots < x_j(i) < x_{j-1}(i) < \dots < x_0(i)$

Update in the time step $i \to i+1$ sequentially for $j = 0, 1, 2, \dots$:
Jump attempt to the right by one with probability $1/2$
Attempt is successful if target is empty: $x_j(i) + 1 < x_{j-1}(i+1)$

Step initial condition: $x_j(0) = -j$ for $j = 0, 1, 2, \dots$

Theorem (Ransford, V., Virág, 2026)

On the left end of the rarefaction fan:

$$x_{n-\sqrt{2}y\sqrt{n}}(n) - x_{n-\sqrt{2}y\sqrt{n}}(0) \overset{\mathrm{d}}{\Longrightarrow} D^{\mathrm{Br}}(0,0;1,[y,\infty))$$

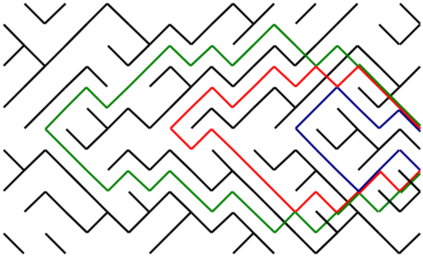
as $n \to \infty$ uniformly in y on compact intervals. In particular,

$$\lim_{n\to\infty} \mathrm{P}\left(x_{n-\sqrt{2}y\sqrt{n}}(n) = x_{n-\sqrt{2}y\sqrt{n}}(0)\right) = \int_y^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,\mathrm{d}z.$$

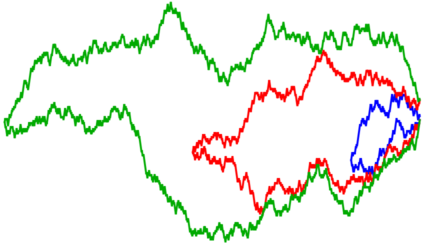
Idea: coupling with the Seppäläinen–Johansson first passage percolation

Boundaries of left neighbourhoods

Regions with blue, red, green boundary: set of starting points with 0, 1 and 2 jumps to the target point on the right



Let $r_k^{\pm}$ and $\rho_k^{\pm}$ be the boundaries of the set of starting points with at most k jumps for D^{RW} and D^{Br}.



Evolution of r_k^{+} given $r_0^{+}, \ldots, r_{k-1}^{+}$: random walk reflected off r_{k-1}^{+} in the discrete Skorokhod sense

Evolution of ρ_k^{+} given $\rho_0^{+}, \ldots, \rho_{k-1}^{+}$: Brownian motion reflected off ρ_{k-1}^{+} in the Skorokhod sense

KPZ limit of Brownian web distance after a shear mapping

Brownian last passage percolation (BLPP):

$$L(t,n) = \sup_{0=t_{-1}\le t_0\le\cdots\le t_n=t} \sum_{i=0}^{n} (W_i(t_i) - W_i(t_{i-1}))$$

where $W_0, W_1, W_2, \ldots$ are independent standard Brownian motions.

Recursion given by Skorokhod reflection:

$$L(t,n) = W_n(t) - \inf_{s\in[0,t]} (W_n(s) - L(t,n-1)).$$

If the target interval is $\{0\}\times\mathbb{R}_-$, then for the boundary

$$\rho_{tn+2zn^{2/3}}(-t) \overset{\mathrm{d}}{=} L(t, tn+2zn^{2/3}) \overset{\mathrm{d}}{=} \frac{1}{\sqrt{n}} L(tn, tn+2zn^{2/3})$$

using the Brownian scaling. The fluctuations of BLPP are known to satisfy

$$\frac{L(tn, tn+2zn^{2/3}) - 2tn - 2zn^{2/3}}{n^{1/3}} \to \mathcal{L}(0,0;z,t).$$

Reason of logarithmic horizontal scaling

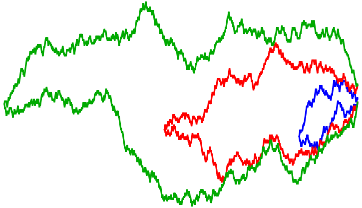
Starting at $(0,0)$, let

$$\tau_k = -\sup\{t \le 0 : \rho_k^+(t) = \rho_k^-(t)\}$$

be the collision time of the kth boundaries. Then

$$M_k = (\rho_k^\pm, \tau_k) \quad k = 0, 1, 2, \ldots$$

forms a Markov chain.



After diffusive rescaling in every step, the Markov chain converges to its unique stationary law under which $\log(\tau_{k+1}/\tau_k)$ has a distribution on $(0,\infty)$ with mean $\mu > 0$.

Hence $\log\tau_k \simeq k\mu$ by LLN which means that for $\tau_k \simeq 2n$ we have
$D^{\mathrm{RW}}(-2n,0;0,0) \simeq k$, that is,
$D^{\mathrm{RW}}(-2n,0;0,0) \simeq \log(n)$.

MŰEGYETEM 1782

HUN REN Ri

The end

Thank you for your attention!