Stability in Gagliardo-Nirenberg-Sobolev inequalities: nonlinear flows, regularity and the entropy method

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A joint project with

Jean Dolbeault
▷ *Ceremade, Université Paris-Dauphine (PSL)*

Bruno Nazaret
▷ *Université Paris 1 Panthéon-Sorbonne and Mokaplan team*

Nikita Simonov
▷ *Laboratoire Jacques-Louis Lions (UMR 7598) Sorbonne Université, Université de Paris, and CNRS*

Stability in Gagliardo-Nirenberg-Sobolev inequalities: nonlinear flows, regularity and the entropy method

1. **Gagliardo-Nirenberg-Sobolev inequalities by variational methods**
   - A special family of Gagliardo-Nirenberg-Sobolev inequalities
   - Stability results by variational methods
2. **The fast diffusion equation and the entropy methods**
   - Rényi entropy powers
   - Improved Spectral gaps and Asymptotics
   - Initial time layer
3. **Constructive Regularity for FDE and Stability for GNS**
   - Global Harnack Principle and Regularity Estimates
   - Uniform convergence in relative error
   - The threshold time
   - Improved entropy-entropy production inequality
   - Constructive Stability Results

A celebrated example: **Sobolev inequality**

$$\|\nabla f\|^2_{L^2(\mathbb{R}^d)} \geq \mathscr{S}_d \|f\|^2_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}$$

for all $f \in C_c^\infty$ (actually all $f \in H^1(\mathbb{R}^d)$).

1. **Validity of the inequality:**
   any constant $\mathscr{S}_d > 0$ would do!
2. **Establish the optimal inequality:**
   find optimal functions (equality) and best constant $\mathscr{S}_d > 0$.
3. **The question of stability:**
   if a function satisfies almost equality, can we say that it is almost an optimal one?

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Heat Equation VS Nash Inequalities

(HE) $$\begin{cases} \partial_t u = \Delta u & \text{in } \mathbb{R}^d \times (0,T), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^d. \end{cases}$$

Solutions satisfy the ultracontractive estimates (smoothing effects)

$$\|u(t)\|_\infty \le C \frac{\|u_0\|_1^\alpha}{t^\beta}$$

the powers α, β are fixed by space-time scalings (and mass cons.).
The **representation formula** makes it easy to prove smoothings

$$|u(x,t)| = \left| \int_{\mathbb{R}^d} u_0(y) H_\Delta(x-y,t)\,\mathrm{d}y \right| \le \overline{\kappa} \frac{\|u_0\|_1}{t^{d/2}}$$

just using the **on diagonal bounds** on H_Δ

$$0 \le H_\Delta(x-y,t) = \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{d/2}} \le \frac{\overline{\kappa}}{t^{d/2}}$$

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The Nash/GNS Inequality via Smoothing Effect.

$$\|f\|_2 \le \mathscr{S} \|\nabla f\|_2^{\theta} \|f\|_1^{1-\theta}$$

Derive the L^2-Norm:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} u(t)^2 \,\mathrm{d}x = -2 \int_{\mathbb{R}^d} |\nabla u(t)|^2 \,\mathrm{d}x \ge -2 \int_{\mathbb{R}^d} |\nabla u_0|^2 \,\mathrm{d}x$$

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$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |\nabla u(t)|^2 \,\mathrm{d}x = 2 \int_{\mathbb{R}^d} \nabla u \cdot \nabla \partial_t u \,\mathrm{d}x = -2 \int_{\mathbb{R}^d} (\Delta u)^2 \,\mathrm{d}x \le 0$$

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Optimizing in t gives the Nash inequality for $f = u_0$.

Smoothing Effects via Nash/GNS inequalities

- Nash proved that the smoothing are implied by “his” inequality, using a nice duality trick, exploiting the symmetry of the heat semigroup.
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Gagliardo-Nirenberg-Sobolev inequalities by variational methods

Consider the following family of inequalities

A special family of Gagliardo-Nirenberg-Sobolev inequalities

(GNS) $$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \geq \mathscr{C}_{\mathrm{GNS}}(p) \, \|f\|_{2p}$$

with

$$\theta = \frac{d\,(p-1)}{(d+2-p\,(d-2))\,p}\,, \qquad p \in \begin{cases} (1,+\infty) & \text{if } d=1,2 \\ (1,p^*] & \text{if } d \geq 3\,, \end{cases} \qquad p^* = \frac{d}{d-2} = \frac{2^*}{2}$$

▷ The validity of the inequality (no sharp constant) is due to [Sobolev 1938], [Gagliardo, Nirenberg 1958], but also DeGiorgi, Hardy, Ladyzenskaya, Littlewood, ...
▷ The family contains the classical Sobolev Inequality: $p = p^*$

$$\boxed{\mathsf{S}_d \, \|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 \geq \|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^2}$$

Optimal functions . . .

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▷ Up to translations, multiplications by a constant and scalings, *there is a unique optimal function* which also provides the value of the optimal constant.

$$\boxed{\mathsf{g}(x) = \left(1+|x|^2\right)^{-\frac{1}{p-1}}}$$

▷ The Sobolev Case $p = p*$ was obtained by [Aubin, Talenti (1976)]. . .

. . . and (before) by [Rodemich (1966)], while the general case was established in 2002

Theorem (Optimal GNS [Del Pino - Dolbeault (2002)]**)**

Equality case in (GNS) *is achieved if and only if*

$$f \in \mathfrak{M} := \left\{ g_{\lambda,\mu,y} : (\lambda,\mu,y) \in (0,+\infty)\times\mathbb{R}\times\mathbb{R}^d \right\}$$

Aubin-Talenti functions: $\boxed{g_{\lambda,\mu,y}(x) := \mu\,\mathsf{g}\left((x-y)/\lambda\right)}$.

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$$\mathscr{C}_{\mathrm{GNS}}(p) = \frac{\left(\frac{4d}{p-1}\pi\right)^{\frac{\theta}{2}} \left(2(p+1)\right)^{\frac{1-\theta}{p+1}}}{(d+2-p(d-2))^{\frac{d-p(d-4)}{2p(d+2-p(d-2))}}} \, \Gamma\left(\frac{2p}{p-1}\right)^{-\frac{\theta}{d}} \Gamma\left(\frac{2p}{p-1} - \frac{d}{2}\right)^{\frac{\theta}{d}} .$$

-1-

Analysis Seminar
Cal. Tech

Spring 1966

The Sobolev inequalities with best possible constants.

by E. Rodemich

1. Introduction

In n-space we define

$$I_p(g) = \int |g|^p dx ,$$

for any scalar or vector function g, with the integral extended over all space. If g is a vector $(g_1, \dots, g_n)$, $|g|$ denotes $\sqrt{\Sigma |g_j|^2}$.

Sobolev's inequality is

$$[I_p(\varphi)]^{1/p} \le C [I_r(\nabla\varphi)]^{1/r}, \tag{1}$$

for any differentiable function φ with compact support and derivatives in L^r, where

$$1 \le r < n ,$$

and

$$\frac{1}{p} + \frac{1}{n} = \frac{1}{r} . \tag{2}$$

The stability result of G. Bianchi and H. Egnell

In Sobolev's inequality (with optimal contant S_d),

$$\delta[f] := \mathsf{S}_d \,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge 0$$

is there a natural way to bound the l.h.s. from below in terms of a "distance" to the set of optimal [Aubin-Talenti] functions when $d \ge 3$?

A question raised in [Brezis-Lieb (1985)]

▷ [Bianchi-Egnell (1991)] There is a positive constant c such that

$$\mathsf{S}_d \,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge \mathsf{c} \inf_{\varphi \in \mathcal{M}} \|\nabla f - \nabla \varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

▷ Various improvements, *e.g.*, [Cianchi, Fusco, Maggi, Pratelli (2009)] there are constants α and κ and $f \mapsto \lambda(f)$ such that

$$\mathsf{S}_d \,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \left(1 + \kappa\, \lambda(f)^\alpha\right) \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

▷ L^q-norm of gradient [Figalli, Maggi, Pratelli (2010,13)], [Figalli, Neumayer (2018)], [Neumayer (2020)], [Figalli, Zhang (2020)]

▷ GNS by [Carlen, Figalli (2013)], [Seuffert (2017)], [Nguyen (2019)]

▷ *However, the question of **constructive** estimates was/is still widely open*

▷ Recent result by [Dolbeault, Esteban, Figalli, Frank, Loss 2023] (Sobolev)

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▷ Various improvements, *e.g.*, [Cianchi, Fusco, Maggi, Pratelli (2009)] there are constants α and κ and $f \mapsto \lambda(f)$ such that

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The stability result of G. Bianchi and H. Egnell

In Sobolev's inequality (with optimal contant S_d),

$$\delta[f] := \mathsf{S}_d \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge 0$$

is there a natural way to bound the l.h.s. from below in terms of a "distance" to the set of optimal [Aubin-Talenti] functions when $d \ge 3$?

A question raised in [Brezis-Lieb (1985)]

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Deficit, scale invariance

Recall the optimal GNS

(GNS) $$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \ge \mathscr{C}_{\rm GNS}(p)\, \|f\|_{2p}$$

▷ Deficit functional *(Non-scale invariant Gagliardo-Nirenberg-Sobolev inequalities)*

$$\delta[f] := \underbrace{(p-1)^2}_{a} \|\nabla f\|_2^2 + 4\underbrace{\frac{d-p\,(d-2)}{p+1}}_{b} \|f\|_{p+1}^{p+1} - \mathscr{K}_{\rm GNS}\, \|f\|_{2p}^{2p\gamma}$$

Lemma

(GNS) *is equivalent to* $\delta[f] \ge 0$ *if and only if*

$$\mathscr{K}_{\rm GNS} = C(p,d)\, \mathscr{C}_{\rm GNS}^{2p\gamma}$$

where $\gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$ *and* $C(p,d)$ *is an explicit positive constant*

[Proof: Take $f_\lambda(x) = \lambda^{\frac{d}{2p}} f(\lambda\, x)$ and optimize on $\lambda > 0$]

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An abstract stability result

Relative entropy

$$\mathscr{F}[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left(f^{p+1} - \mathrm{g}^{p+1} - \frac{1+p}{2p}\, \mathrm{g}^{1-p} \left(f^{2p} - \mathrm{g}^{2p} \right) \right) dx$$

Deficit functional

$$\delta[f] := a\, \|\nabla f\|_2^2 + b\, \|f\|_{p+1}^{p+1} - \mathscr{K}_{\mathrm{GN}}\, \|f\|_{2p}^{2p\gamma} \geq 0$$

Theorem (Abstract Stability for GNS [BDNS (2020)])

Let $d \geq 1$ *and* $p \in (1, p^*)$. *There is a* $\mathscr{C} > 0$ *such that*

$$\delta[f] \geq \mathscr{C}\, \mathscr{F}[f]$$

for any $f \in \mathscr{W} := \left\{ f \in \mathrm{L}^1(\mathbb{R}^d, (1+|x|)^2\, dx) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, dx) \right\}$ *such that*

$$\int_{\mathbb{R}^d} f^{2p}(1,x)\, dx = \int_{\mathbb{R}^d} |\mathrm{g}|^{2p}(1,x)\, dx$$

Relative entropy, relative Fisher information

Idea of the proof of the Abstract Stability result:

▷ *Free energy* or *relative entropy functional*

$$\mathcal{E}[f|g] := \frac{2\,p}{1-p}\int_{\mathbb{R}^d}\left(f^{p+1}-g^{p+1}-\tfrac{1+p}{2\,p}\,g^{1-p}\left(f^{2\,p}-g^{2\,p}\right)\right)\mathrm{d}x$$

▷ *Relative Fisher information* or *Entropy production*

$$\mathcal{J}[f|g] := \frac{p+1}{p-1}\int_{\mathbb{R}^d}\left|(p-1)\,\nabla f+f^p\,\nabla g^{1-p}\right|^2\mathrm{d}x$$

It turns out that the GNS is nothing but a *Entropy - Entropy Production inequality*:

Lemma (Entropy - Entropy Production inequality [Del Pino - Dolbeault (2002)])

$$\frac{p+1}{p-1}\,\delta[f] = \mathcal{J}\left[f|g_f\right]-4\,\mathcal{E}\left[f|g_f\right]\geq 0$$

A weak stability result and the entropy controls L^1 distance

Lemma (A weak stability result [Dolbeault-Toscani (2016)])

$$\delta[f] \gtrapprox \mathcal{E}[f|g]^2$$

If $\int_{\mathbb{R}^d} f^{2p}(1, x, |x|^2)\, dx = \int_{\mathbb{R}^d} g^{2p}(1, x, |x|^2)\, dx, \quad g \in \mathfrak{M}$

then $\mathcal{E}[f|g] = \frac{2p}{1-p} \int_{\mathbb{R}^d} \left(f^{p+1} - g^{p+1} \right) dx$ and $\delta[f] \gtrapprox \mathcal{E}[f|g]^2$

Lemma (Csiszár-Kullback inequality [BDNS (2020)])

Let $d \geq 1$ and $p > 1$. There exists a constant $C_p > 0$ such that

$$\left\| f^{2p} - g^{2p} \right\|^2_{L^1(\mathbb{R}^d)} \leq C_p \, \mathcal{E}[f|g] \quad \textit{if} \quad \|f\|_{2p} = \|g\|_{2p}$$

▷ The proof uses also:

- the Carré du Champ method (nonlinear version of Bakry-Emery)
- Concentration Compactness (that is where "we lose the constant"). □

A constructive stability result by the “flow method”

The relative entropy

$$\mathscr{F}[f] := \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(f^{p+1} - \mathrm{g}^{p+1} - \frac{1+p}{2p}\,\mathrm{g}^{1-p}\left(f^{2p} - \mathrm{g}^{2p}\right)\right)\mathrm{d}x$$

The deficit functional

$$\delta[f] := a\,\|\nabla f\|_2^2 + b\,\|f\|_{p+1}^{p+1} - \mathcal{K}_{\mathrm{GN}}\,\|f\|_{2p}^{2p\gamma} \ge 0$$

Theorem (Constructive Stability for GNS — BDNS (2020))

Let $d \ge 1$, $p \in (1, p^*)$, $A > 0$ and $G > 0$. There is an explicit constant $\mathscr{C} = \mathscr{C}(d, p, A, G) > 0$ such that

$$\boxed{\delta[f] \ge \mathscr{C}\,\mathscr{F}[f]}$$

for any $f \in \mathscr{W} := \left\{f \in \mathrm{L}^1(\mathbb{R}^d, (1+|x|)^2\,\mathrm{d}x) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x)\right\}$ such that

$$\int_{\mathbb{R}^d} f^{2p}\,\mathrm{d}x = \int_{\mathbb{R}^d} |\mathrm{g}|^{2p}\,\mathrm{d}x\,, \quad \int_{\mathbb{R}^d} x\,f^{2p}\,\mathrm{d}x = 0$$

$$\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} f^{2p}\,dx \le A \quad \text{and} \quad \mathscr{F}[f] \le G$$

The fast diffusion equation and the entropy methods

$$\frac{\partial u}{\partial t} = \Delta u^m$$

Letting

$$u = f^{2p} \qquad \text{so that} \qquad u^m = f^{p+1}$$

we have

$$p = \frac{1}{2m-1} \in (1, p^*] \iff m = \frac{p+1}{2p} \in [m_1, 1)$$

- ▷ The Rényi entropy powers and the Gagliardo-Nirenberg inequalities: Nonlinear Carré du Champ method in original variables.
- ▷ Selfsimilar variables: the Nonlinear Fokker-Plank FDE Self-similar solutions and the entropy-entropy production method
- ▷ Large time asymptotics: spectral analysis (Hardy-Poincaré inequality) and improved rates of convergence to equilibrium. Constructive regularity estimates needed.
- ▷ The initial time layer improvement: backward estimate. Bringing the asymptotic improvement as $t \to \infty$ back to $t = 0$.

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The fast diffusion equation in original variables

Consider the fast diffusion equation in $\mathbb{R}^d$, $d \geq 1$, $m \in (0,1)$

(FDE) $$\frac{\partial u}{\partial t} = \Delta u^m$$

with initial datum $u(t=0,x) = u_0(x) \geq 0$ such that

$$\int_{\mathbb{R}^d} u_0 \, \mathrm{d}x = \mathscr{M} > 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 \, u_0 \, \mathrm{d}x < +\infty$$

The large time behavior is governed by the self-similar Barenblatt solutions

$$B(t,x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \mathscr{B}\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where $\mu := 2 + d\,(m-1)$, $\kappa := \left|\frac{2\,\mu\,m}{m-1}\right|^{1/\mu}$ and $\mathscr{B}$ is the Barenblatt profile

$$\mathscr{B}(x) := \left(C + |x|^2\right)^{-\frac{1}{1-m}}$$

▷ Existence and uniqueness has been proven by [Herrero-Pierre (1981)] see also [Vazquez (2006,07)]

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Mass, moment, entropy and Fisher information

(i) *Mass conservation.* With $m \geq m_c := (d-2)/d$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\,\mathrm{d}x = 0$$

(ii) *Second moment.* With $m > d/(d+2)$ and $u_0 \in \mathrm{L}^1_+\big(\mathbb{R}^d,(1+|x|^2)\,\mathrm{d}x\big)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |x|^2\,u(t,x)\,\mathrm{d}x = 2\,d\int_{\mathbb{R}^d} u^m(t,x)\,\mathrm{d}x$$

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Entropy functional and *Fisher information functional*

$$\mathsf{E}[u] := \int_{\mathbb{R}^d} u^m\,\mathrm{d}x \quad \text{and} \quad \mathsf{I}[u] := \frac{m^2}{(1-m)^2}\int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,\mathrm{d}x$$

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$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |x|^2\, u(t,x)\,\mathrm{d}x = 2\,d \int_{\mathbb{R}^d} u^m(t,x)\,\mathrm{d}x$$

(iii) *Entropy estimate*. With $m \geq m_1 := (d-1)/d$, $u_0^m \in \mathrm{L}^1(\mathbb{R}^d)$ and $u_0 \in \mathrm{L}^1_+\big(\mathbb{R}^d, (1+|x|^2)\,\mathrm{d}x\big)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} u^m(t,x)\,\mathrm{d}x = \frac{m^2}{1-m} \int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,\mathrm{d}x$$

Entropy functional and *Fisher information functional*

$$\mathsf{E}[u] := \int_{\mathbb{R}^d} u^m\,\mathrm{d}x \quad \text{and} \quad \mathsf{I}[u] := \frac{m^2}{(1-m)^2} \int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,\mathrm{d}x$$

From the carré du champ method to stability results

▷ ***Nonlinear Carré du champ method*** (adapted from D. Bakry and M. Emery)

$$\frac{\partial u}{\partial t} = \Delta u^m , \quad \frac{\mathrm{d}\mathsf{E}}{\mathrm{d}t} = -\mathsf{I} , \quad \frac{\mathrm{d}\mathsf{I}}{\mathrm{d}t} \leq -\Lambda\,\mathsf{I}$$

deduce that $\mathsf{I} - \Lambda\,\mathsf{F}$ is monotone non-increasing with limit 0

$$\boxed{\mathsf{I}[u] \geq \Lambda\,\mathsf{F}[u]}$$

Consequence: $\boxed{\mathsf{I} - \Lambda\,\mathsf{F} \geq 0}$ is equivalent to sharp GNS $\boxed{\delta[f] \geq 0}$

▷ ***Improved constant*** means ***stability***

Under some restrictions on the functions, there is some $\Lambda_\star \geq \Lambda$ such that

$$\boxed{\mathsf{I} - \Lambda\,\mathsf{F} \geq (\Lambda_\star - \Lambda)\,\mathsf{F}}$$

We use linearization and *improved Hardy-Poincaré Inequalities*

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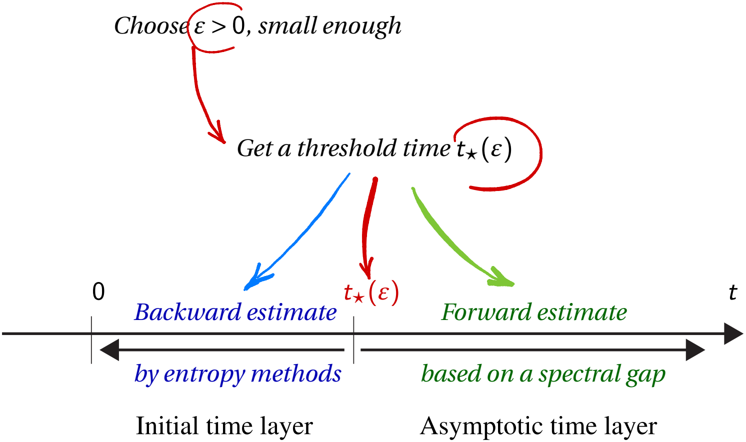
Stability in (subcritical) Gagliardo-Nirenberg inequalities: The Flow Method

*Our strategy: a deep **constructive** analysis of the FDE flow for all times*



Self-similar variables: entropy-entropy production inequality

With a time-dependent rescaling based on *self-similar variables*

$$u(t,x)=\frac{1}{\kappa^d\,R^d}\,v\Big(\tau,\frac{x}{\kappa\,R}\Big)\quad\text{where}\quad \frac{dR}{dt}=R^{1-\mu}\,,\quad \tau(t):=\tfrac12\log R(t)$$

$\frac{\partial u}{\partial t}=\Delta u^m$ is changed into *a Fokker-Planck type equation*

(1) $$\boxed{\frac{\partial v}{\partial \tau}+\nabla\cdot\Big[v\Big(\nabla v^{m-1}-2\,x\Big)\Big]=0}\quad\text{and}\quad \mathcal{B}(x):=\big(C+|x|^2\big)^{-\frac{1}{1-m}}$$

Generalized entropy (free energy) and *Fisher information*

$$\mathcal{F}[v]:=-\frac{1}{m}\int_{\mathbb{R}^d}\big(v^m-\mathcal{B}^m-m\,\mathcal{B}^{m-1}(v-\mathcal{B})\big)\,dx$$

$$\mathcal{I}[v]:=\int_{\mathbb{R}^d}v\,\big|\nabla v^{m-1}+2\,x\big|^2\,dx$$

are such that $\boxed{\mathcal{I}[v]\ge 4\,\mathcal{F}[v]}$ by (GNS) [Del Pino-Dolbeault (2002)] so that

$$\boxed{\mathcal{F}[v(t,\cdot)]\le\mathcal{F}[v_0]\,e^{-4\,t}}$$

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$$u(t,x) = \frac{1}{\kappa^d R^d}\, v\Big(\tau, \frac{x}{\kappa R}\Big) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}, \quad \tau(t) := \tfrac{1}{2}\log R(t)$$

$\frac{\partial u}{\partial t} = \Delta u^m$ is changed into *a Fokker-Planck type equation*

$$(1) \quad \boxed{\frac{\partial v}{\partial \tau} + \nabla \cdot \Big[v\big(\nabla v^{m-1} - 2\,x\big)\Big] = 0} \quad \text{and} \quad \mathscr{B}(x) := \big(C + |x|^2\big)^{-\frac{1}{1-m}}$$

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Spectral gap: sharp asymptotic rates of convergence

[Blanchet, MB, Dolbeault, Grillo, Vázquez, BBDGV (2009) and BDGV (2010)]

(H) $$\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}}$$

Let $\Lambda_{\alpha,d}>0$ be the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d} f^2\,\mathrm{d}\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}\mu_\alpha \quad \forall\, f\in \mathrm{H}^1(\mathrm{d}\mu_\alpha)\,,\quad \int_{\mathbb{R}^d} f\,\mathrm{d}\mu_{\alpha-1}=0$$

with $\mathrm{d}\mu_\alpha := (1+|x|^2)^\alpha\,\mathrm{d}x$, for $\alpha<0$

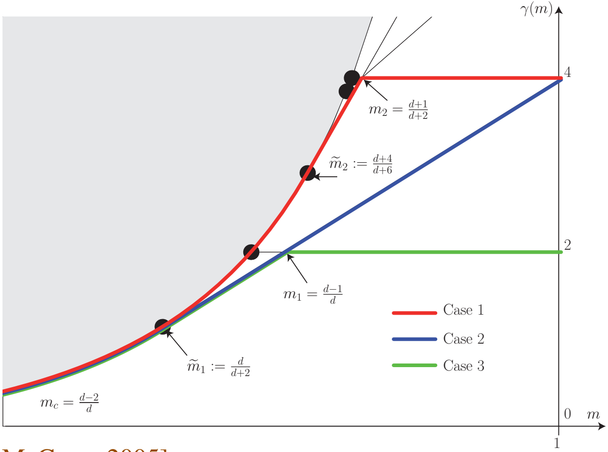
Lemma ([BBDGV (2009), BDGV (2010)])

Under assumption (H), *for all* $m\in(0,1)$

$$\mathscr{F}[v(t,\cdot)] \le C\,e^{-2\gamma(m)\,t} \quad \forall\, t\ge 0\,,\quad \gamma(m):=(1-m)\,\Lambda_{1/(m-1),d}$$

Moreover $\boxed{\gamma(m):=2}$ *if* $\frac{d-1}{d}=m_1\le m<1$ *(the case under consideration here)*

Spectral gap



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [Dolbeault, Toscani, 2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

The asymptotic time layer improvement

▷ *Linearized free energy* and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} g^2\,\mathscr{B}^{2-m}\,dx \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathscr{B}\,dx$$

[*Weighted linearization: consider* $v = \mathscr{B} + \mathsf{h}\mathscr{B}^{2-m} g$ *as* $\mathsf{h}\to 0$]

▷ *Hardy-Poincaré inequality*. Let $d\ge 1$, $m\in(m_1,1)$ and $g\in L^2(\mathbb{R}^d,\mathscr{B}^{2-m}\,dx)$ such that $\nabla g\in L^2(\mathbb{R}^d,\mathscr{B}\,dx)$, $\int_{\mathbb{R}^d} g\,\mathscr{B}^{2-m}\,dx = 0$ and $\int_{\mathbb{R}^d} x\,g\,\mathscr{B}^{2-m}\,dx = 0$

$$\mathsf{I}[g] \ge 4\,\alpha\,\mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d\,(1-m)$$

Proposition (Asymptotic time layer improvement [BDNS (2021)])

Let $m\in(m_1,1)$ *if* $d\ge 2$, $m\in(1/3,1)$ *if* $d=1$, $\eta = 2\,(d\,m-d+1)$ *and* $\chi = m/(266+56\,m)$. *If* $\int_{\mathbb{R}^d} v\,dx = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v\,dx = 0$ *and*

$$(1-\varepsilon)\,\mathscr{B} \le v \le (1+\varepsilon)\,\mathscr{B}$$

for some $\varepsilon\in(0,\chi\,\eta)$, *then*

$$\mathscr{I}[v] \ge (4+\eta)\,\mathscr{F}[v]$$

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The initial time layer improvement: backward estimate

▷ Rephrasing the *nonlinear carré du champ* method:

$$\mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]} \quad \text{is such that} \quad \boxed{\frac{d\mathcal{Q}}{dt} \le \mathcal{Q}\,(\mathcal{Q}-4)}$$

Lemma (Initial time layer improvement [BDNS (2021)])

Assume that $m > m_1$ and v is a solution to (1) with nonnegative initial datum v_0. If for some $\eta > 0$ and $T > 0$, we have

$$\mathcal{Q}[v(T,\cdot)] \ge 4+\eta, \quad \text{then} \quad \boxed{\mathcal{Q}[v(t,\cdot)] \ge 4 + \frac{4\,\eta\, e^{-4T}}{4+\eta-\eta\, e^{-4T}} \quad \forall\, t \in [0,T]}$$

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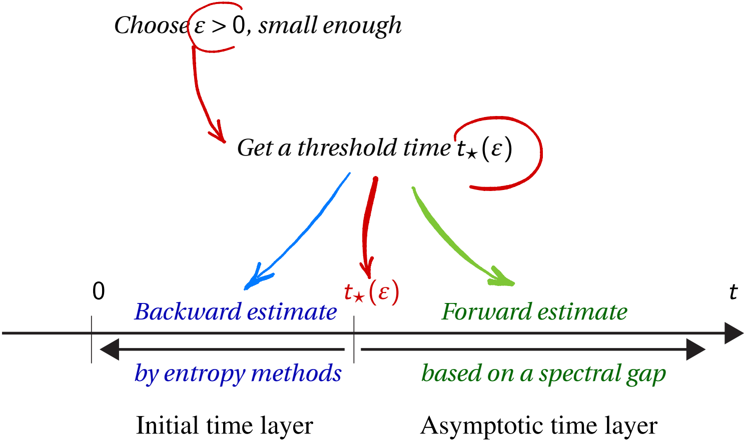
Stability in (subcritical) Gagliardo-Nirenberg inequalities

Our strategy



The threshold time and the uniform convergence in relative error (UCRE)

Theorem (Uniform convergence in relative error [BDNS (2021)])

Assume that $m \in [m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$ be given. There exists an explicit threshold time $t_\star \geq 0$ such that, if u is a solution of

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{2}$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} u_0\, dx \leq A < \infty \tag{H_A}$$

$\int_{\mathbb{R}^d} u_0 \, \mathrm{d}x = \int_{\mathbb{R}^d} \mathscr{B} \, \mathrm{d}x = \mathscr{M}$ *and* $\boxed{\mathscr{F}[u_0] \leq G}$, *then*

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t,x)}{\mathscr{B}(t,x)} - 1 \right| \leq \varepsilon \quad \forall\, t \geq t_\star$$

The Explicit Threshold Time: a Journey into Constructive Regularity

Proposition (Explicit threshold time [BDNS (2021)])

Let $m \in (m_1, 1)$ *if* $d \geq 2$, $m \in (1/3, 1)$ *if* $d = 1$, $\varepsilon \in (0, \varepsilon_{m,d})$, $A > 0$ *and* $G > 0$

$$t_\star = \mathsf{c}_\star \frac{1 + A^{1-m} + G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}$$

where $\mathsf{a} = \frac{\alpha}{\vartheta} \frac{2-m}{1-m}$, $\alpha = d\,(m - m_c)$ *and* $\vartheta = \nu/(d+\nu)$

$$\mathsf{c}_\star = \mathsf{c}_\star(m,d) = \sup_{\varepsilon \in (0,\varepsilon_{m,d})} \max\left\{\varepsilon\,\kappa_1(\varepsilon,m), \varepsilon^{\mathsf{a}}\kappa_2(\varepsilon,m), \varepsilon\,\kappa_3(\varepsilon,m)\right\}$$

$$\kappa_1(\varepsilon,m) := \max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m} - 1}, \frac{2^{3-m}\,\kappa_\star}{1-(1-\varepsilon)^{1-m}}\right\}$$

$$\kappa_2(\varepsilon,m) := \frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\,\frac{\alpha}{\vartheta}}} \quad \text{and} \quad \kappa_3(\varepsilon,m) := \frac{8\,\alpha^{-1}}{1-(1-\varepsilon)^{1-m}}$$

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Global Harnack Principle and Uniform Convergence in Relative Error

The proof of UCRE requires various constructive regularity estimates:

Theorem (Characterization of GHP and UCRE [MB-Simonov (2021)]**)**

Assume that $m \in (m_c, 1)$ *where* $m_c := \frac{d-2}{d}$*, and if* u *is a solution to the Cauchy problem for* (FDE)*. Then the following assertions are equivalent*
(i) The initial datum satisfies the tail condition H_A*, namely*

$$(\mathrm{H}_A) \qquad \sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} u_0\,dx < \infty$$

(i') The solution satisfies the tail condition H_A*, at some time* $t_0 \in [0,\infty)$*.*
(ii) The Global Harnack Principle holds true: $\exists \tau_1, \tau_2, M_1, M_2 > 0$ *such that*

$$(\mathrm{GHP}) \qquad \mathcal{B}_{M_1}(t-\tau_1, x) \le u(t,x) \le \mathcal{B}_{M_2}(t+\tau_2, x)\,,$$

(iii) The solution "converges" uniformly in relative error to the Barenblatt solution with the same mass:

$$(\mathrm{UCRE}) \qquad \lim_{t\to\infty} \left\| \frac{u(t)-\mathcal{B}_M(t)}{\mathcal{B}_M(t)} \right\|_{L^\infty(\mathbb{R}^d)} = 0$$

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(iii) The solution "converges" uniformly in relative error to the Barenblatt solution with the same mass:

$$(\mathrm{UCRE}) \qquad \lim_{t\to\infty} \left\| \frac{u(t) - \mathcal{B}_M(t)}{\mathcal{B}_M(t)} \right\|_{L^\infty(\mathbb{R}^d)} = 0$$

More about Global Harnack Principle

▷ If the tail condition H_A is not satisfied, GHP and UCRE are not true:

$$\mathcal{B}_M(t,x) \lneqq u_0(x) = \frac{1}{(1+|x|^2)^{\frac{m}{1-m}}},$$

then the solution $u(t,x)$ with initial data u_0 satisfies

$$\mathcal{B}_M(t,x) \lneqq \frac{1}{\left[(c\,t+1)^{\frac{1}{1-m}}+|x|^2\right]^{\frac{m}{1-m}}} \le u(t,x) \le \frac{(1+t)^{\frac{m}{1-m}}}{\left(1+t+|x|^2\right)^{\frac{m}{1-m}}},$$

Recall that $\mathcal{B}_M(t,x) \sim |x|^{-\frac{2}{1-m}}$

▷ The GHP was first proven by [Vazquez (2003)] for radial functions, then by [MB-Vazquez (2006)] under non-sharp conditions on the data. [Carrillo-Vazquez (2003)] introduced a condition a posteriori equivalent to H_A and conjectured that it was sharp.
▷ Our result, together with the results of [Denzler-Koch-McCann (2015)] provide an optimal answer to the conjecture about sharp rates of decay for UCRE:

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Ideas of the proof of the constructive UCRE and of the threshold time

▷ The GHP implies UCRE on "outer cylinders" of the type $\{|x| \geq C t^{\vartheta}\}$.

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Improved entropy – entropy production inequality: already a stability result

Theorem (Improved entropy – entropy production inequality [BDNS (2021)])

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. Then there is a positive number ζ such that

$$\mathscr{I}[v] \geq (4+\zeta)\,\mathscr{F}[v]$$

for any nonnegative function $v \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v] = G$, $\int_{\mathbb{R}^d} v\,\mathrm{d}x = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v\,\mathrm{d}x = 0$ and v satisfies (H_A)

We have the *asymptotic time layer estimate*

$$\varepsilon \in (0, 2\varepsilon_\star)\,, \quad \varepsilon_\star := \frac{1}{2}\min\left\{\varepsilon_{m,d}, \chi\,\eta\right\} \quad \text{with} \quad T = \frac{1}{2}\log R(t_\star)$$

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Two consequences

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▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary (Improved rates of convergence [BDNS (2021)])

Let $m \in (m_1, 1)$ *if* $d \ge 2$, $m \in (1/2, 1)$ *if* $d = 1$, $A > 0$ *and* $G > 0$. *If* v *is a solution of* (1) *with nonnegative initial datum* $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ *such that* $\mathscr{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,\mathrm{d}x = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v_0\,\mathrm{d}x = 0$ *and* v_0 *satisfies* (H_A), *then*

$$\mathscr{F}[v(t,.)] \le \mathscr{F}[v_0]\,e^{-(4+\zeta)\,t} \quad \forall\, t \ge 0$$

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Constructive stability results I - Subcritical Case - Entropy Version

Theorem (Constructive stability I for Gagliardo-Nirenberg [BDNS (2020)])

Let $d \geq 1$, $p \in (1, p^*)$, *where* $p^* = +\infty$ *if* $d = 1$ *or* 2, $p^* = \frac{d}{d-2}$ *if* $d \geq 3$.

If $f \in \mathscr{W}_p(\mathbb{R}^d) := \left\{ f \in \mathrm{L}^{2p}(\mathbb{R}^d) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d),\ |x|\, f^p \in \mathrm{L}^2(\mathbb{R}^d) \right\}$,

$$\left(\|\nabla f\|_2^{\theta} \, \|f\|_{p+1}^{1-\theta} \right)^{2p\gamma} - \left(\mathscr{C}_{\mathrm{GN}} \, \|f\|_{2p} \right)^{2p\gamma} \geq \mathfrak{S}[f] \, \|f\|_{2p}^{2p\gamma} \, \mathsf{E}[f]$$

where

$$\mathfrak{S}[f] := \frac{\mathscr{M}^{\frac{p-1}{2p}}}{p^2 - 1} \, \frac{\mathsf{Z}\big(\mathsf{A}[f], \mathsf{E}[f]\big)}{C(p,d)} = \frac{\mathsf{k}_{p,d} \, \zeta_\star}{1 + \mathsf{A}[f]^{(1-m)\frac{2}{\alpha}} + \mathsf{E}[f]}$$

$$\mathsf{E}[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2p}}} \, f^{p+1} - \mathrm{g}^{p+1} - \frac{1+p}{2p} \, \mathrm{g}^{1-p} \left(\frac{\kappa[f]^{2p}}{\lambda[f]^2} \, f^{2p} - \mathrm{g}^{2p} \right) \right) dx$$

$$\mathsf{A}[f] := \frac{\mathscr{M}}{\lambda[f]^{\frac{d-p(d-4)}{p-1}} \, \|f\|_{2p}^{2p}} \, \sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} |f(x + x_f)|^{2p} \, dx$$

$$\lambda[f] := \left(\frac{2d\,\kappa[f]^{p-1}}{p^2-1} \, \frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2} \right)^{\frac{2p}{d-p(d-4)}}, \qquad \kappa[f] := \frac{\mathscr{M}^{\frac{1}{2p}}}{\|f\|_{2p}}$$

Constructive stability results I - Subcritical Case - Entropy Version

Theorem (Constructive stability I for Gagliardo-Nirenberg [BDNS (2020)])

Let $d \ge 1$, $p \in (1, p^*)$, where $p^* = +\infty$ if $d = 1$ or 2, $p^* = \frac{d}{d-2}$ if $d \ge 3$.

$$\text{If } f \in \mathcal{W}_p(\mathbb{R}^d) := \left\{ f \in \mathrm{L}^{2p}(\mathbb{R}^d) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d),\ |x|\, f^p \in \mathrm{L}^2(\mathbb{R}^d) \right\},$$

$$\left(\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \right)^{2p\gamma} - \left(\mathscr{C}_{\rm GN} \, \|f\|_{2p} \right)^{2p\gamma} \ge \mathfrak{S}[f] \, \|f\|_{2p}^{2p\gamma} \, \mathsf{E}[f]$$

where

$$\mathfrak{S}[f] := \frac{\mathscr{M}^{\frac{p-1}{2p}}}{p^2-1} \, \frac{\mathsf{Z}\big(\mathsf{A}[f], \mathsf{E}[f]\big)}{C(p,d)} = \frac{\mathsf{k}_{p,d}\, \zeta_\star}{1 + \mathsf{A}[f]^{\,(1-m)\,\frac{2}{\alpha}} + \mathsf{E}[f]}$$

$$\mathsf{E}[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2p}}} f^{p+1} - \mathsf{g}^{p+1} - \frac{1+p}{2p}\, \mathsf{g}^{1-p} \left(\frac{\kappa[f]^{2p}}{\lambda[f]^2} f^{2p} - \mathsf{g}^{2p} \right) \right) dx$$

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where

$$\mathfrak{S}[f] := \frac{\mathcal{M}^{\frac{p-1}{2p}}}{p^2 - 1} \, \frac{\mathsf{Z}\left(\mathsf{A}[f], \mathsf{E}[f]\right)}{C(p,d)} = \frac{\mathsf{k}_{p,d}\, \zeta_\star}{1 + \mathsf{A}[f]^{\,(1-m)\,\frac{2}{\alpha}} + \mathsf{E}[f]}$$

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$$\lambda[f] := \left(\frac{2\,d\,\kappa[f]^{p-1}}{p^2-1} \, \frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2} \right)^{\frac{2p}{d-p(d-4)}}, \qquad \kappa[f] := \frac{\mathcal{M}^{\frac{1}{2p}}}{\|f\|_{2p}}$$

Constructive stability results II - Subcritical Case - Gradient Version

With $\mathscr{K}_{\rm GNS} = C(p,d)\,\mathscr{C}_{\rm GNS}^{2p\gamma}$, $\gamma = \frac{d+2-p(d-2)}{d-p(d-4)}$, consider the *deficit functional*

$$\delta[f] := (p-1)^2\,\|\nabla f\|_2^2 + 4\,\frac{d-p(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\rm GNS}\,\|f\|_{2p}^{2p\gamma}$$

Theorem (Constructive stability II for Gagliardo-Nirenberg [BDNS (2020)])

Let $d \ge 1$ and $p \in (1,p^*)$. There is an explicit $\mathscr{C} = \mathscr{C}[f]$ such that, for any $f \in \mathrm{L}^{2p}\big(\mathbb{R}^d,(1+|x|^2)\,dx\big)$ such that $\nabla f \in \mathrm{L}^2(\mathbb{R}^d)$ and $\mathrm{A}\big[f^{2p}\big] < \infty$,

$$\delta[f] \ge \mathscr{C}[f]\,\inf_{\varphi\in\mathfrak{M}} \int_{\mathbb{R}^d} \left|(p-1)\,\nabla f + f^p\,\nabla\varphi^{1-p}\right|^2 \mathrm{d}x$$

▷ The dependence of $\mathscr{C}[f]$ on $\mathrm{A}\big[f^{2p}\big]$ and $\mathscr{F}\big[f^{2p}\big]$ is explicit and does not degenerate if $f \in \mathfrak{M}$

▷ Can we remove the condition $\mathrm{A}\big[f^{2p}\big] < \infty$? Not with this method :(

Constructive stability results II - Subcritical Case - Gradient Version

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$$\delta[f] \ge \mathscr{C}[f] \inf_{\varphi\in\mathfrak{M}} \int_{\mathbb{R}^d} \left|(p-1)\,\nabla f + f^p\,\nabla\varphi^{1-p}\right|^2 \mathrm{d}x$$

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$$\delta[f] \ge \mathscr{C}[f]\,\inf_{\varphi\in\mathfrak{M}} \int_{\mathbb{R}^d} \left|(p-1)\,\nabla f + f^p\,\nabla\varphi^{1-p}\right|^2 \mathrm{d}x$$

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Constructive stability results II - Subcritical Case - Gradient Version

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▷ Can we remove the condition $\mathsf{A}\big[f^{2p}\big] < \infty$? Not with this method :(

A constructive stability result by the "flow method" (from the beginning)

The relative entropy

$$\mathcal{F}[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left(f^{p+1} - g^{p+1} - \frac{1+p}{2p}\, g^{1-p} \left(f^{2p} - g^{2p} \right) \right) dx$$

The deficit functional

$$\delta[f] := a \, \|\nabla f\|_2^2 + b \, \|f\|_{p+1}^{p+1} - \mathcal{K}_{\mathrm{GN}} \, \|f\|_{2p}^{2p\gamma} \geq 0$$

Theorem (Constructive Stability for GNS BDNS (2020))

Let $d \geq 1$, $p \in (1, p^*)$, $A > 0$ and $G > 0$. There is an explicit constant $\mathcal{C} > 0$ such that

$$\boxed{\delta[f] \geq \mathcal{C}\, \mathcal{F}[f]} \quad \text{with} \quad \boxed{\mathcal{C} = \frac{\mathsf{k}_{p,d}}{1 + \mathsf{A}^a + G}}$$

for any $f \in \mathcal{W} := \left\{ f \in \mathrm{L}^1(\mathbb{R}^d, (1+|x|)^2\, dx) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, dx) \right\}$ such that

$$\int_{\mathbb{R}^d} f^{2p}\, dx = \int_{\mathbb{R}^d} |g|^{2p}\, dx\,, \qquad \int_{\mathbb{R}^d} x\, f^{2p}\, dx = 0$$

$$\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} f^{2p}\, dx \leq A \quad \text{and} \quad \mathcal{F}[f] \leq G$$

Constructive Stability in Sobolev's inequality (critical case)

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \ge 3$ and

$$\mathscr{W}_{p^\star}(\mathbb{R}^d) = \left\{ f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d),\ |x|\, f^{p^\star} \in \mathrm{L}^2(\mathbb{R}^d) \right\}$$

Theorem (Constructive stability for Sobolev [BDNS (2021)])

Let $d \ge 3$ and $A > 0$. Then for any nonnegative $f \in \mathscr{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\, \mathrm{d}x = \int_{\mathbb{R}^d} (1, x, |x|^2)\, \mathrm{g}\, \mathrm{d}x \quad \text{and} \quad \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\, \mathrm{d}x \le A$$

we have

$$\delta[f] := \|\nabla f\|_2^2 - \mathsf{S}_d^2\, \|f\|_{2^*}^2 \ge \frac{\mathscr{C}_\star(A)}{4 + \mathscr{C}_\star(A)} \int_{\mathbb{R}^d} \left| \nabla f + \tfrac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla \mathrm{g}^{-\frac{2}{d-2}} \right|^2 \mathrm{d}x$$

$\mathscr{C}_\star(A) = \mathfrak{C}_\star \left(1 + A^{1/(2d)}\right)^{-1}$ *and* $\mathfrak{C}_\star > 0$ *depends only on* d

We can remove the normalization of f, use the r.h.s. to measure the distance to the Aubin-Talenti manifold of optimal functions (in relative Fisher information) and obtain for

$$A[f] := \sup_{r>0} r^d \int_{r>0} |f|^{2^*}(x+x_f) \quad \text{and} \quad Z[f] := \left(1+\mu[f]^{-d}\,\lambda[f]^d\,A[f]\right)$$

the *Bianchi-Egnell type result*

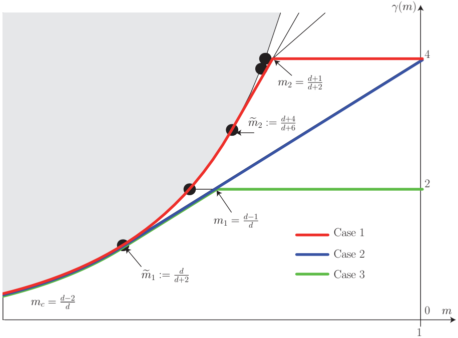
$$\delta[f] \geq \frac{\mathfrak{C}_\star\, Z[f]}{4+Z[f]} \inf_{g\in\mathfrak{M}} \mathscr{J}[f|g]$$

with x_f, $\lambda[f]$ and $\mu[f]$ as in the subcritical case

Idea of the proof: Extending the subcritical result in the critical case

To improve the spectral gap for $m = m_1$, we need to adjust the Barenblatt function $\mathscr{B}_\lambda(x) = \lambda^{-d/2}\mathscr{B}\left(x/\sqrt{\lambda}\right)$ in order to match $\int_{\mathbb{R}^d}|x|^2\,v\,dx$ where the function v solves (1) or to further rescale v according to

$$v(t,x) = \frac{1}{\mathfrak{R}(t)^d}\,w\left(t+\tau(t),\frac{x}{\mathfrak{R}(t)}\right),$$



$$\frac{d\tau}{dt} = \left(\frac{1}{\mathcal{K}_\star}\int_{\mathbb{R}^d}|x|^2\,v\,dx\right)^{-\frac{d}{2}(m-m_c)} - 1\,,\quad \tau(0)=0 \quad\text{and}\quad \mathfrak{R}(t) = e^{2\tau(t)}$$

Lemma (Delay estimates [BDNS (2021)]**)**

$t \mapsto \tau(t)$ *is bounded on* $\mathbb{R}^+$ *(explicit estimates)*

The End

Thank You!!!

Grazie Mille!!!

Muchas Gracias!!!

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Download slides and papers at: http://verso.mat.uam.es/~matteo.bonforte

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The constant in Moser's Harnack inequality 1/3

Let Ω be an open domain and let us consider a nonnegative *weak solution* to

$$\frac{\partial v}{\partial t} = \nabla \cdot \big(A(t,x)\,\nabla v\big) \tag{2}$$

on $\Omega_T := (0,T)\times\Omega$, where $A(t,x)$ is a real symmetric matrix with bounded measurable coefficients satisfying the *uniform ellipticity condition*

$$0 < \lambda_0\,|\xi|^2 \le \xi\cdot(A\xi) \le \lambda_1\,|\xi|^2 \quad \forall\,(t,x,\xi)\in\mathbb{R}^+\times\Omega_T\times\mathbb{R}^d\,, \tag{3}$$

where $\xi\cdot(A\xi) = \sum_{i,j=1}^d A_{i,j}\xi_i\xi_j$ and λ_0, λ_1 are positive constants.

The constant in Moser's Harnack inequality 2/3

Let us consider the neighborhoods

(4)
$$D_R^+(t_0,x_0) := \left(t_0+\tfrac{3}{4}R^2,\, t_0+R^2\right)\times B_{R/2}(x_0)\,,$$
$$D_R^-(t_0,x_0) := \left(t_0-\tfrac{3}{4}R^2,\, t_0-\tfrac{1}{4}R^2\right)\times B_{R/2}(x_0)\,,$$

We claim that the following *Harnack inequality* holds [Moser (1964,71)]:

Theorem (Parabolic Harnack inequality [BDNS (2020,21)])

Let $T>0$, $R\in(0,\sqrt{T})$, and take $(t_0,x_0)\in(0,T)\times\Omega$ such that $\left(t_0-R^2,t_0+R^2\right)\times B_{2R}(x_0)\subset\Omega_T$. Under Assumption (3), if v satisfies

(5)
$$\iint_{(0,T)\times\Omega}\left(-\varphi_t\, v+\nabla\varphi\cdot(A\nabla v)\right)\mathrm{d}x\,\mathrm{d}t=0$$

for any $\varphi\in C_c^\infty((0,T)\times\Omega)$, then

(6)
$$\sup_{D_R^-(t_0,x_0)} v \le \overline{\mathsf{h}} \inf_{D_R^+(t_0,x_0)} v\,.$$

▷ This result is known from [Moser (1964,71)]. However, to the best of our knowledge, a complete constructive proof and an expression of $\overline{\mathsf{h}}$ was still missing.

The constant in Moser's Harnack inequality 3/3

The constant in Moser's Harnack inequality has the expression

(7) $$\overline{\mathsf{h}} := \mathsf{h}^{\lambda_1 + \lambda_0^{-1}} .$$

where

(8) $$\mathsf{h} := \exp\left[2^{d+4}\,3^d\,d + c_0^3\,2^{2\,(d+2)+3}\left(1 + \frac{2^{d+2}}{(\sqrt{2}-1)^{2\,(d+2)}}\right)\sigma\right]$$

where

(9) $$c_0 = 3^{\frac{2}{d}}\,2^{\frac{(d+2)\,(3\,d^2+18\,d+24)+13}{2\,d}}\left(\frac{(2+d)^{1+\frac{4}{d^2}}}{d^{1+\frac{2}{d^2}}}\right)^{(d+1)(d+2)}\mathcal{K}^{\frac{2\,d+4}{d}} ,$$

(10) $$\sigma = \sum_{j=0}^{\infty}\left(\tfrac{3}{4}\right)^j\left((2+j)\,(1+j)\right)^{2\,d+4} .$$

The constant $\mathcal{K}$ is the constant in Sobolev embedding (explicit).

Explicit Hölder continuity exponent

▷ It is well known that Harnack inequalities imply Hölder continuity of solutions.
▷ We obtain a quantitative expression of the Hölder continuity exponent, which only depends on the Harnack constant, *i.e.* on d, λ_0 and λ_1.
▷ Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^d$ be bounded domains and let $Q_1 := (T_2, T_3) \times \Omega_1 \subset (T_1, T_4) \times \Omega_2 =: Q_2$, where $0 \le T_1 < T_2 < T_3 < T < 4$. Define the *parabolic distance:*

(11)
$$\mathrm{dist}(Q_1, Q_2) := \inf_{\substack{(t,x)\in Q_1 \\ (s,y)\in[T_1,T_4]\times\partial\Omega_2\cup\{T_1,T_4\}\times\Omega_2}} |x-y| + |t-s|^{\frac{1}{2}}\,.$$

Theorem (Hölder Continuity with explicit exponents [BDNS (2020,21)])

Let v be a nonnegative solution of (2) *on Q_2 which satisfies* (5) *and assume that $A(t,x)$ satisfies* (3)*. Then we have*

(12)
$$\sup_{(t,x),(s,y)\in Q_1} \frac{|v(t,x)-v(s,y)|}{\left(|x-y|+|t-s|^{1/2}\right)^{\nu}} \le 2\left(\frac{128}{\mathrm{dist}(Q_1,Q_2)}\right)^{\nu} \|v\|_{\mathrm{L}^\infty(Q_2)}\,.$$

where

(13)
$$\nu := \log_4\left(\frac{\overline{\mathsf{h}}}{\overline{\mathsf{h}}-1}\right),$$

and $\overline{\mathsf{h}}$ *is as in* (7)*.*

From the expression of h in (8) it is clear that $\overline{\mathsf{h}} \ge \frac{4}{3}$, so that $\nu \in (0,1)$.

Local Sobolev embeddings and optimal constants

Let us denote by B_R the ball of radius $R > 0$ centered at the origin, and define

$$
\begin{aligned}
\mathsf{p} &:= \frac{2\,d}{d-2} \quad \text{if} \quad d \ge 3\,,\\
\mathsf{p} &:= 4 \quad \text{if} \quad d = 2\,,\\
\mathsf{p} &\in (4,+\infty) \quad \text{if} \quad d = 1\,.
\end{aligned}
$$

Theorem (Sobolev Inequality)

Let $d \ge 1$, $R > 0$. For $d = 1,\,2$, we further assume that $R \le 1$. Then

(14) $$\|f\|^2_{\mathrm{L}^{\mathsf{p}}(B_R)} \le \mathscr{K}\left(\|\nabla f\|^2_{\mathrm{L}^2(B_R)} + \frac{1}{R^2}\,\|f\|^2_{\mathrm{L}^2(B_R)}\right) \quad \forall\, f \in \mathrm{H}^1(B_R)$$

holds for some constant

(15) $$\mathscr{K} \le \begin{cases} \dfrac{4\,\Gamma\left(\frac{d+1}{2}\right)^{2/d}}{2^{\frac{2}{d}}\,\pi^{1+\frac{1}{d}}} & \text{if} \quad d \ge 3\,,\\[2mm] \dfrac{4}{\sqrt{\pi}} & \text{if} \quad d = 2\,,\\[2mm] 2^{1+\frac{2}{\mathsf{p}}}\,\max\left\{\dfrac{\mathsf{p}-2}{\pi^2},\,\dfrac{1}{4}\right\} & \text{if} \quad d = 1\,. \end{cases}$$

Gagliardo Interpolation inequalities

Lemma

Let $d \ge 1$, $p \ge 1$ and $\nu \in (0,1)$. Then there exists a positive constant $C_{d,\nu,p}$ such that, for any $u \in \mathrm{L}^p(B_{2R}(x)) \cap C^\nu(B_{2R}(x))$, $R>0$ and $x \in \mathbb{R}^d$

$$\text{(16)} \qquad \|u\|_{\mathrm{L}^\infty(B_R(x))} \le C_{d,\nu,p} \left(\lfloor u \rfloor_{C^\nu(B_{2R}(x))}^{\frac{d}{d+p\nu}} \|u\|_{\mathrm{L}^p(B_{2R}(x))}^{\frac{p\nu}{d+p\nu}} + R^{-\frac{d}{p}} \|u\|_{\mathrm{L}^p(B_{2R}(x))} \right).$$

Analogously, we have

$$\text{(17)} \qquad \|u\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le C_{d,\nu,p} \lfloor u \rfloor_{C^\nu(\mathbb{R}^d)}^{\frac{d}{d+p\nu}} \|u\|_{\mathrm{L}^p(\mathbb{R}^d)}^{\frac{p\nu}{d+p\nu}} \quad \forall\, u \in \mathrm{L}^p(\mathbb{R}^d) \cap C^\nu(\mathbb{R}^d).$$

In both cases, the inequalities hold with the constant

$$C_{d,\nu,p} = 2^{\frac{(p-1)(d+p\nu)+dp}{p(d+p\nu)}} \left(1 + \frac{d}{\omega_d}\right)^{\frac{1}{p}} \left(1 + \left(\frac{d}{p\nu}\right)^{\frac{1}{p}}\right)^{\frac{d}{d+p\nu}} \left(\left(\frac{d}{p\nu}\right)^{\frac{p\nu}{d+p\nu}} + \left(\frac{p\nu}{d}\right)^{\frac{d}{d+p\nu}}\right)^{\frac{1}{p}}.$$

Mass displacement estimates

Lemma (BDNS-2021)

Let $m \in (0,1)$ *and* $u(t,x)$ *be a nonnegative solution to the FDE. Then, for any* $t, \tau \geq 0$ *and* $r, R > 0$ *such that* $\varrho_0 \, r \geq 2\,R$ *for some* $\varrho_0 > 0$*, we have*

$$\int_{B_{2R}(x_0)} u(t,x)\,\mathrm{d}x \leq 2^{\frac{m}{1-m}} \int_{B_{2R+r}(x_0)} u(\tau,x)\,\mathrm{d}x + \mathsf{c}_3\,\frac{|t-\tau|^{\frac{1}{1-m}}}{r^{\frac{2-d(1-m)}{1-m}}}\,, \tag{18}$$

where

$$\mathsf{c}_3 := 2^{\frac{m}{1-m}}\,\omega_d \left(\frac{16\,(d+1)\,(3+m)}{1-m}\right)^{\frac{1}{1-m}} (\varrho_0+1)\,. \tag{19}$$

Under the same assumptions, we have that

$$\int_{\mathbb{R}^d \setminus B_{2R+r}(x_0)} u(t,x)\,\mathrm{d}x \leq 2^{\frac{m}{1-m}} \int_{\mathbb{R}^d \setminus B_{2R}(x_0)} u(\tau,x)\,\mathrm{d}x + \mathsf{c}_3\,\frac{|t-\tau|^{\frac{1}{1-m}}}{r^{\frac{2-d(1-m)}{1-m}}}\,. \tag{20}$$

Local Upper Bounds

Lemma (BDNS-2021)

Let $d \geq 1$, $m \in [m_1, 1)$. Then there exists a positive constant $\overline{\kappa}$ such that for any solution u of FDE with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfies for all $(t, R) \in (0, +\infty)^2$ the estimate

$$\sup_{y \in B_{R/2}(x)} u(t,y) \leq \overline{\kappa} \left(\frac{1}{t^{d/\alpha}} \left(\int_{B_R(x)} u_0(y)\, \mathrm{d}y \right)^{2/\alpha} + \left(\frac{t}{R^2} \right)^{\frac{1}{1-m}} \right). \tag{21}$$

▷ This is a particular case (but with explicit constants computed) of the Local Smoothing Effects proven by many authors:
Daskalopoulos-Kenig (Moser Iteration), DiBenedetto (DeGiorgi method), [...] and constructive proof by MB-Vazquez [2010], MB-Simonov [2019] (CKN-weights).

Local Lower Bounds in the good FDE range $m \in (m_c, 1)$

Lemma (BDNS-2021)

Let $d \ge 1$ *and* $m \in \left[m_1, 1\right)$. *Let* $x_0 \in \mathbb{R}^d$, $u(t,x)$ *be a solution to FDE with nonnegative initial datum* $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ *and let* $R > 0$ *such that* $M_R(x_0) := \|u_0\|_{\mathrm{L}^1(B_R(x_0))} > 0$. *Then the inequality*

$$\inf_{|x-x_0|\le R} u(t,x) \ge \kappa \left(R^{-2}\, t\right)^{\frac{1}{1-m}} \qquad \forall\, t \in [0, 2\,\underline{t}] \tag{22}$$

holds with

$$\underline{t} = \tfrac{1}{2}\,\kappa_\star\, M_R^{1-m}(x_0)\, R^{\alpha}\,.$$