Aspects of the cutoff phenomenon

for diffusions

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Functional inequalities
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Joint works with Jeanne Boursier (Columbia), Cyril Labbé (Paris-Cité), Max Fathi (Paris-Cité), and Nikita Simonov (Sorbonne Université)

Outline

Langevin process

- $V \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, $V - \frac{\rho}{2} |\cdot|^2$ convex for some $\rho > 0$ (ρ-convex)

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- Lichnerowicz, Bakry – Émery, Klartag : $\lambda \geq \rho$

Rigid Langevin : cutoff and critical time

Theorem (C. – Fathi 2024)

$$\lim_{n\to\infty} \sup_{x_0\in S} \mathrm{dist}(\mu_t \mid \mu_\infty) = \begin{cases} \max & \textit{if } t \leq (1-\varepsilon)T \\ 0 & \textit{if } t \geq (1+\varepsilon)T \end{cases}$$

- max = maximum value of dist
- dist = Wasserstein, Entropy, Fisher, total variation

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where $S = \{x \in \mathbb{R}^n : |x - m| \leq c\sqrt{n}\}$ *and* $T = \frac{\log(n)}{2\lambda}$.

- max = maximum value of dist
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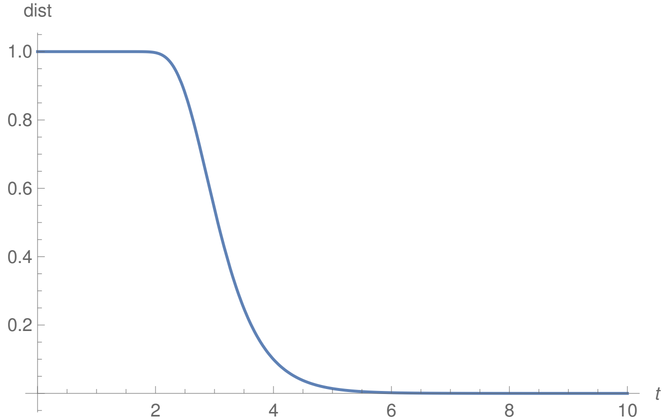
- m = mean of μ_∞, $c > 0$ arbitrary constant

Cutoff for OU in total variation distance



$$\rho = 1,\ n = 50,\ \frac{|x_0|^2}{n} = 1,\ \log(50) \approx 3.91$$

Example beyond OU : Dyson !

$$V = \frac{\rho}{2}|\cdot|^2 + W, \quad \rho > 0$$

- $W : \mathbb{R}^n \to \mathbb{R}$ convex, constant on $x + \mathbb{R}(1, \ldots, 1)$, $x \in \mathbb{R}^n$

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- Gaussian factorization using $\pi(x) = \mathrm{proj}_{\mathbb{R}(1,\ldots,1)}(x)$

$$\mathrm{e}^{-V(x)} = \mathrm{e}^{-\frac{\rho}{2}|\pi(x)|^2} \mathrm{e}^{-(W(\pi^\perp(x)) + \frac{\rho}{2}|\pi^\perp(x)|^2)}$$

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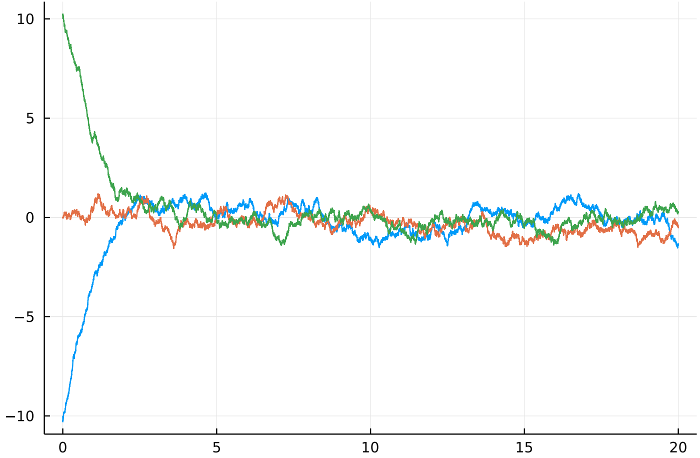
- Spectral gap of $\mathcal{L} = \Delta - \nabla V \cdot \nabla$ is $\lambda = \rho$, free of W!
- $x_1 + \cdots + x_n$ is an eigenfunction of $-\mathcal{L}$ associated to λ

Dyson : numerical experiments



$n = 3$, $\beta = 0$: confinement and independence (OU)

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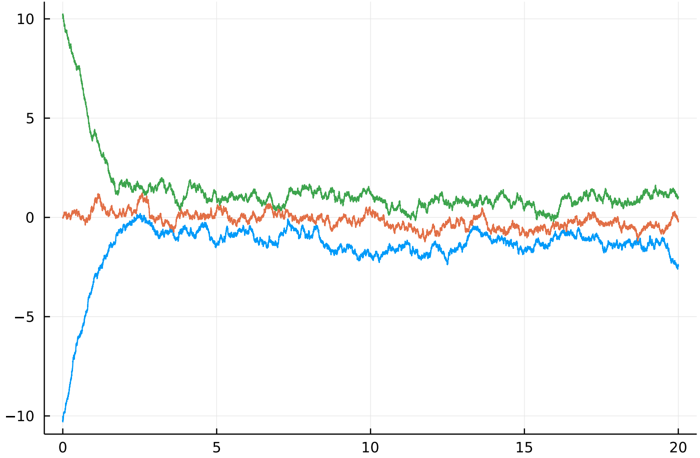

$n = 3$, $\beta = 2$: confinement and repulsion (DOU)

Some distances or divergences

$$\chi^2(\nu \mid \mu) = \mathrm{Var}_\mu\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) = \left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu} - 1\right\|^2_{L^2(\mu)}$$

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OU : emergence of cutoff phenomenon

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If $|x_0|^2 = n$ then cutoff at $t = \frac{\log(n)}{2\rho}$ for $\mathrm{dist}(\mu_t \mid \mu_\infty)$: n versus $\mathrm{e}^{-2\rho t}$

Some universal comparisons

- Functional inequalities à la Pinsker

$$
\begin{aligned}
\|\mu - \nu\|_{\mathrm{TV}}^2 &\leq 2\mathrm{Entropy}(\nu \mid \mu) \\
\mathrm{Entropy}(\nu \mid \mu) &\leq 2\chi(\nu \mid \mu) + \chi^2(\nu \mid \mu) \\
\mathrm{Hellinger}^2(\mu, \nu) &\leq \|\mu - \nu\|_{\mathrm{TV}} \\
\|\mu - \nu\|_{\mathrm{TV}}^2 &\leq \mathrm{Hellinger}^2(\mu, \nu)(2 - \mathrm{Hellinger}^2(\mu, \nu))
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- Useful for cutoff upper/lower bounds, but not enough !

Contraction from variational formula

- If $\mathrm{dist} \in \{\mathrm{TV}, \mathrm{Entropy}, \chi^2\}$ then

$$\mathrm{dist}(\nu \circ f^{-1} \mid \mu \circ f^{-1}) \leq \mathrm{dist}(\nu \mid \mu)$$

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- If f affine eigenfunction of $-\mathcal{L}$ then $(f(X_t))_{t \geq 0}$ is 1D OU

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$$\Phi(u) = \begin{cases} u^2 - 1 & \text{if dist} = \chi^2 \\ u\log(u) & \text{if dist} = \text{Entropy} \\ \frac{1}{2}|u-1| & \text{if dist} = \text{TV} \\ \frac{1}{2}(1-\sqrt{u}) & \text{if dist} = \text{Hellinger}^2 \end{cases}$$

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$$\mathrm{dist}(\nu \mid \mu) = \int \Phi\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\mathrm{d}\mu$$

$$\Phi(u) = \begin{cases} u^2 - 1 & \text{if dist} = \chi^2 \\ u\log(u) & \text{if dist} = \text{Entropy} \\ \frac{1}{2}|u-1| & \text{if dist} = \text{TV} \\ \frac{1}{2}(1-\sqrt{u}) & \text{if dist} = \text{Hellinger}^2 \end{cases}$$

- Fisher and Wasserstein : involve also convexity of V

Curvature $\Rightarrow$ Functional inequalities

Thanks to ρ-convexity of V :

- Log-Sobolev and Talagrand inequalities

$$\text{Entropy}(\mu \mid \mu_\infty) \leq \frac{1}{2\rho}\text{Fisher}(\mu \mid \mu_\infty)$$

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- Monotonicity (à la Boltzmann), convexity (à la Cercignani)

$$\partial_t \mathrm{Entropy}(\mu_t \mid \mu_\infty) = -\mathrm{Fisher}(\mu_t \mid \mu) \leq 0$$

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Curvature $\Rightarrow$ Exponential decays and regularization

Thanks to ρ-convexity of V :

- Exponential decays via Grönwall

$$\mathrm{Entropy}(\mu_t \mid \mu_\infty) \leq \mathrm{e}^{-2\rho t}\mathrm{Entropy}(\mu_0 \mid \mu_\infty)$$
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- Regularization in short-time (or short-time blowup control)

$$\mathrm{Entropy}(\mu_t \mid \mu_\infty) \leq \frac{\rho \mathrm{e}^{-2\rho t}}{1-\mathrm{e}^{-2\rho t}}\mathrm{Wasserstein}^2(\mu_0, \mu_\infty)$$

Rigidity $\Rightarrow$ Gaussian factorization

If $\lambda = \rho$ (rigidity) then :

- Up to rotation and translation

$$\mu_\infty = \mathrm{e}^{-V} = \mathcal{N}(0, \tfrac{1}{\lambda}) \otimes \text{factor with curvature } \lambda$$

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- If f is such an eigenfunction then $(f(X_t))_{t\geq 0}$ is 1D OU !
- Allows cutoff lower bound by contraction (OU comparison)

Rigidity ⇒ Double sided Wasserstein estimate

If $\lambda = \rho$ then for all $S \subset \mathbb{R}^n$:

$$\sup_{x_0 \in S} \mathrm{Wasserstein}^2(\mu_t, \mu_\infty) \begin{cases} \leq \mathrm{e}^{-2\lambda t} \sup_{x_0 \in S} \int |x - x_0|^2 \mathrm{d}\mu_\infty(x) \\ \geq \dfrac{\mathrm{e}^{-2\lambda t}}{\lambda} \sup_{x_0 \in S} \Big(k + \sup_{f_1,\ldots,f_k} \sum_{i=1}^{k} |f_i(x_0)|^2 \Big) \end{cases}$$

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- Rest of proof of theorem with functional inequalities □

Outline

Cutoff for high-dimensional nonlinear diffusions

- Fast diffusion ($m < 1$) and porous medium ($m > 1$) in $\mathbb{R}^n$

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- Barenblatt, Pattle, Vázquez, Demange, Dolbeault, ...

Nonlinear diffusion : cutoff and critical time

Theorem (C. – Fathi – Simonov 2025)

For all $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \sup_{x_0 \in S} \operatorname{dist}(v_t \mid v_\infty) = \begin{cases} +\infty & \text{if } t \leq (1-\varepsilon)T \\ 0 & \text{if } t \geq (1+\varepsilon)T \end{cases}$$

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- Alternative fixed $m > 1$ regime: cutoff at $\frac{\log(n)}{2}$

Outline

Cutoff people (among others)

Persi Diaconis

David Aldous

Laurent Saloff-Coste

Justin Salez

From cards shuffling to Brownian motion and beyond!

General setting

- Family of ergodic Markov processes

$$(X_t^{(n)})_{t\geq 0} \quad \text{on state space } E^{(n)}$$

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- Markov semigroup

$$\mu P_t(f) = \mu_t(f) = \mathbb{E}(f(X_t)), \quad X_0 \sim \mu$$

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There exists a critical time T such that for all $\varepsilon \in (0,1)$

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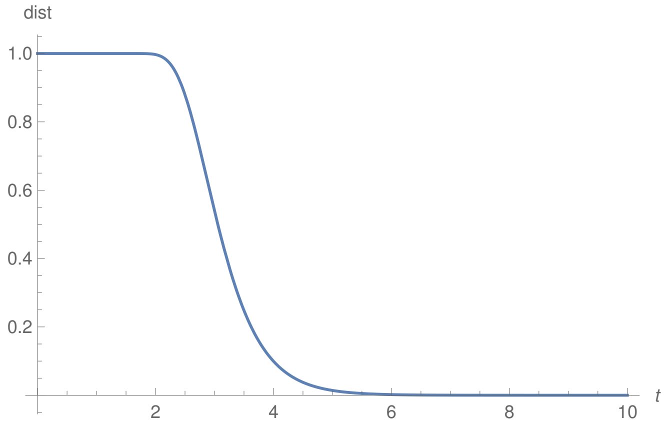
- Counter example : random walk on circle $\mathbb{Z}/n\mathbb{Z}$, CLT

Threshold phenomenon



Analogy with concentration of measure
High-dimensional phenomenon
Critical value not necessarily known

More examples

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- Worst initial condition versus trend to equilibrium

Mixing time and relaxation

- Mixing time : for $\mathcal{P}_0, \mathrm{dist}, \eta \ll \max$

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- When $P_t = \mathrm{e}^{t\mathcal{L}}$ is reversible then

$$\lambda = \text{spectral gap of infinitesimal generator } \mathcal{L}$$

Peres product condition $\Rightarrow$ cutoff at mixing time

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- $p = 1$ (TV) : counter examples by Aldous and by Pak
- $p = 1$ (TV) : fix by Salez via reverse Pinsker and varent

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$$\lim_{n\to\infty} \sup_{\mu_0 \in \mathcal{P}_0} \|\mu_t - \mu_\infty\|_p = \begin{cases} \max & \text{if } t \leq (1-\varepsilon)T_{\mathrm{mix}} \\ 0 & \text{if } t \geq (1+\varepsilon)T_{\mathrm{mix}} \end{cases}$$

Still works for $p = 1$ *for non-negatively curved diffusions.*

- Simple, via $d_p(s+t) \leq d_p(s)d_p(t)$ and $d_p(t) \searrow \leq C_q \mathrm{e}^{-c_q \lambda t}$
- $p = 1$ (TV) : counter examples by Aldous and by Pak
- $p = 1$ (TV) : fix by Salez via reverse Pinsker and varent
- Critical time T_{mix} not provided by this approach

Curvature product condition $\Rightarrow$ cutoff at mixing time

Theorem (C. – Fathi 2025 – Beyond rigidity, for diffusions)

For a diffusion $\mathcal{L} = \Delta - \nabla V \cdot \nabla$ *with* $\mathrm{Hess}\, V \geq \rho$, *if*

$$\rho \times T_{\mathrm{mix}}^{\mathrm{TV}} \xrightarrow[n\to\infty]{} +\infty$$

then for all $\varepsilon \in (0,1)$ *and* $\mathrm{dist} \in \{\mathrm{Entropy}, \mathrm{Fisher}, \mathrm{Wass.}, \mathrm{TV}\}$,

$$\lim_{n\to\infty} \sup_{\mu_0 \in \mathcal{P}_0} \mathrm{dist}(\mu_t \mid \mu_\infty) = \begin{cases} \max & \text{if } t \leq (1-\varepsilon)\, T_{\mathrm{mix}} \\ 0 & \text{if } t \geq (1+\varepsilon)\, T_{\mathrm{mix}} \end{cases}$$

- Proof : [Chen – Saloff-Coste] + [Salez] + [C. – Fathi]

Thank you for your attention!