Hardy-Sobolev interpolation inequalities

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Hardy's inequality

$$\boxed{\int_{\mathbb{R}^d} |\nabla u|^2 \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2}}$$

for all $u \in C_c^\infty(\mathbb{R}^d)$, $d \geq 3$

Q: What can we say about the error?

$H^s \hookrightarrow L^q$

$$\left(\int_{\mathbb{R}^d} |\nabla u|^2 - \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \right)^\theta \left(\int_{\mathbb{R}^d} |u|^2 \right)^{1-\theta} \geq C \, \|(-\Delta)^{s/2} u\|_{L^2}^2 \geq C \, \|u\|_{L^q}^2$$

Frank

Brezis-Vazquez

for $d \geq 3$, $0 < s < 1$, $q = \frac{2d}{d-2s}$, $\theta = d\left(\frac{1}{2} - \frac{1}{q}\right)$

Note: $2 < q < 2^* = \frac{2d}{d-2}$. The range of q is optimal, i.e. the result by Brezis-Vazquez does not hold for $q = 2^*$.

Q: Can we prove an improved version of Hardy's inequality involving the L^{2^*}-norm?

Theorem If $d=3$ and $\theta=\frac{1}{3}$, then there exists a constant $C=C(d,\theta)>0$ such that for all $u\in \dot{H}^1(\mathbb{R}^d)$,

$$\boxed{\left(\int_{\mathbb{R}^d}|\nabla u|^2-\frac{(d-2)^2}{4}\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)|^2}{|x-y|^2}dx\right)^{\theta}\left(\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)|^2}{|x-y|^2}dx\right)^{1-\theta}\geq C\|u\|_{L^{2^*}}^2 \qquad (*)}$$

Moreover, $(*)$ does <u>not</u> hold if $d\geq 4$ or $\theta\neq\frac{1}{3}$.

$$\boxed{\left(\int_{\mathbb{R}^d} |\nabla u|^2 - \frac{(d-2)^2}{4} \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2}dx\right)^{\theta} \left(\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2}dx\right)^{1-\theta} \geq C\|u\|^2_{L^{2^*}} \quad (*)}$$

only for $d=3$, $\theta=\frac{1}{3}$

Remarks:

- It is important to include $\sup_{y\in\mathbb{R}^d}$ in the second term. ($\to$ translations of u)
- All terms scale in the same way when replacing u by $a\cdot u(b(x-z))$ for $a,b>0$, $z\in\mathbb{R}^d$.
- Compare with Morrey-type inequality by Palatucci-Pisante

$$\boxed{\left(\int_{\mathbb{R}^d}|\nabla u|^2\right)^{\theta}\left(\sup_{\substack{y\in\mathbb{R}^d\\ R>0}} \frac{1}{R^2}\int_{B_R(y)} |u|^2\right)^{1-\theta} \geq C\|u\|^2_{L^{2^*}} \quad \text{for all } d\geq 3,\ 1-\frac{2}{d}\leq\theta\leq 1}$$

Proof idea:

$$\boxed{\left(\int_{\mathbb{R}^d} |\nabla u|^2 - \frac{(d-2)^2}{4} \sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2}dx\right)^{\theta} \left(\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2}dx\right)^{1-\theta} \geq C\|u\|_{L^{2^*}}^2}$$

only for $d=3, \theta=\frac{1}{3}$

- If $\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2}dx \leq c \int_{\mathbb{R}^d} |\nabla u|^2$, use $\boxed{\left(\int_{\mathbb{R}^d} |\nabla u|^2\right)^{\theta} \left(\sup_{\substack{y\in\mathbb{R}^d\\ R>0}} \frac{1}{R^2}\int_{B_R(y)} |u|^2\right)^{1-\theta} \geq C\|u\|_{L^{2^*}}^2}$
- If $\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^2}dx > c \int_{\mathbb{R}^d} |\nabla u|^2$, then we can replace u by its radial decreasing rearrangement u^* ($\rightarrow$ reduction to 1D problem). Then use an explicit computation and the ground state representation

For $u^*(x) = \frac{f(|x|)}{|x|^{\frac{d-2}{2}}}$, $\boxed{\int_{\mathbb{R}^d} |\nabla u^*|^2 - \frac{(d-2)^2}{4}\int_{\mathbb{R}^d} \frac{|u^*|^2}{|x|^2} = \int_{\mathbb{R}^d} \frac{|\nabla f|^2}{|x|^{d-2}}}$.

Why does it only hold for $d=3$, $\theta=\frac{1}{3}$?

$$\boxed{\left(\int_{\mathbb{R}^d}|\nabla u|^2-\frac{(d-2)^2}{4}\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)|^2}{|x-y|^2}dx\right)^{\theta}\left(\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)|^2}{|x-y|^2}dx\right)^{1-\theta}\geq C\|u\|_{L^{2^*}}^2}$$

- In the case where we use the ground state representation, $\boxed{}$ reduces to

$$\left(\int_0^\infty dr\, r|f'(r)|^2\right)^{\theta}\left(\int_0^\infty dr\,\frac{1}{r}|f(r)|^2\right)^{1-\theta}\geq C\left(\int_0^\infty dr\,\frac{|f(r)|^{2^*}}{r}\right)^{2/2^*}$$

and this requires $\theta\leq\frac{1}{d}$. To see this one can consider

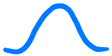
$$f(r)=\begin{cases} r^{\varepsilon}, & 0<r\leq 1\\ r^{-\varepsilon}, & r>1\end{cases}$$

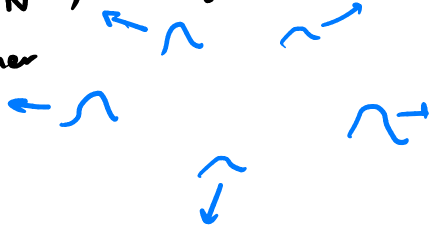
and let $\varepsilon\to 0$.

$$\rightarrow \qquad \varepsilon^{\theta}\left(\frac{1}{\varepsilon}\right)^{1-\theta}\geq C\left(\frac{1}{\varepsilon}\right)^{2/2^*}$$

$$\boxed{\left(\int_{\mathbb{R}^d}|\nabla u|^2-\frac{(d-2)^2}{4}\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)|^2}{|x-y|^2}dx\right)^{\theta}\left(\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)|^2}{|x-y|^2}dx\right)^{1-\theta}\geq C\|u\|_{L^{2^*}}^2}$$

- For the Morrey-type inequality, we need $\theta\geq 1-\frac{2}{d}$. We can also construct a counterexample to $\boxed{}$ for $\theta<1-\frac{2}{d}$:

Let $\varphi\in C_c^\infty(\mathbb{R}^d)$  be fixed. Now define $u=u_N$ by taking N copies of φ and sending them far away from each other



$\rightarrow \quad N^{\theta}\, 1^{1-\theta}\geq C\, N^{2/2^*}$ and let $N\to\infty$

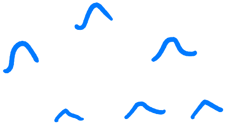
bubbling phenomenon

Open question:

Existence of optimisers? $(d=3,\ \theta=\frac{1}{3})$.

$$\boxed{\left(\int_{\mathbb{R}^d} |\nabla u|^2 - \frac{(d-2)^2}{4}\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)|^2}{|x-y|^2}dx\right)^{\theta}\left(\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)|^2}{|x-y|^2}dx\right)^{1-\theta} \geq C\|u\|_{L^{2^*}}^2}$$

Main difficulties:

- All 3 terms scale in the same way → can only fix the value of one
- bubbling phenomenon: $u_N =$  with φ almost optimiser → $\underline{u_N}$ is optimising sequence

→ "vanishing" minimising sequence → standard concentration compactness techniques do not apply

→ Completely new techniques needed.