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Global minimizers of Interaction Energies

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European Research Council
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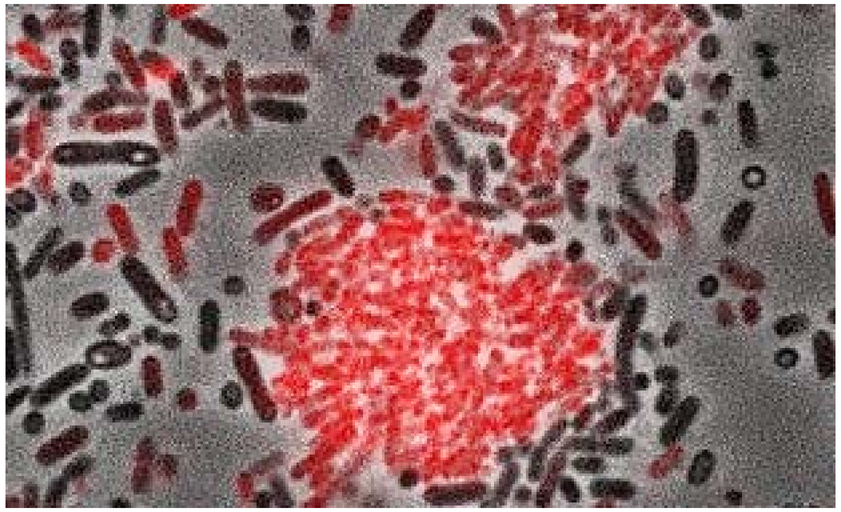
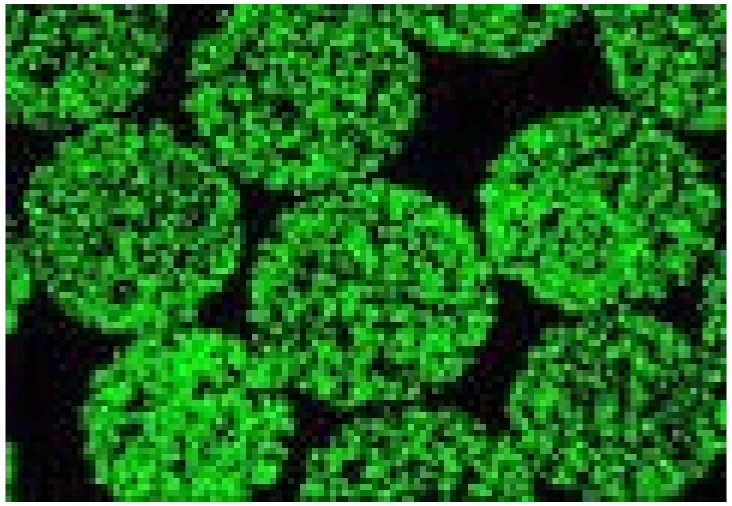
Collective behavior

macroscopic
individuals:
animals

microscopic
individuals:
bacteria





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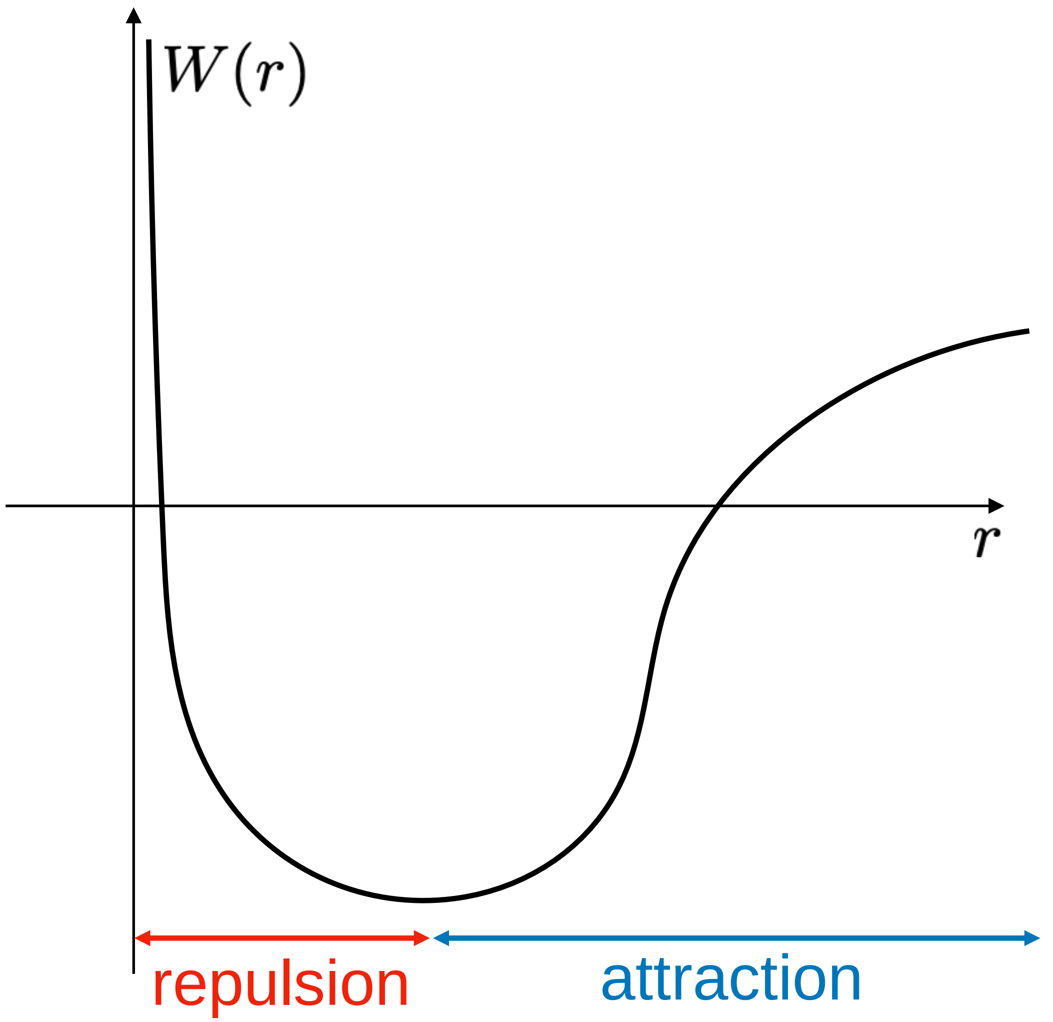
pictures from the internet

First-order models with attractive-repulsive pairwise interactions

$$\dot{\mathbf{x}}_i = -\frac{1}{N} \sum_{j=1,\ldots,N,\, j\neq i} \nabla W(\mathbf{x}_i - \mathbf{x}_j) \qquad\qquad i = 1,\ldots,N$$

- $\mathbf{x}_i(t)$: location of the i-th individual
- $W = W(r),\ r = |\mathbf{x}|$ radial interaction potential
- Usually used for micro-organisms, where inertia can be neglected.



Energy structure

$$\dot{\mathbf{x}}_i = -\frac{1}{N} \sum_{j=1,\ldots,N,\, j\neq i} \nabla W(\mathbf{x}_i - \mathbf{x}_j) \qquad i = 1, \ldots, N$$

- It is the gradient flow of the energy functional

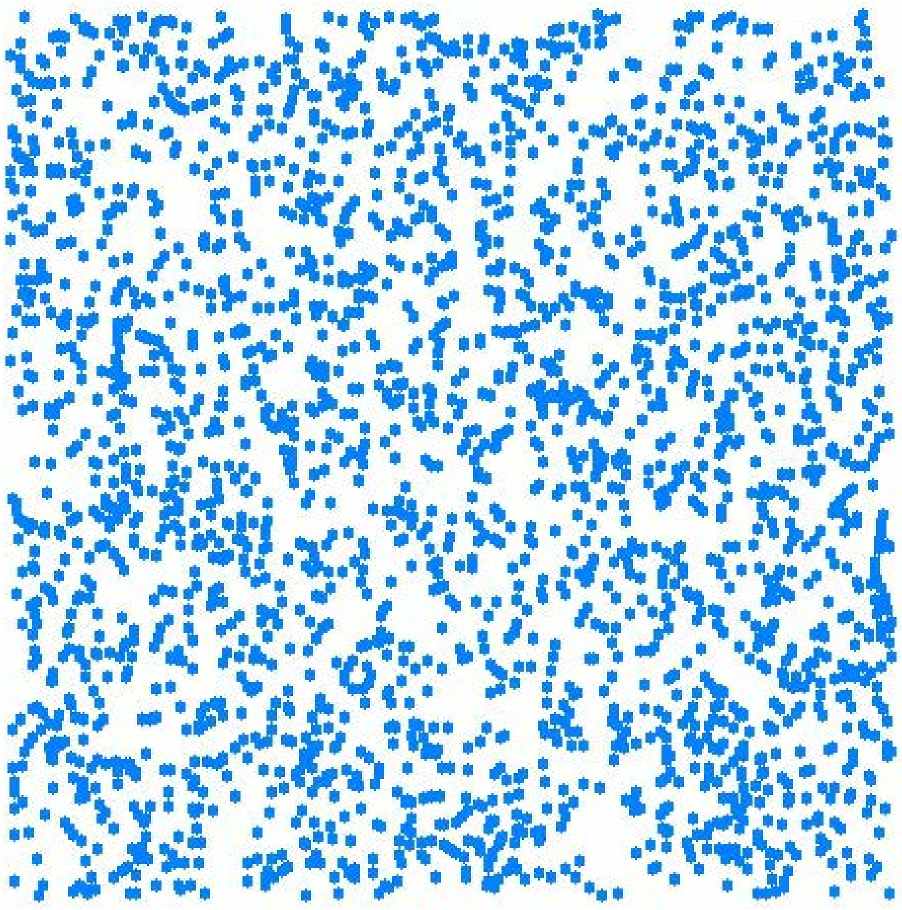
$$E(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \frac{1}{2N^2} \sum_{i,j:i\neq j} W(\mathbf{x}_i - \mathbf{x}_j)$$

- Energy dissipation rate $\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{1}{N} \sum_i |\dot{\mathbf{x}}_i|^2$

- Expected large time behavior: convergence to local energy minimizers

Question: Existence/Uniqueness of global/local energy minimizers?

How do they look like?



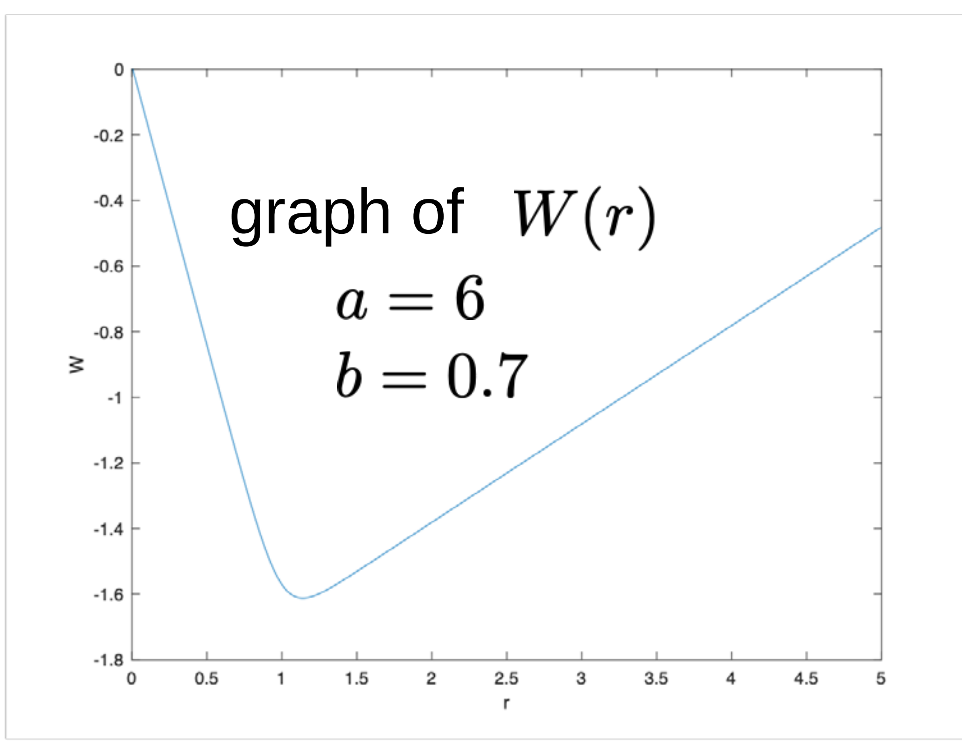
Energy minimizers

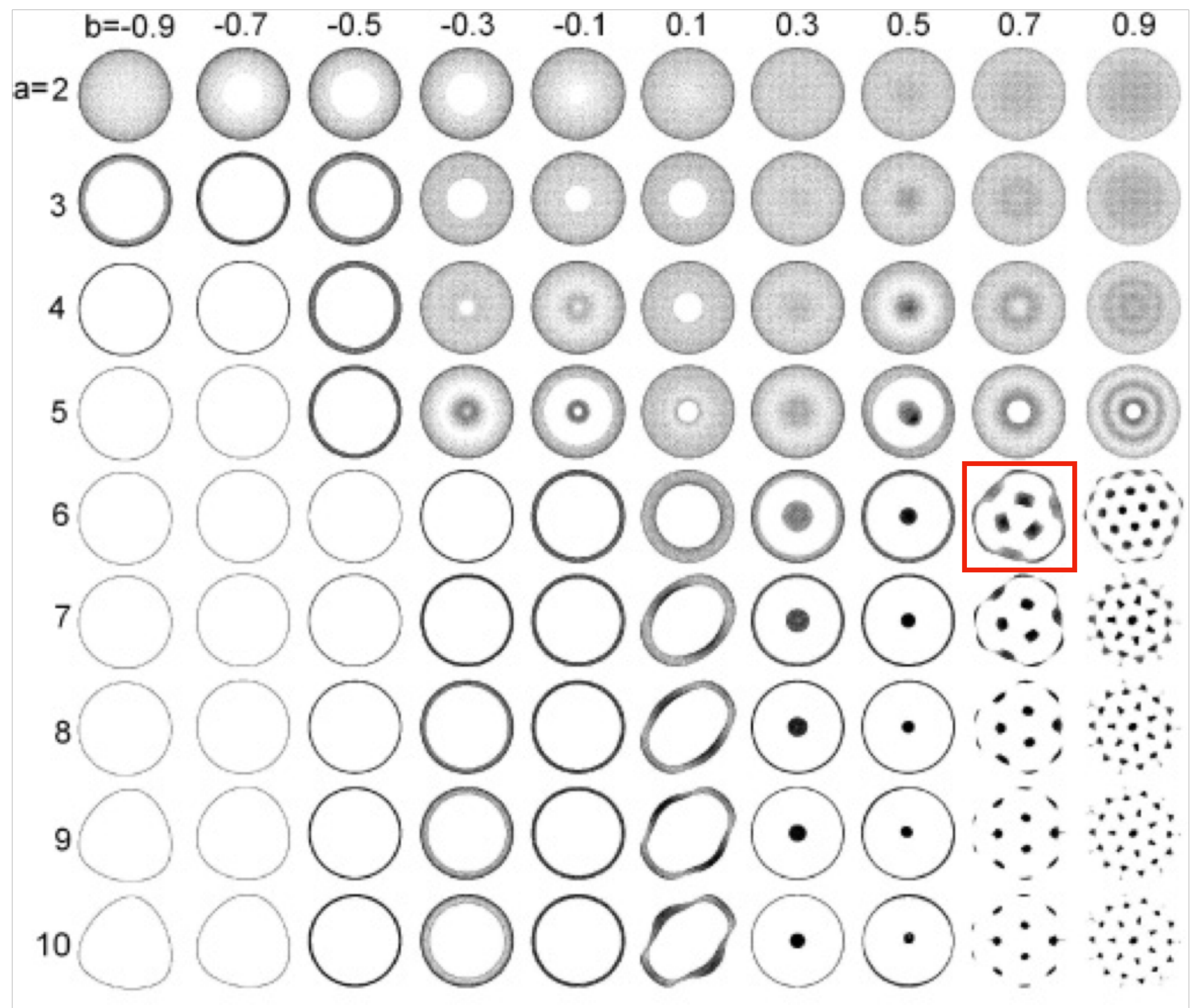
- Large time simulation: Kolokolnikov-Sun-Uminsky-Bertozzi 11'

$$-W'(r) = F(r) = \tanh\left[(1-r)a\right] + b;\ 0 < a;\ -\tanh(a) < b < 1.$$

- **Very complicated**

- Break of symmetry





Continuum formulation

- Particle density function $\rho(t, \mathbf{x})$
- Energy $\boxed{E = \frac{1}{2} \int \int W(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}) \mathrm{d}\mathbf{y} \rho(\mathbf{x}) \mathrm{d}\mathbf{x}}$
- Wasserstein-2 gradient flow

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \ \mathbf{u}(t, \mathbf{x}) = -\int \nabla W(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}) \mathrm{d}\mathbf{y}$$

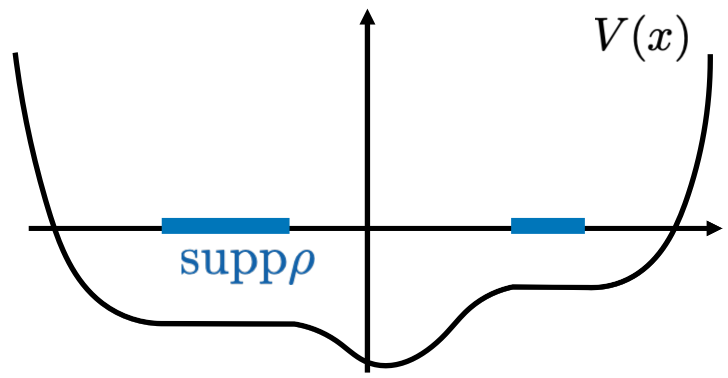
$\mathbf{u}(t, \mathbf{x})$: transport velocity

- Energy dissipation rate

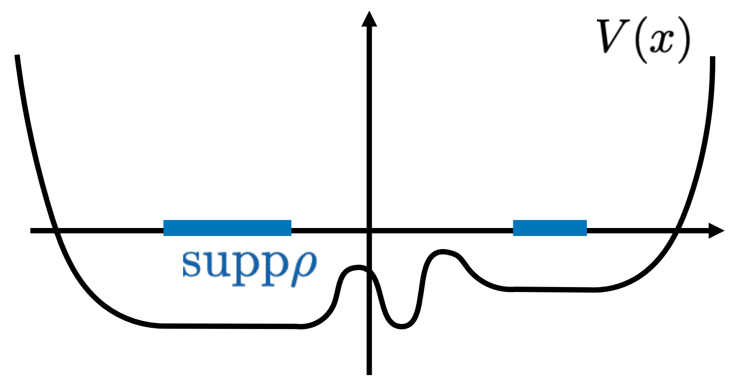
$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int |\mathbf{u}|^2 \rho \mathrm{d}\mathbf{x}$$

"Characterization" of Energy minimizers

- Steady states: $\nabla V = 0$ on $\mathrm{supp}\rho$, or $V = W * \rho$ is constant on each connected component of $\mathrm{supp}\rho$



- (Wasserstein-∞) local minimizers: further, V is larger nearby



- Global minimizers

$V = W * \rho$: potential generated by ρ

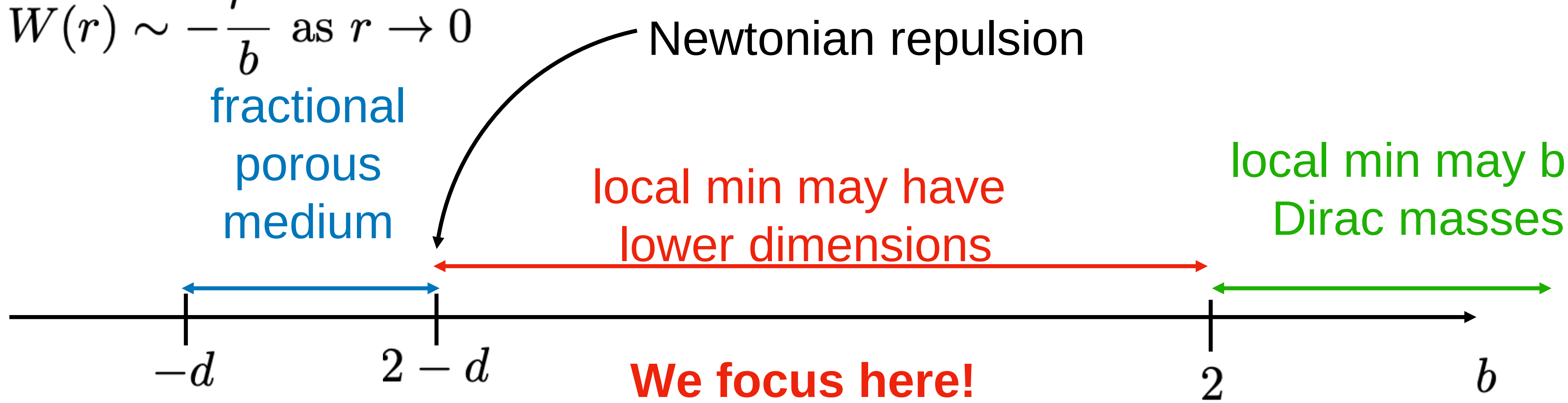
Dimensionality of minimizers

- Balagué-C.-Laurent-Raoul 13': if

$$\Delta W(x) \sim -\frac{1}{|x|^{\beta}} \quad \text{as } x \to 0 \qquad 0 < \beta < d$$

then the Hausdorff dimension of any Wasserstein-∞ local minimizer is $\geq \beta$

- If $W(r) \sim -\frac{r^b}{b}$ as $r \to 0$



Part 1: Linear interpolation convexity: uniqueness of local minimizers

E as a quadratic form

$$E[\rho] = \frac{1}{2}\int\int W(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})\mathrm{d}\mathbf{y}\rho(\mathbf{x})\mathrm{d}\mathbf{x}$$

Define an associated bilinear form

$$\begin{aligned} E[\rho_1, \rho_2] &= \int\int W(\mathbf{x}-\mathbf{y})\rho_2(\mathbf{y})\mathrm{d}\mathbf{y}\rho_1(\mathbf{x})\mathrm{d}\mathbf{x} \\ &= \int (W * \rho_2)(\mathbf{x})\rho_1(\mathbf{x})\mathrm{d}\mathbf{x} \\ &= \int (W * \rho_1)(\mathbf{x})\rho_2(\mathbf{x})\mathrm{d}\mathbf{x} \end{aligned}$$

ρ_1 in the potential generated by ρ_2

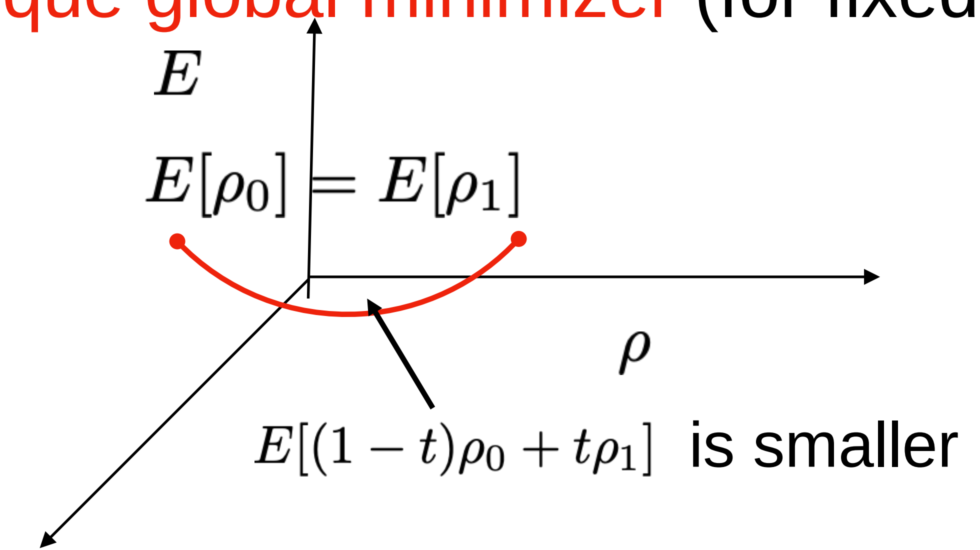
ρ_2 in the potential generated by ρ_1

Linear interpolation convexity (LIC)

- **Definition**: W has LIC property if for given ρ_0 and ρ_1 with the same total mass and center of mass, the function:
  $$t \mapsto E[(1-t)\rho_0 + t\rho_1]$$
  is strictly convex on $0 \leq t \leq 1$

- LIC is basically the positive-definiteness of E as a quadratic form.
  $$\frac{\mathrm{d}^2}{\mathrm{d}t^2} E[(1-t)\rho_0 + t\rho_1] = 2E[\rho_1 - \rho_0]$$

- LIC potentials have unique global minimizer (for fixed total mass and center of mass).



Linear interpolation convexity (LIC)

- LIC property can be guaranteed by Fourier transform:

$$E = \frac{1}{2}\int\int W(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})\mathrm{d}\mathbf{y}\rho(\mathbf{x})\mathrm{d}\mathbf{x} = \frac{1}{2}\int \underline{\hat{W}(\xi)}|\hat{\rho}(\xi)|^2\mathrm{d}\xi$$

diagonalize the quadratic form in ρ

Eigenvalues $\hat{W}(\xi) > 0$

Lopes 19' uses this to prove the LIC property of power-law potentials

$$W(\mathbf{x}) = \frac{|\mathbf{x}|^a}{a} - \frac{|\mathbf{x}|^b}{b} \qquad 2 \le a \le 4,\ -d < b < 0$$

and therefore obtain uniqueness and radial symmetry of global minimizers.
Similar use for uniqueness by Frank 21', Davies-Lim-McCann 21'.

- Equivalent to: $E[\mu] > 0$ for any signed measure μ with $\int \mu = \int \mathbf{x}\mu = 0$
- Almost equivalent to $\hat{W}(\xi) \ge 0, \quad \forall \xi$

LIC applied to local minimizers: radial symmetry

- **Theorem (C.-Shu 21')**: in 2D, assume W has LIC property. Then compactly supported local minimizers ρ are radially-symmetric.

- **Proof**: suppose ρ is a local minimizer which is not radially-symmetric.

- (1) By LIC, $\rho_\theta := \frac{1}{2}(\rho + \mathcal{R}_\theta \rho)$ has lower energy than ρ.

  $\mathcal{R}_\theta$: rotate by θ around its center of mass 0

- (2) ρ_θ has higher energy than ρ by minimizing property, since ρ_θ is close to ρ in Wasserstein-∞. Contradiction!

Part 2: Linear interpolation

concavity: small scales and fractals

Dimensionality of minimizers

- Balagué-C.-Laurent-Raoul 13': if

$$\Delta W(x) \sim -\frac{1}{|x|^{\beta}} \quad \text{as } x \to 0 \qquad 0 < \beta < d$$

then the Hausdorff dimension of any local minimizer is $\geq \beta$

- β needs not to be an integer, but it was never observed that a local minimizer has non-integer dimension.

- We will show that non-integer dimension is possible.

Concavity

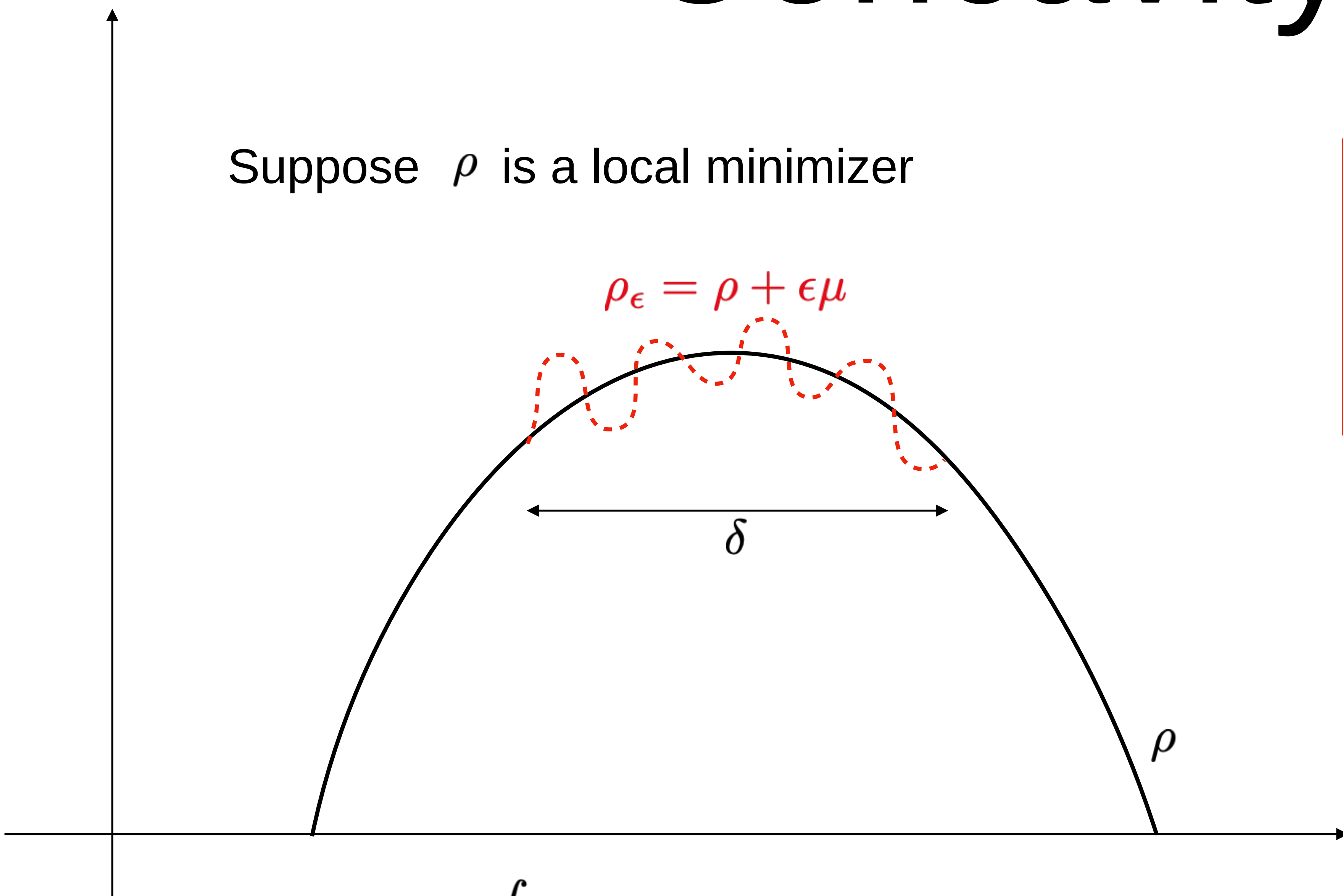
- **Definition**: W is infinitesimal-concave if for any $\delta > 0$ there exists $\mu \in L^\infty$ with $\int \mu \mathrm{d}\mathbf{x} = 0$ and $\mathrm{diam}(\mathrm{supp}\ \mu) \le \delta$ such that $E[\mu] < 0$

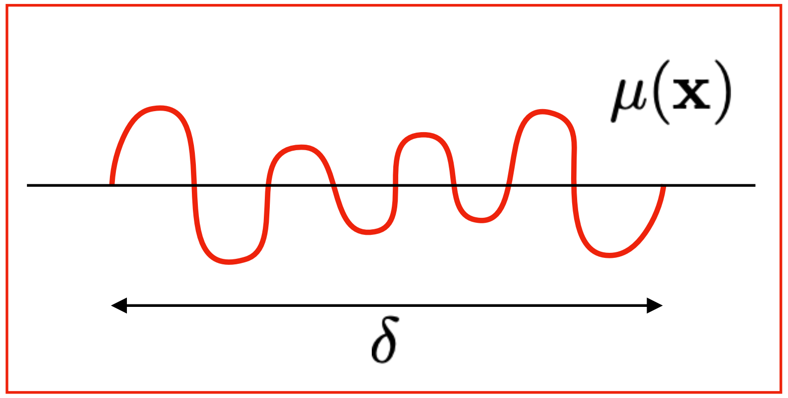
recall that LIC basically says $E[\mu] > 0$

- **Lemma (C.-Shu 21')**: if W is infinitesimal-concave, then for any local minimizer ρ , “$\mathrm{supp}\rho$” does not contain any interior point.

Concavity

Suppose ρ is a local minimizer





$E[\mu] < 0$

$$E[\rho_\epsilon] = E[\rho] + \epsilon \int \underbrace{\mu(\mathbf{x})}_{\text{mean zero}} \underbrace{(W * \rho)(\mathbf{x})}_{\text{constant}} \mathrm{d}\mathbf{x} + \epsilon^2 E[\mu] \quad < E[\rho]$$

contradiction!

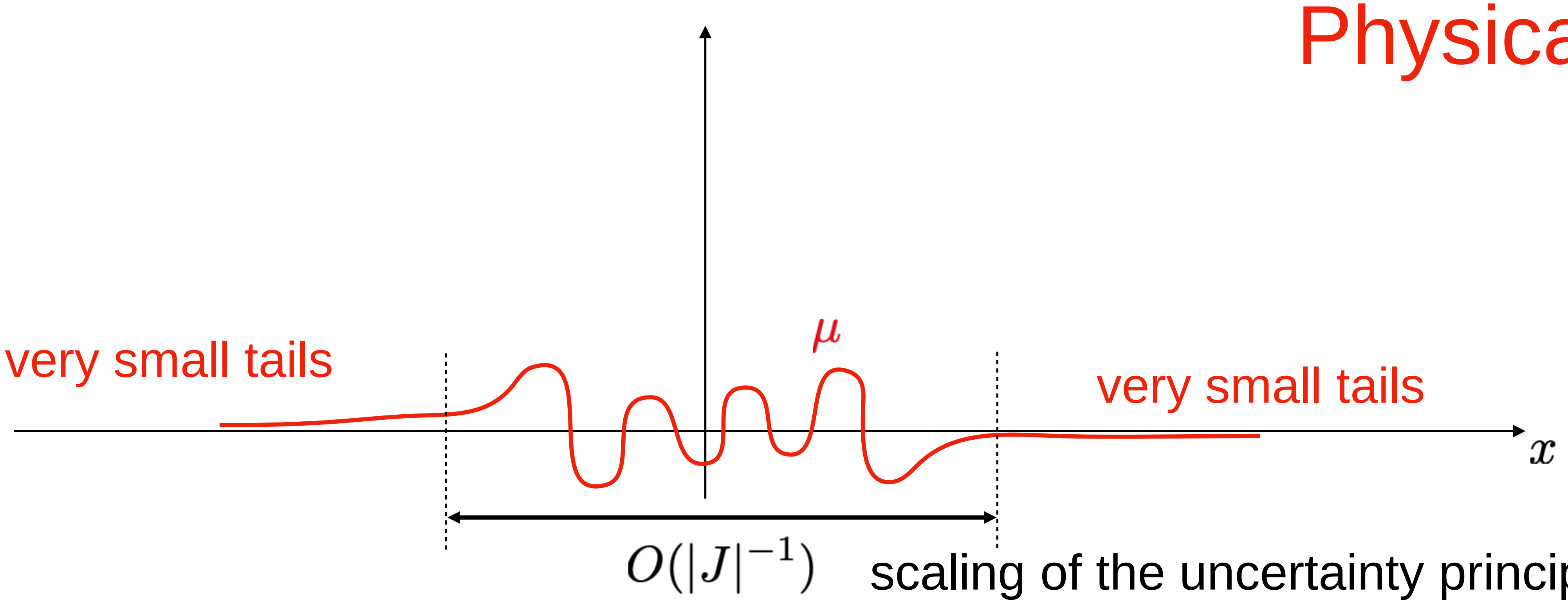
Concavity by Fourier transform

- Recall that LIC corresponds to $\hat{W}(\xi) > 0, \quad \forall \xi \neq 0$
- **Lemma (C.-Shu 21')**: Suppose W has certain regularity+growth, and for any $R > 0$, there exists such that $J \subset \mathbb{R}_+, |J| = R$

$$\hat{W}(\xi) < -c_1 R^{-\alpha}, \forall |\xi| \in J$$

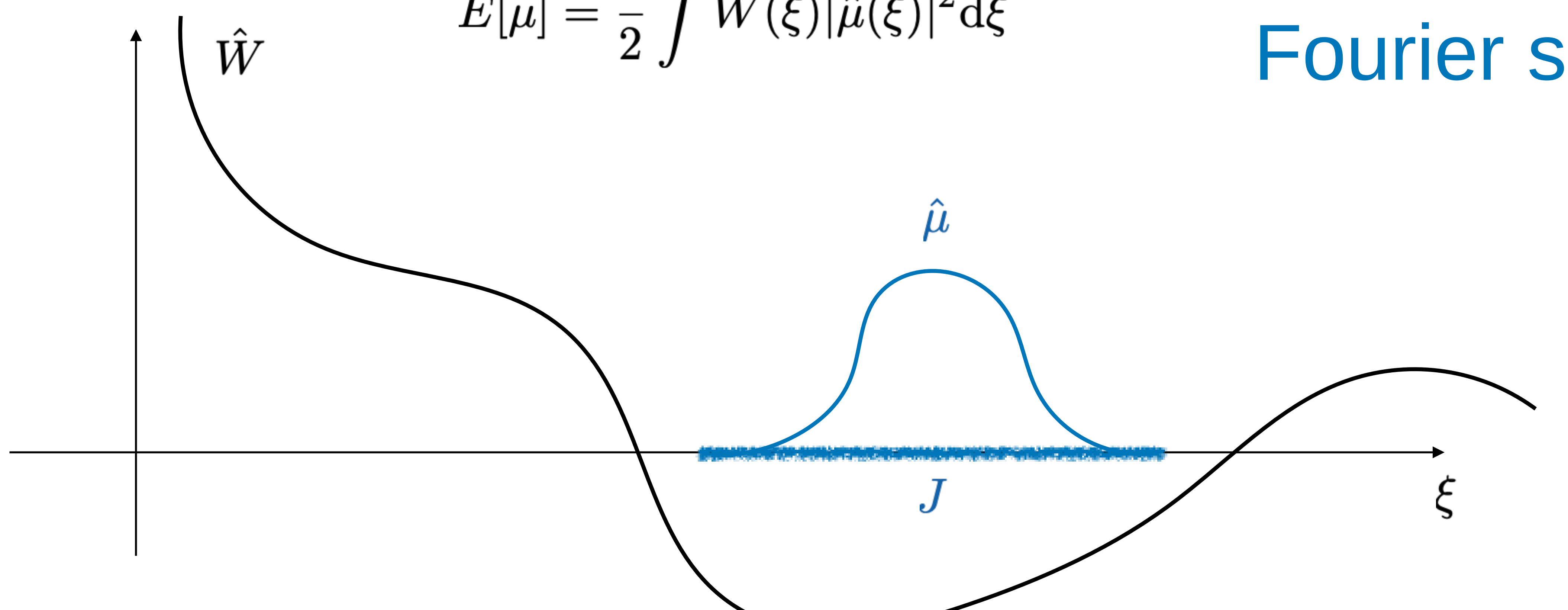
Then W is infinitesimal-concave.

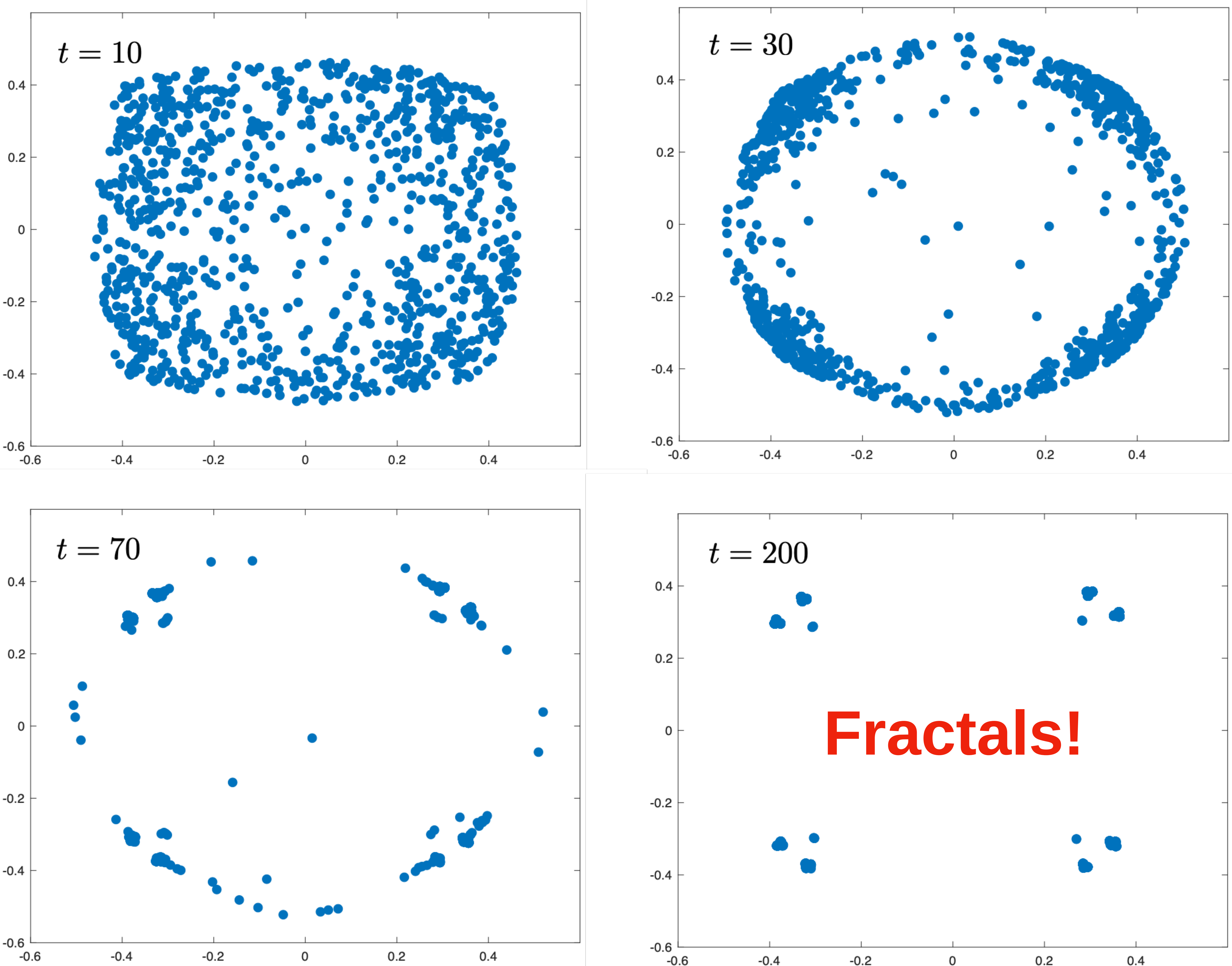
Physical side



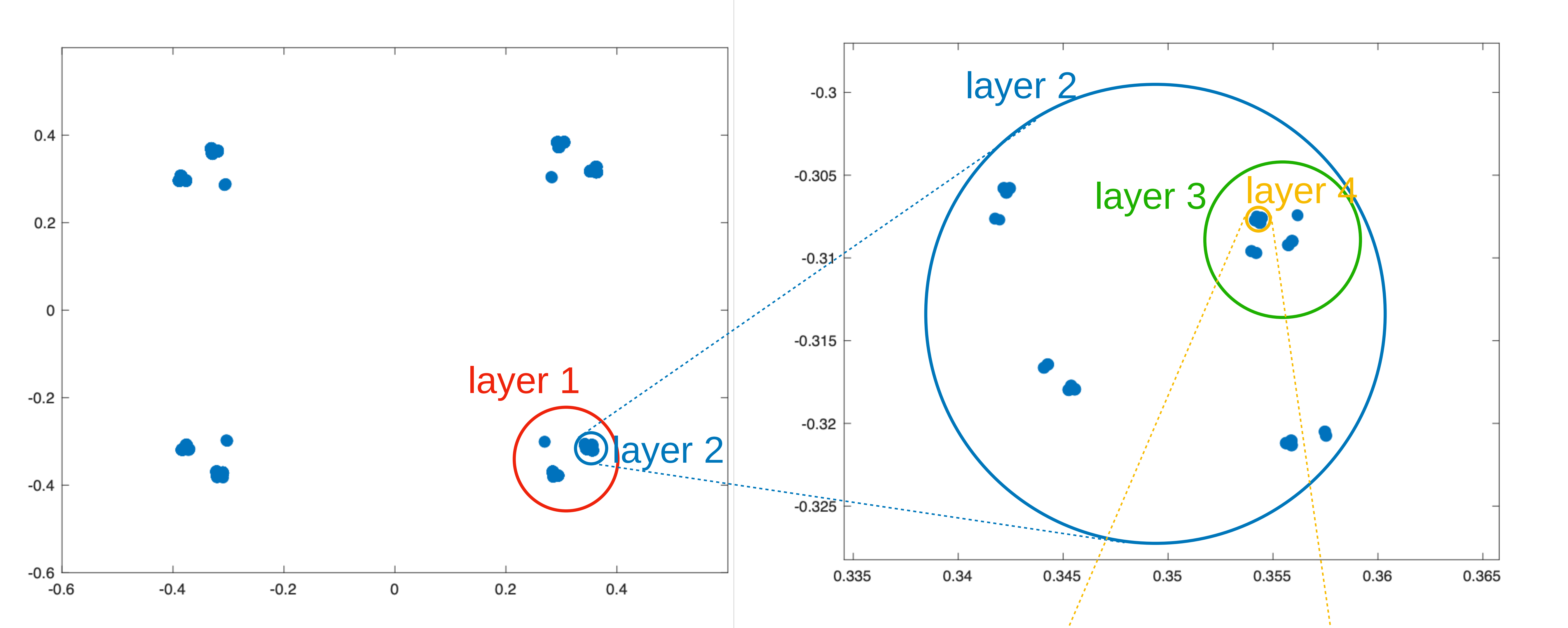
$$E[\mu] = \frac{1}{2}\int \hat{W}(\xi)|\hat{\mu}(\xi)|^2 \mathrm{d}\xi$$

Fourier side

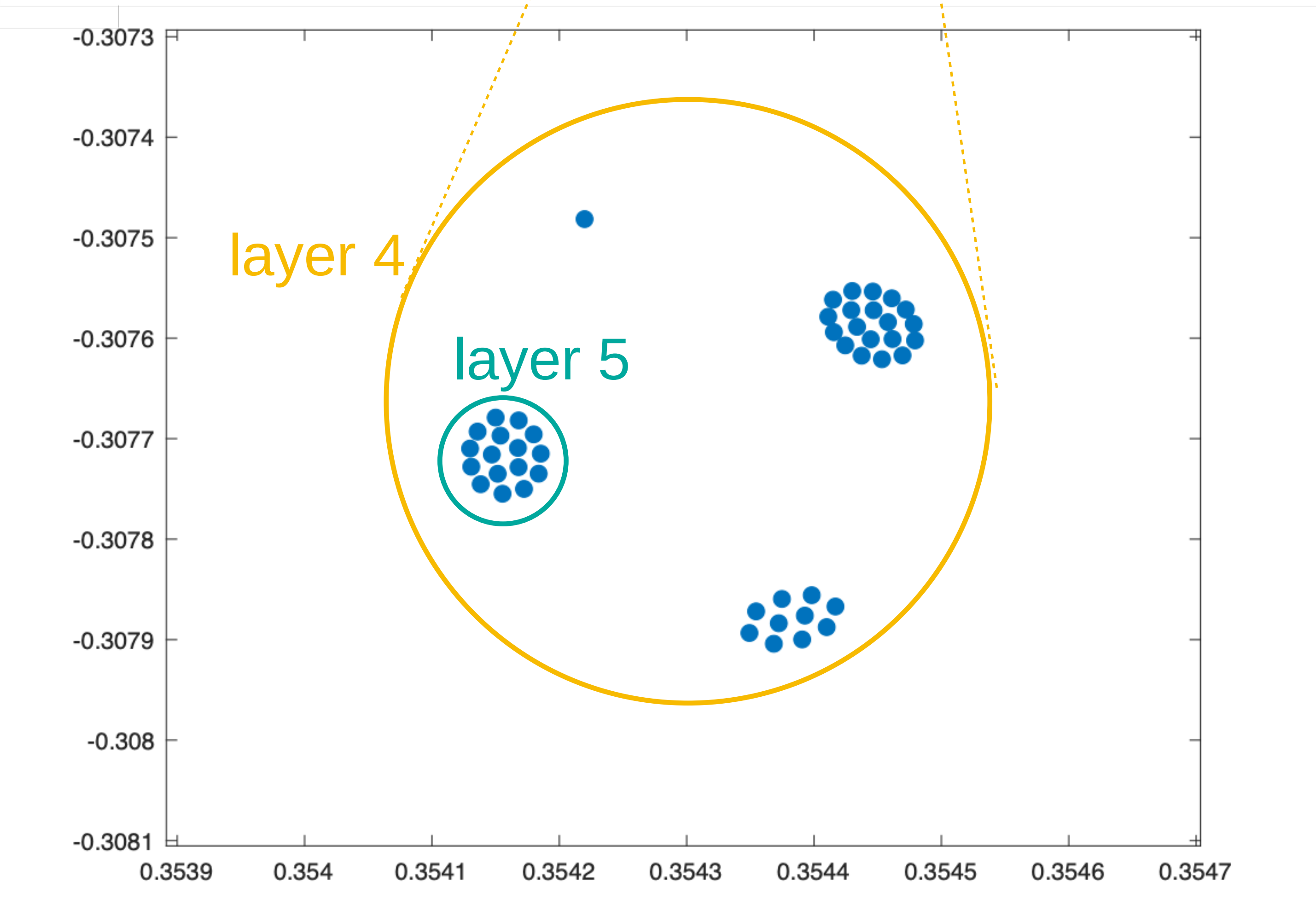




$t = 10$
$t = 30$
$t = 70$
$t = 200$
Fractals!



$$W(\mathbf{x}) = W_0(\mathbf{x}) - c_W \sum_{k=1}^{\color{red}{K}} \lambda^{(\alpha-d)k} \exp(-\frac{|\mathbf{x}|^2}{2\lambda^{2k}})$$

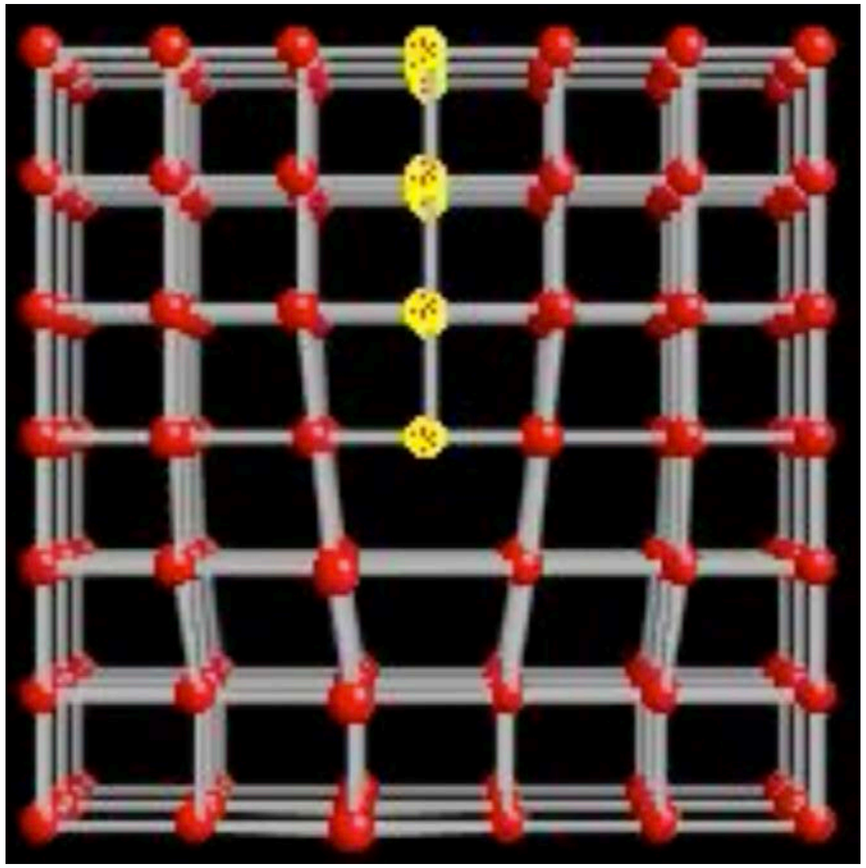


$\lambda = 0.15$ size ratio between two layers

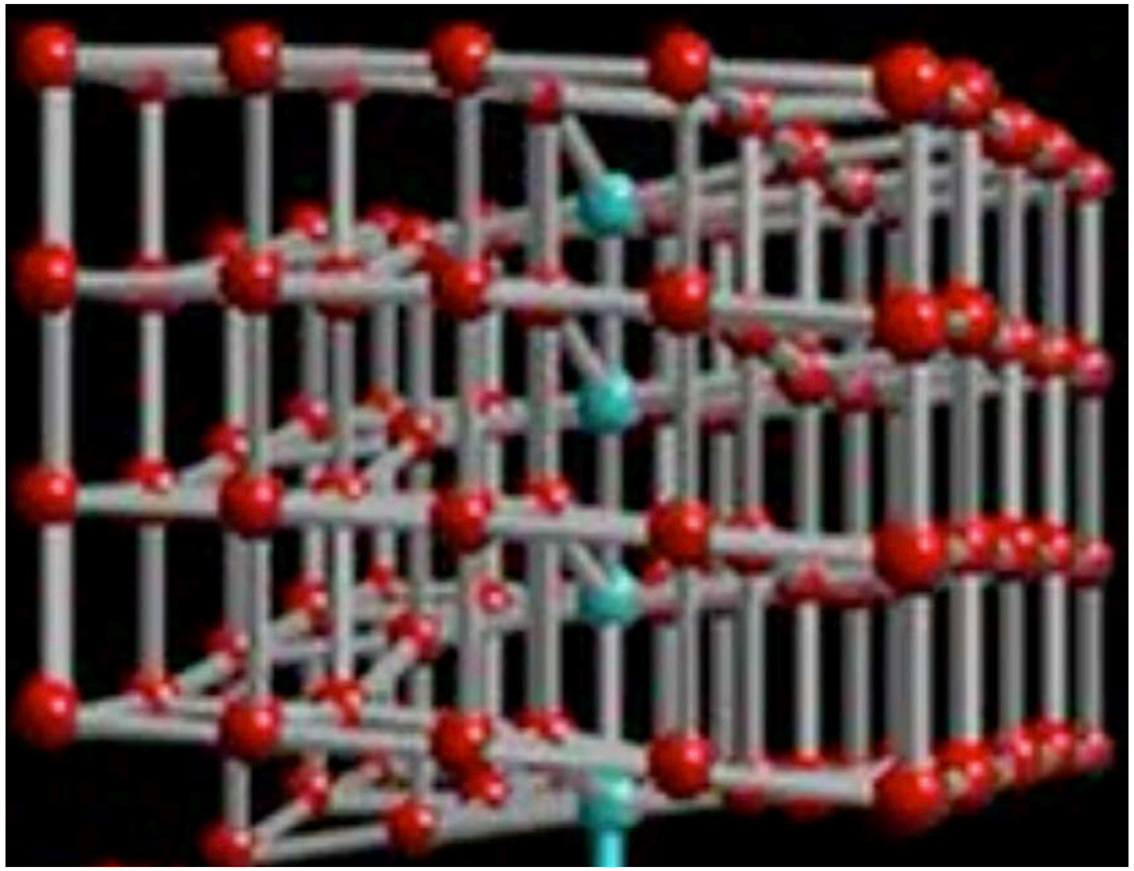
$\color{red}{K = 5}$ number of perturbation terms
—> number of layers

Part 3: Anisotropic Potentials

Dislocations in crystals



Edge dislocation



Screw dislocation

Pictures from https://www.doitpoms.ac.uk/tlplib/dislocations

Dislocations in crystals

- Phenomenological dislocation models have been derived in literature
- Each dislocation can be viewed as a particle
- Number of dislocations can be huge (say, $10^{15}\mathrm{m}^{-2}$)
- It is typical to use a continuum model to describe the spatial configuration of dislocations, i.e., use a density function
- Roy-Peerlings-Geers-Kasyanyuk 08', Geers-Peerlings-Peletier-Scardia 13'

Minimization of 2D interaction energy

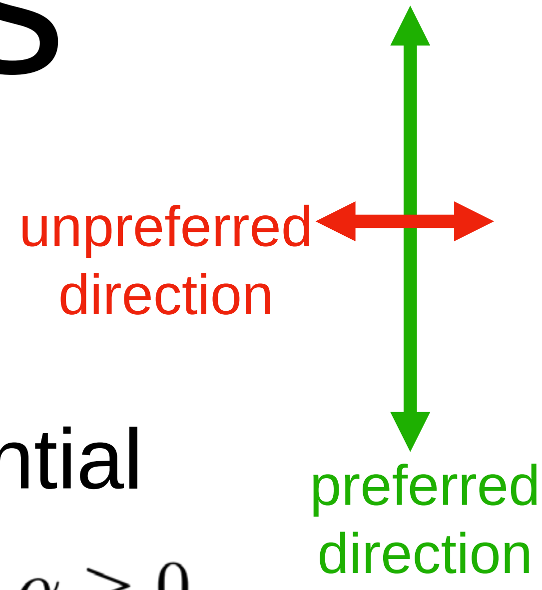
$$E[\rho] = \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} W(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}\,\rho(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

- ρ : density of dislocations, a probability measure on $\mathbb{R}^2$
- W : anisotropic interaction potential;
- Ω describes the anisotropy

$$W(\mathbf{x}) = \underbrace{|\mathbf{x}|^{-s}\Omega(\theta)}_{\text{anisotropic repulsion}} + \underbrace{|\mathbf{x}|^2}_{\text{attraction/confinement}}, \quad 0 < s < 2 \qquad \frac{\mathbf{x}}{|\mathbf{x}|} = (\cos\theta, \sin\theta)$$

- Logarithmic (s=0) case: $W_{\log}(\mathbf{x}) = -\ln|\mathbf{x}| + \Omega(\theta) + |\mathbf{x}|^2$

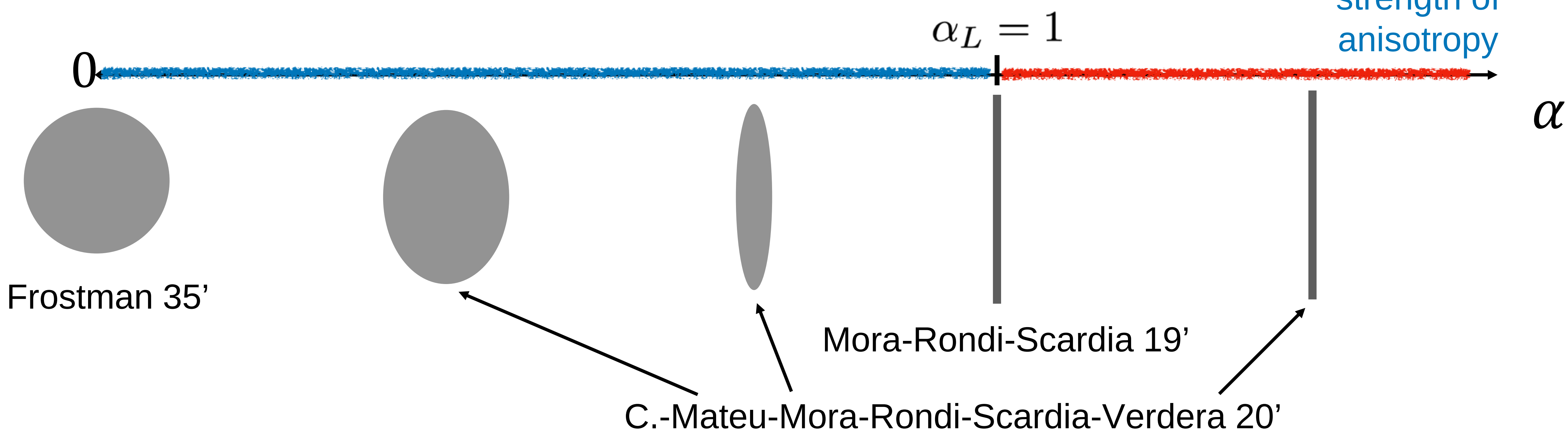
Previous results



- Previous results on the minimizers for the potential

$$W_{\log,\alpha}(\mathbf{x}) = -\ln|\mathbf{x}| + \alpha\omega(\theta) + |\mathbf{x}|^2, \quad \omega(\theta) = \cos^2\theta, \quad \alpha \geqslant 0.$$

strength of anisotropy

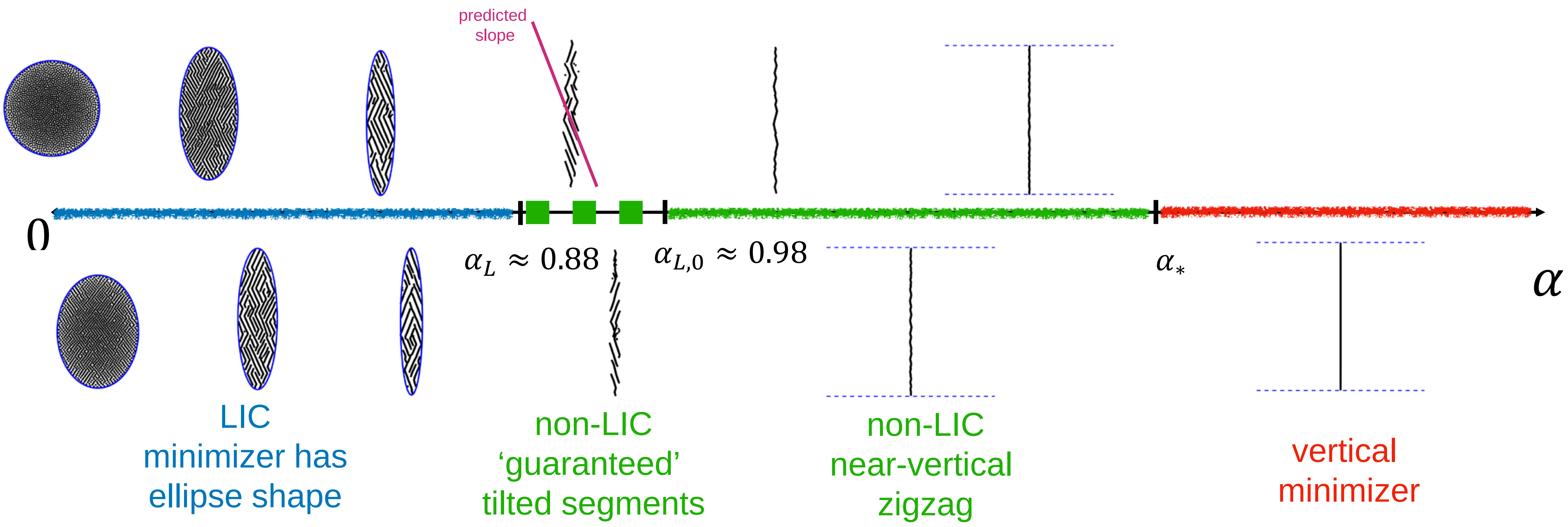


Previous results

- Similar ellipse-shaped minimizers are obtained in 3D in C.-Mateu-Mora-Rondi-Scardia-Verdera 21'
- Ellipse-shaped minimizers are obtained in 2D for a perturbed logarithmic potential in Mateu-Mora-Rondi-Scardia-Verdera 21'
- Questions: General singularity? General ω? Ellipses / Collapse to lower-dimensions?

A generic example in 2D

$W_\alpha(\mathbf{x}) = |\mathbf{x}|^{-s}(1 + \alpha\omega(\theta)) + |\mathbf{x}|^2$ $\quad s = 0.4,\ \omega(\theta) = \cos^4\theta + 0.1\cos^2\theta$



LIC and Euler-Lagrange condition

$$E[\rho] = \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} W(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}\,\rho(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

- Euler-Lagrange condition for energy minimizers (Balagué-Carrillo-Laurent-Raoul 13'):

$$(W*\rho)(\mathbf{x}) \leqslant \operatorname{essinf}(W*\rho), \quad \rho \ a.e.$$

($W*\rho$ achieves min on $\operatorname{supp}\rho$)

- **Theorem** (CMMSRV 20', C**.**-Shu 21'): for LIC potentials, the unique energy minimizer is the only probability measure satisfying the E-L condition (up to translation).
- The ellipse-shaped minimizers in CMMSRV 20' are proved by verifying E-L condition explicitly

Fourier transform of anisotropic potential

- Assume Ω is smooth and $\Omega(\theta+\pi)=\Omega(\theta)$

- **Lemma (C.-Shu** 22'):

$$\mathcal{F}[|\mathbf{x}|^{-s}\Omega(\theta)] = |\xi|^{-2+s}\tilde{\Omega}(\varphi), \quad 0<s<2$$

$$\tilde{\Omega}(\varphi) = \tau_{2-s}\int_{-\pi}^{\pi} |\cos(\varphi-\theta)|^{-2+s}\Omega(\theta)\,\mathrm{d}\theta, \quad 1<s<2.$$

- Applying reversely, we get

$$\Omega(\theta) = \tau_s\int_{-\pi}^{\pi} |\cos(\theta-\varphi)|^{-s}\tilde{\Omega}(\varphi)\,\mathrm{d}\varphi, \quad 0<s<1.$$

LIC for anisotropic energy

$$W(\mathbf{x}) = |\mathbf{x}|^{-s}\Omega(\theta) + |\mathbf{x}|^2, \quad 0 < s < 2$$

- **Theorem** (C.-Shu 22'): W has the LIC property if and only if $\mathcal{F}[|\mathbf{x}|^{-s}\Omega(\theta)] = |\xi|^{-2+s}\tilde{\Omega}(\varphi)$ with $\tilde{\Omega} \geq 0$

$$|\mathbf{x}|^{-s}\Omega(\theta) = \tau_s \int_{-\pi}^{\pi} |\mathbf{x}\cdot\vec{e}_\varphi|^{-s}\tilde{\Omega}(\varphi)\,\mathrm{d}\varphi$$

- Notice that LIC is strict convexity but the Fourier condition is non-strict inequality. This comes from a complex analysis argument.

- If LIC fails, then W is infinitesimally concave (C.-Shu 21'), implying superlevel sets of any Wasserstein-infinity local minimizer do not contain interior points. This strongly suggests that minimizers collapse to lower dimensions.

LIC for anisotropic energy

$$W_\alpha(\mathbf{x}) = |\mathbf{x}|^{-s}(1+\alpha\omega(\theta)) + |\mathbf{x}|^2 \qquad \hat{W}_\alpha(\xi) = |\xi|^{-2+s}(c_s + \alpha\tilde{\omega}(\theta))$$

- If $0 < s < 1$, then there exists a threshold

$$\alpha_L = \frac{c_s}{-\min\tilde{\omega}} > 0$$

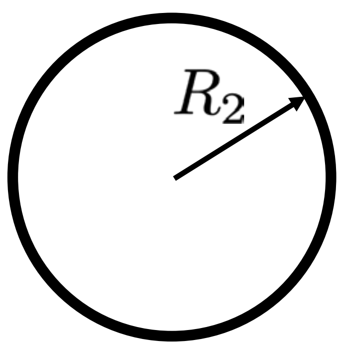
- If $1 \leq s < 2$, then LIC always holds.

Part 3.1: Ellipse-shaped minimizers for the LIC case

Ellipse-shaped minimizers

- For $0 < s < 2$, the minimizer for the isotropic potential $|\mathbf{x}|^{-s} + |\mathbf{x}|^2$ is

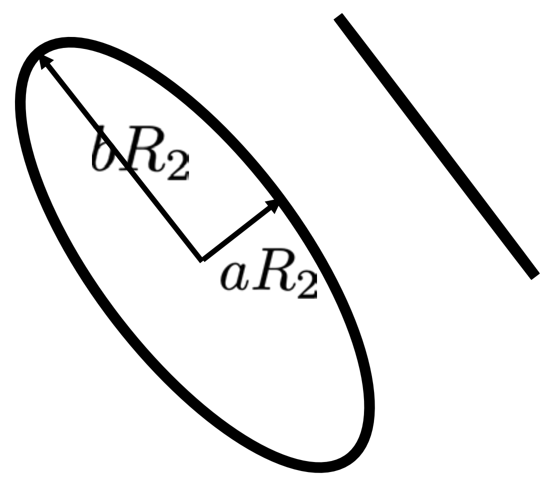
$$\rho_2(\mathbf{x}) = C_2(R_2^2 - |\mathbf{x}|^2)_+^{s/2}$$



(Caffarelli-Vázquez 11’, C.-Huang 17’, C.-Shu 21’)

- Rescaled and rotated version:

$$\rho_{a,b}(\mathbf{x}) = \frac{1}{ab}\rho_2\left(\frac{x_1}{a}, \frac{x_2}{b}\right), \quad \rho_{a,b,\eta}(\mathbf{x}) = \rho_{a,b}(\mathcal{R}_{-\eta}\mathbf{x})$$



- These densities are supported on possibly degenerate ellipses

Ellipse-shaped minimizers

- **Theorem** (C.-S**hu** 22'): For $0 < s < 1$, $W(\mathbf{x}) = |\mathbf{x}|^{-s}\Omega(\theta) + |\mathbf{x}|^2$
    - If $\tilde{\Omega} \geq c > 0$, then the unique energy minimizer is $\rho_{a,b,\eta}$ some $a, b > 0$ with $(a,b) \in [0,\infty)^2 \backslash \{(0,0)\}$
    - If $\tilde{\Omega} \geq 0$, then it is some $\rho_{a,b,\eta}$ with a=0 possibly.
- This result covers all LIC cases

0 LIC, ellipses α_L non-LIC, "lower-dimension minimizers" α

$$W_\alpha(\mathbf{x}) = |\mathbf{x}|^{-s}(1 + \alpha\omega(\theta)) + |\mathbf{x}|^2$$

Key lemma

- By LIC, it suffices to verify the Euler-Lagrange condition, i.e., carefully calculate the potential generated by $\rho_{a,b,\eta}$
- **Lemma** (C.-Shu 22'): For $0 < s < 1$,

$$(|\mathbf{x}|^{-s}\Omega(\theta) + Ax_1^2 + Bx_2^2 + 2Dx_1x_2) * \rho_{a,b} = C_{\Omega,a,b}, \quad \mathbf{x} \in \operatorname{supp} \rho_{a,b}$$

$$\begin{pmatrix} A(a,b) \\ B(a,b) \\ D(a,b) \end{pmatrix} = \tau_s (R_1/R_2)^{2+s} \int_{-\pi}^{\pi} (a^2\cos^2\varphi + b^2\sin^2\varphi)^{-(2+s)/2} \begin{pmatrix} \cos^2\varphi \\ \sin^2\varphi \\ \cos\varphi\sin\varphi \end{pmatrix} \tilde{\Omega}(\varphi)\,\mathrm{d}\varphi$$

If $\tilde{\Omega} \geq 0$, then the generated potential is larger outside support

- This implies theorem by finding correct a, b, η to make

$$A = B = 1, D = 0$$

Sketch of proof

$$\mathcal{F}[|\mathbf{x}|^{-s}\Omega(\theta)] = |\xi|^{-2+s}\tilde{\Omega}(\varphi) \qquad \Omega(\theta) = \tau_s \int_{-\pi}^{\pi} |\cos(\theta-\varphi)|^{-s}\tilde{\Omega}(\varphi)\,\mathrm{d}\varphi, \quad 0 < s < 1.$$

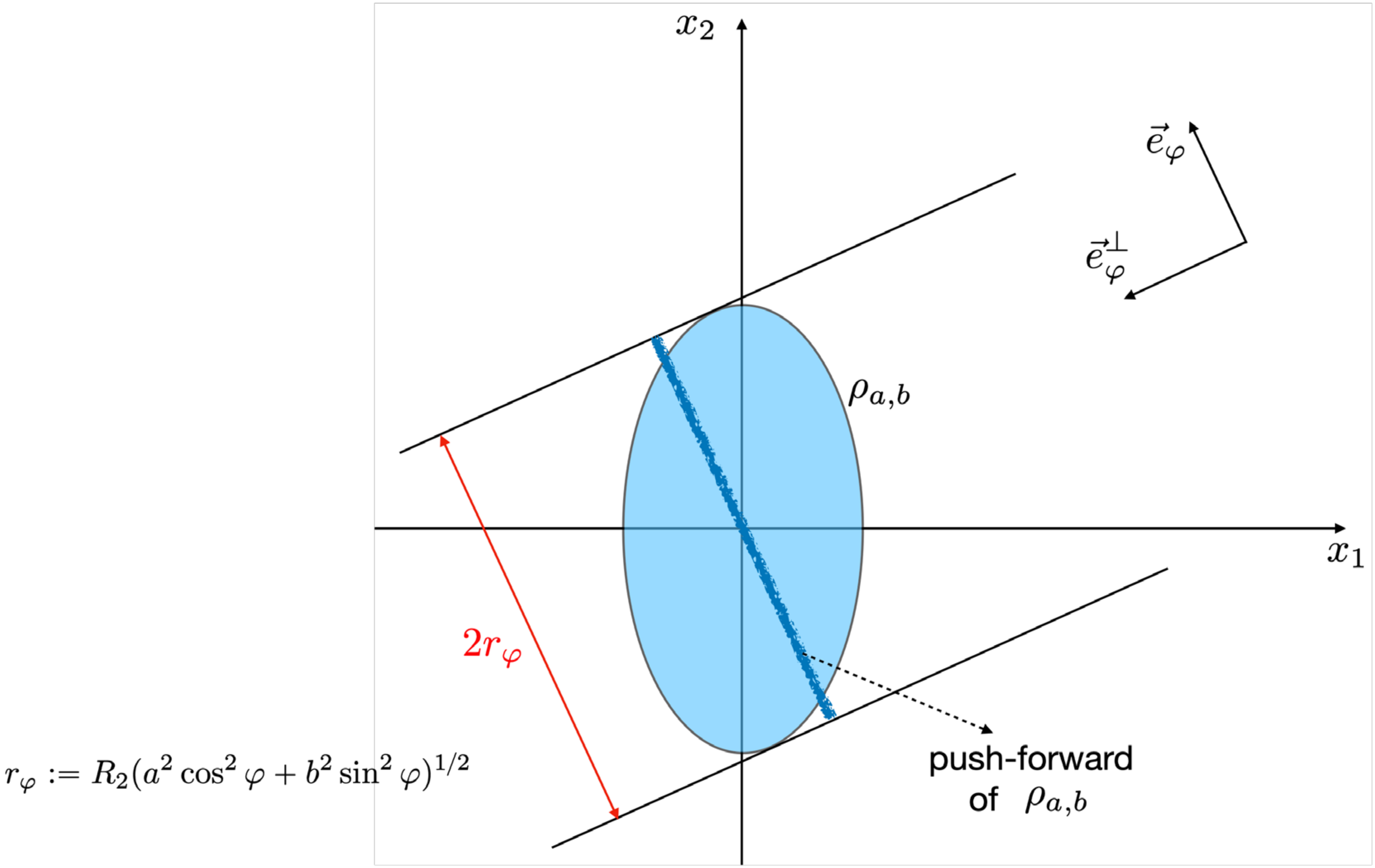
- Decomposition of anisotropic potential into 1D potentials

$$|\mathbf{x}|^{-s}\Omega(\theta) = \tau_s \int_{-\pi}^{\pi} \underbrace{|\mathbf{x}\cdot\vec{e}_\varphi|^{-s}}_{\text{1D potential along } \varphi}\tilde{\Omega}(\varphi)\,\mathrm{d}\varphi.$$

- To calculate $|\mathbf{x}|^{-s}\Omega(\theta) * \rho_{a,b}$, we calculate $|\mathbf{x}\cdot\vec{e}_\varphi|^{-s} * \rho_{a,b}$ and assembly

- The push-forward of $\rho_{a,b}$ by the projection operator onto $\vec{e}_\varphi$ is a rescaling of the 1D energy minimizer for the potential

$$|x|^{-s} + |x|^2$$


x_2
x_1
$\vec{e}_\varphi$
$\vec{e}_\varphi^{\perp}$
$\rho_{a,b}$
$2r_\varphi$
push-forward
of $\rho_{a,b}$
$r_\varphi := R_2(a^2\cos^2\varphi + b^2\sin^2\varphi)^{1/2}$

Sketch of proof

$$\left(\left(|\mathbf{x}\cdot\vec{e}_\varphi|^{-s}+\left(\frac{R_1}{r_\varphi}\right)^{2+s}|\mathbf{x}\cdot\vec{e}_\varphi|^2\right)*\rho_{a,b}\right)(y_1\vec{e}_\varphi+y_2\vec{e}_\varphi^{\perp})\begin{cases}=V_1\left(\frac{R_1}{r_\varphi}\right)^s, & y_1\in[-r_\varphi,r_\varphi]\\ >V_1\left(\frac{R_1}{r_\varphi}\right)^s, & y_1\notin[-r_\varphi,r_\varphi]\end{cases}$$

- Integrated together, we see the following is constant on support

$$\left(|\mathbf{x}|^{-s}\Omega(\theta)+\tau_s\int_{-\pi}^{\pi}\left(\frac{R_1}{r_\varphi}\right)^{2+s}|\mathbf{x}\cdot\vec{e}_\varphi|^2\tilde{\Omega}(\varphi)\,\mathrm{d}\varphi\right)*\rho_{a,b}$$

- If $\tilde{\Omega}\geq 0$ then this generated potential is larger outside because the integration is a positive linear combination

Part 3.2: Collapse to 1D for strong anisotropy

Collapse to 1D

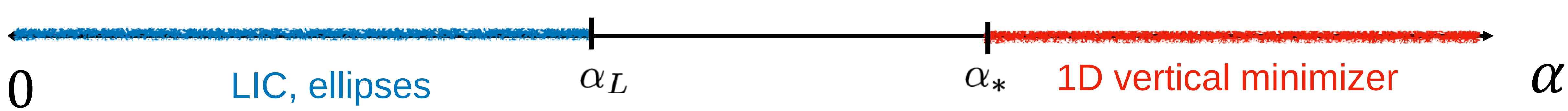
$$W_\alpha(\mathbf{x}) = |\mathbf{x}|^{-s}(1+\alpha\omega(\theta)) + |\mathbf{x}|^2$$

- Assume ω achieves minimum at $\omega(\frac{\pi}{2}) = 0$

- **Theorem** (C.-Shu 22'): For $0 < s < 1$, if ω satisfies the non-degeneracy condition

$$\omega(\theta) \geqslant c_\omega \left|\theta - \frac{\pi}{2}\right|^2, \quad \forall\theta \in [0,\pi].$$

then minimizers collapse to 1D vertical distribution for sufficiently large α



- The non-degeneracy is necessary

Proof by comparison

- **Lemma** (C.-Shu 22’): there exists a smooth Ω_* such that the potential $W_*(\mathbf{x}) = |\mathbf{x}|^{-s}\Omega_*(\theta) + |\mathbf{x}|^2$ satisfies
  - $\hat{W}_*(\xi) \geqslant 0$ (LIC)

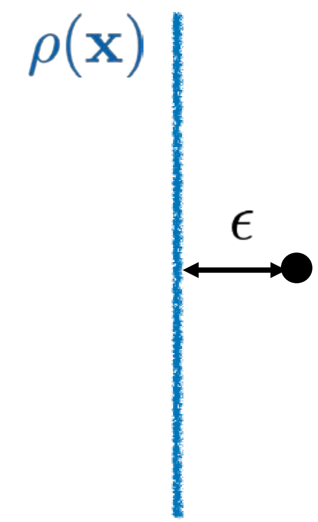
  - $W_* * \rho_{1\mathrm{D}}$ achieves minimum on support
- If viewing W_* as an element in the family W_α , one has no ‘gap’ between ellipses and 1D vertical minimizer. In the logarithmic potential case, CMMSRV 20’ considered exactly such a potential
- The theorem follows from a comparison similar to CMMSRV 20’

Part 3.3: Zigzag formation in between

Local expansion around 1D segment

- **Theorem** (C.-Shu 22'): for $0 < s < 1$ and a 1D segment

$$\rho(\mathbf{x}) = \psi(x_2)\delta(x_1)$$



the generated potential has the expansion

$$(W * \rho)(\epsilon, 0) - (W * \rho)(0, 0) = \frac{1}{2\tau_{2-s}}\underline{\tilde{\Omega}(0)}\psi(0)\epsilon^{1-s} + O(\epsilon)$$

$\tau_{2-s} < 0$

- The local stability of a 1D segment is related the negativity of the Fourier transform of W in the direction of its normal.

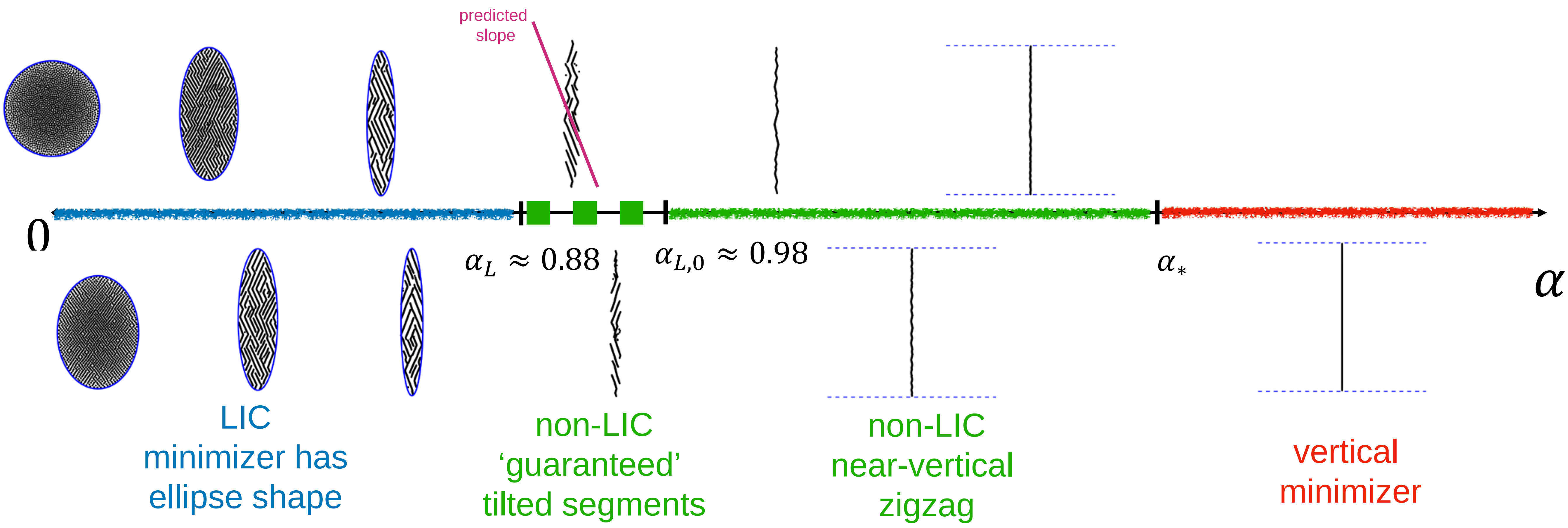
Local expansion around 1D segment

$$W_\alpha(\mathbf{x}) = |\mathbf{x}|^{-s}(1 + \alpha\omega(\theta)) + |\mathbf{x}|^2 \qquad \hat{W}_\alpha(\xi) = |\xi|^{-2+s}(c_s + \alpha\tilde{\omega}(\varphi))$$

- Assume $\tilde{\omega}$ achieves the most negative value NOT at $\varphi = 0$
  As α increases, the first “allowed” direction of 1D segment is not the vertical direction.
- This guarantees the appearance of zigzags.

A generic example

$$W_\alpha(\mathbf{x}) = |\mathbf{x}|^{-s}(1 + \alpha\omega(\theta)) + |\mathbf{x}|^2 \qquad s = 0.4,\ \omega(\theta) = \cos^4\theta + 0.1\cos^2\theta$$



Remarks

- This framework can be generalized to higher dimensions (C.-Shu 23' in 3D)
- The logarithmic potentials can be derived as a limit $s \to 0^+$. Most results can be carried over. This generalizes the result in CMMSRV 20'.

- The set of ellipse-shaped distributions is closed under the associated Wasserstein-2 gradient flows. Long time convergence to the minimizer can be justified in the LIC case.
- Open question: $\omega(\theta) = \cos^2\theta\sin^2\theta$