Heating Laplace and Legendre

Based on two papers

Heat flow and duality

D. Cordero-Erausquin, N. Gozlan, S. Nakamura and H. Tsuji

Advances in Math. (2025)

On a Santaló point for Nakamura-Tsuji's Laplace transform inequality

D. Cordero-Erausquin, M. Fradelizi and D. Langharst

Forum of Math., Sigma (2025)

General principle

Obtain new/improved inequalities under the assumption of symmetry or centering

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Two manifestations:

- functional correlation/convolution inequalities (Brascamp-Lieb, gaussian correlation, Laplace inequalisites)
- improved Brunn-Minkowski theory in convexity/log-concavity (log-Brunn-Minkowski, dimensional Brunn-Minkowski, B-inequalities). Several conjectures, and a few results.

Laplace Transform

For nonnegative $f : \mathbb{R}^n \to \mathbb{R}^+$, define:

$$Lf(y) := \int_{\mathbb{R}^n} f(x)\, e^{x \cdot y}\, dx, \quad \forall y \in \mathbb{R}^n$$

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We are interested in (reverse) sharp $L^p - L^q$ bounds, in the regime

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- Note that $Lf : \mathbb{R}^n \to [0, \infty]$ is convex
- Note that, if $f \not\equiv 0$, then $Lf(y) \in (0, \infty]$ and by Hölder's inequality, we get that $\log Lf$ convex.
  In particular, $(Lf)^q$ is log-concave.

Reverse L^p bound

Nakamura-Tsuji (2023) Laplace inequality

If $f \geq 0$ is **even** then

$$\|Lf\|_{L^q(\mathbb{R}^n)} \geq C_p \|f\|_{L^p(\mathbb{R}^n))}$$

where the best constant C_p is attained for $f(x) = e^{-|x|^2/2}$.

'Improvement of positivity' in the terminology of C. Borell.

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- The quantity $\dfrac{\|Lf\|_{L^q(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}}$ is linear invariant.
- The inequality cannot hold for general $f \geq 0$. Note for instance that

$$L(\tau_z f)(x) = e^{z \cdot x} Lf(x)$$

where $\tau_z(f)(x) := f(x-z)$, so $L(\tau_z f)(x)^q = e^{qz \cdot x} Lf(x)^q$ can degenerate.

Reverse L^p Bound Theorem

Theorem (C-F-L 2024)

For any $f \geq 0$,

$$\sup_{z \in \mathbb{R}^n} \|L(\tau_z f)\|_{L^q(\mathbb{R}^n)} \geq C_p \|f\|_{L^p(\mathbb{R}^n)}$$

where $C_p = [p^{1/p}(-q)^{-1/q}]^{n/2}(2\pi)^{n/q}$ *is attained for Gaussians.*

Moreover if $\int x f(x)^p \, dx = 0$, *then*

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Theorem

If $f \not\equiv 0$ *then there exists* z *such that* $\|L(\tau_z f)\|_q > 0$ *and*

- *either* $\sup_z \|L(\tau_z f)\|_q = \infty$
- *or else the supremum is attained at a unique point, characterized by...*

Reverse L^p Bound Theorem

Theorem (C-F-L 2024)

For any nonnegative $f \in L^p(\mathbb{R}^n)$ *define*

$$f_t := P_t(f^p)^{1/p}$$

where P_t *is the Fokker-Planck semi-group, so that* $\|f_t\|_p = \|f\|_p$*. Then*

$$t \to \sup_{z \in \mathbb{R}^n} \|L(\tau_z f_t)\|_{L^q(\mathbb{R}^n)}$$

decreases on $[0, \infty)$*.*

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$L^p - L^q$ **inequalities for Laplace transform admit interesting consequences and reformulations.**

Equivalent formulation 1: Hypercontractivity

Consider the Ornstein-Uhlenbeck semi-group

$$P_\theta F(x) = \int_{\mathbb{R}^n} F(\cos(\theta)x + \sin(\theta)y)\, d\gamma(y)$$

for $\theta \in [0, \pi/2]$. Recall that for $1 \le p \le r$ and $\cos(\theta) = \sqrt{\frac{r-1}{q-1}}$,

$$\|P_\theta F\|_{L^r(\gamma)} \le \|F\|_{L^p(\gamma)}$$

Inequality is reversed for $r \le p < 1$ (C. Borell).

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Under the change of function $f(x) = F(x)\, e^{-x^2/2p}$, we get

Theorem (Improved hypercontractivty under centering)

For $F \geq 0$ with $\int xF(x)^p\, d\gamma(x) = 0$ we have for $p = 1 - \cos^2(\xi)$, $q = \frac{p}{p-1}$, that

$$\|P_\xi F\|_{L^q(\gamma)} \geq \|F\|_{L^p(\gamma)}.$$

Note that $\cos(\xi) = \sqrt[4]{\frac{p-1}{q-1}} \geq \sqrt{\frac{p-1}{q-1}}$.

Equivalent formulation 2: Inverse Brascamp-Lieb

The inequality $\|Lf\|_q \geq C_p \|f\|_p$ is equivalent, using that

$$\|g\|_q = \inf_h \frac{\int gh}{\|h\|_p}$$

replacing f by $f^{1/p}$ and setting $\lambda = \frac{1}{p} \geq 1$ to the inequality,

Theorem

For $\lambda \geq 0$, and $f, g \geq 0$, with $\int x f(x)\, dx = 0$,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} e^{x \cdot y} f(x)^\lambda g(y)^\lambda \, dxdy \geq \tilde{C}_p \left(\int_{\mathbb{R}^n} f \right)^\lambda \left(\int_{\mathbb{R}^n} g \right)^\lambda .$$

Equivalent formulation 3: Symmetric transport-entropy inequality

Theorem (Symmetric entropic-transport inequality)

Given two probability measures μ_1, μ_2 with $\int x \, d\mu_1(x) = 0$, we have

$$\frac{1}{2}\tilde{W}_2^{(p)}(\mu_1, \mu_2) \leq \mathrm{Ent}_\gamma(\mu_1) + \mathrm{Ent}_\gamma(\mu_2) + D_p.$$

Here γ is the standard Gaussian measure, Ent the (relative) entropy and

$$\tilde{W}_2^{(p)}(\mu_1, \mu_2) := \inf_\pi \int |y - x|^2 \, d\pi(x, y) + p\mathrm{Ent}_{\mu_1 \otimes \mu_2}(\pi).$$

The $p \to 0^+$ limit case (so $q \sim -p$)

To perform the limit in $\|Lf\|_q \geq C_p \|f\|_p$, let us rescale a bit. Replacing f by $f^{1/p}$ we have

$$1 \geq \tilde{C}_p \Big(\int f \Big) \|L(f^{1/p})(\frac{\cdot}{-p})\|_q^{-p}.$$

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Note that, $\frac{-p}{q} \to 1$, and writing $f = e^{-\varphi}$,

$$\begin{aligned} L(f^{1/p})(x/-p)^q &= \left(\left(\int \left(e^{-x \cdot y} f(y) \right)^{1/p} dy \right)^p \right)^{\frac{1}{p-1}} \\ &= \left(\left(\int \left(e^{x \cdot y - \varphi(y)} \right)^{1/p} dy \right)^p \right)^{\frac{1}{p-1}} \\ &\longrightarrow e^{-\varphi^*(x)} \end{aligned}$$

where

$$\varphi^*(x) = \sup_{y \in \mathbb{R}^n} x \cdot y - \varphi(y).$$

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So the limit of $\|Lf\|_q \geq C_p \|f\|_p$ as $p \to 0^+$ is

Theorem (Centered Blaschke-Santaló)

Given $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, *with* $\int x e^{-\varphi(x)}\, dx = 0$, *we have*

$$\int e^{-\varphi} \int e^{-\varphi^*} \leq (2\pi)^n.$$

This consequence of our result is well known (the novelty is in the semi-group approach).

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- Linear invariant
- there is equality when $\varphi(x) = |x|^2/2$, which is the fixed point of the Legendre transform.
- It implies the geometric form

$$\mathrm{vol}(K)\mathrm{vol}(K^\circ) \leq \mathrm{vol}(B_2^n)^2.$$

- It suffices to prove the inequality when φ is convex.

Remark on centering

We want to study the maximum of $\int e^{-\varphi} \int e^{-\varphi^*}$.
Need to assume some centering. Note, since

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Attained at a unique point', which is $z = 0$ iff $\int x e^{-\varphi^*(x)}\, dx = 0$. We can ensure this by translation, noting that $e^{(\varphi(x) - z \cdot x)^*} = e^{\tau_z(\varphi^*)}$.
If $\int x e^{-\varphi(x)}\, dx = 0$, we have

$$\int e^{-z \cdot x - \varphi(x)}\, dx \geq \int e^{-\varphi}.$$

since $e^t \geq 1 + t$.

Heat flow approach to Laplace inequality

Let P_t be either the **Heat semi-group**, $P_t = e^{t\Delta}$, or the **Fokker-Plank semi-group**, $P_t = e^{tL}$ with $Lu = \Delta u + \operatorname{div}(xu)$.

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We will study the $p = 0$ case (which is of independent interest)

Heat flow approach to Blaschke-Santaló

Recall $e^{-(\tau_z\varphi)^*(x)} = e^{z\cdot x}e^{-\varphi^*(x)}$.

Theorem (C-F-L '24 , C-G-N-T '24)

For $\varphi : \mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ *define* $\varphi_t := -\log P_t(e^{-\varphi})$ *so that* $\int e^{-\varphi_t} = \int e^{-\varphi}$*. Then, if we set*

$$\psi_t := (\varphi_t)^*,$$

we have that

$$t \to \inf_{z\in\mathbb{R}^n} \int e^{-(\tau_z\varphi_t)^*} = \inf_{z\in\mathbb{R}^n} \int e^{z\cdot x} e^{-\psi_t(x)}\, dx$$

increases in time $[0,\infty)$.

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Comparing $t=0$ and $t=\infty$,

$$\int e^{-\varphi} \inf_{z\in\mathbb{R}^n} \int e^{z\cdot x} e^{-\varphi^*(x)}\, dx \le \text{gaussian} = (2\pi)^n.$$

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For $\varphi_t = -\log P_t(e^{-\varphi_t})$, it is classical. From $\partial_t e^{-\varphi_t} = \Delta e^{-\varphi_t}$ we find

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Evolution of the Legendre's transform ψ_t.

Theorem (CE-Gozlan-Nakamura-Tsuji '24)

Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a (super-linear) convex function, and $\varphi_t = -\log P_t(e^{-\phi})$ where P_t is the heat semi-group. Let $\psi_t = (\varphi_t)^*$. Then for every $z \in \mathbb{R}^n$ and $t > 0$:

$$\partial_t \psi_t(z) = |z|^2 - \mathrm{Tr}\left[(D^2 \psi_t(z))^{-1}\right]$$

Proof of $\partial_t \psi_t(z) = |z|^2 - \mathrm{Tr}\left[(D_z^2 \psi_t)^{-1}\right]$

General property for perturbations: $\boxed{\partial_t \psi_t(z) = -\partial_t \phi_t(\nabla \psi_t(z))}$.

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- Then for $\varepsilon > 0$ small, we have

$$\psi_{t_0+\varepsilon}(z_0) \quad = \quad \sup_y z_0 \cdot (x_0 + \varepsilon y) - \phi_{t_0+\varepsilon}(x_0 + \varepsilon y)$$

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$$\begin{aligned}
\psi_{t_0+\varepsilon}(z_0) &= \sup_y z_0 \cdot (x_0 + \varepsilon y) - \phi_{t_0+\varepsilon}(x_0 + \varepsilon y) \\
&= \psi_{t_0}(z_0) + \sup_y \Big\{ \varepsilon z_0 \cdot y \\
&\qquad - \varepsilon (\partial_t \phi)_{t=t_0}(x_0) - \varepsilon \nabla \phi_{t_0}(x_0) \cdot y + O(\varepsilon^2) \Big\}
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$$\begin{aligned}\partial_t \psi_t(z) &= -\Delta \phi_t(\nabla \psi_t(z)) + |\nabla \phi_t|(\nabla \psi_t(z)^2 \\ &= -\mathrm{Tr}\left[D^2 \phi_t(\nabla \psi_t(z))\right] + |z|^2\end{aligned}$$

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&= -\mathrm{Tr}\left[(D^2 \psi_t(z))^{-1}\right] + |z|^2.
\end{aligned}$$

as wanted.

Back to our aim: prove the

Theorem

Given ϕ convex (super-linear) and $e^{-\phi_t} = P_t(e^{-\phi})$, the function

$$\alpha(t) := \inf_z \int e^{-(\tau_z \phi_t)^*} = \inf_{z\in\mathbb{R}^n} \int e^{z\cdot x} e^{-\psi_t(x)}\, dx$$

increases in $t > 0$, where $\psi_t = (\varphi_t)^*$.

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$$\alpha(t) := \inf_z \int e^{-(\tau_z \phi_t)^*} = \inf_{z \in \mathbb{R}^n} \int e^{z \cdot x} e^{-\psi_t(x)}\, dx$$

increases in $t > 0$, where $\psi_t = (\varphi_t)^*$.

Note that

$$\alpha(t) = \inf L(e^{-\psi_t}).$$

Back to our aim: prove the

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We shall rather work with $\log(\alpha)$. Introduce

$$Q(t,z) := \log L(e^{-\psi_t})(z) = \log \int e^{-\psi_t(y)} e^{z \cdot y}\, dy,$$

so that

$$\log \alpha(t) = \inf_z Q(t,z)$$

.

For $Q(t,z) = \log L(e^{-\psi_t})(z) = \log \int e^{-\psi_t(y)} e^{z \cdot y}\, dy$ we prove that

Theorem

At any $t > 0$ *and* $z \in \mathbb{R}^n$,

$$\partial_t Q + |\nabla_z Q|^2 \geq 0.$$

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Proof of the monotonicty of α. Let z_t be such that $\inf_z Q(t,z) = Q(t,z_t) = \log \alpha(t)$, so

$$\begin{aligned}(\log \alpha)'(t) &= \partial_t Q(t,z_t) + \partial_t z_t \cdot \nabla_z Q(t,z_t) \\ &\geq \partial_t z_t \cdot \nabla_z Q(t,z_t) - |\nabla_z Q(t,z_t)|^2.\end{aligned}$$

But by construction:

$$\nabla_z Q(t,z_t) = 0.$$

Proof of $\partial_t Q + |\nabla_z Q|^2 \geq 0$ for $Q(t,z) = \log L(e^{-\psi_t})(z)$

Proof of $\partial_t Q + |\nabla_z Q|^2 \geq 0$ for $Q(t,z) = \log L(e^{-\psi_t})(z)$

We have

$$\begin{aligned}
\partial_t Q(z) &= \frac{1}{L(e^{-\psi_t})(z)} \int -\partial_t \psi_t(y)\, e^{z\cdot y - \psi_t(y)}\, dy \\
&= \int -\partial_t \psi_t(y)\, d\mu_V(y)
\end{aligned}$$

where

$$V(y) := V_{t,z}(y) := \psi_t(y) - z\cdot y \quad \text{and} \quad d\mu_V(y) = \frac{e^{-V(y)}\, dy}{\int e^{-V}}.$$

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Recall that

$$-\partial_t \psi_t(y) = \mathrm{Tr}\big[(D^2\psi_t(y))^{-1}\big] - |y|^2 = \mathrm{Tr}\big[(D^2 V(y))^{-1}\big] - |y|^2.$$

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So

$$\partial_t Q(z) = \int \left(\mathrm{Tr}\left[(D^2 V(y))^{-1}\right] - |y|^2\right) d\mu_V(y)$$

Proof of $\partial_t Q + |\nabla_z Q|^2 \geq 0$ for $Q(t,z) = \log L(e^{-\psi_t})(z)$

Theorem (Brascamp-Lieb '76)

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a smooth convex function with $\int e^{-V} < \infty$, and denote $d\mu_V(x) := e^{-V(x)} \frac{dx}{\int e^{-V}}$. Then for any $u \in L^2(\mu_V)$, we have

$$\int u^2 \, d\mu_V - \left(\int u \, d\mu \right)^2 \leq \int (D^2 V)^{-1} \nabla u \cdot \nabla u \, d\mu_V$$

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$$\int |y|^2 \, d\mu_V(y) - \left| \int y \, d\mu_V(y) \right|^2 \leq \int \mathrm{Tr}\left[(D^2 V(y))^{-1} \right] d\mu_V(y).$$

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This translates to

$$\partial_t Q(t,z) \geq - \left| \int y \, d\mu_V(y) \right|^2 = -|\nabla_z Q(z,t)|^2$$

recalling that $Q(t,z) = \log \int e^{z \cdot y} e^{-\psi_t(y)}$.

Local Maximizers

The argument gives actually a description of local maximizers:

Theorem

Let ϕ be an even convex (super-linear) function and $\varepsilon > 0$ such that

$$M(e^{-\phi}) = \sup\left\{M(e^{-\psi})\ ;\ \psi \textit{ even convex with } \|e^{-\phi} - e^{-\psi}\|_{L^1(\mathbb{R}^n)} \leq \varepsilon\right\}$$

Then $e^{-\phi}$ is a centered Gaussian function.

- Must have $M(e^{-\phi_t}) = M(e^{-\phi})$ for $t \in [0, t_0]$ and so $\alpha'(\frac{t_0}{2}) = 0$.
- Equality cases in Brascamp-Lieb inequality $\in \mathrm{span}\{\partial_i \psi_{t_0/2}\}$.
- $\partial_i \psi_{t_0/2}$ is a linear combination of $y \to y_j$
- $D^2 \psi_{t_0/2}$ is constant on $\mathbb{R}^n$
- $\psi_{t_0/2}$ and $\phi_{t_0/2}$ are quadratic, i.e. $P_{t_0/2}(e^{-\phi})$ is a centered Gaussian.
- $e^{-\phi}$ must be a centered Gaussian function

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Celebrating the solution of the slicing/hyperplane conjecture !

SLICING DAY IN JUSSIEU
June, Wednesday 25, Room 15-25-104

Four lectures by Apostolos Giannopoulos (Athens) and Joseph Lehec (Poitiers).

- 10h - 12h30 : A. Giannopoulos : The isotropic constant in the theory of high-dimensional convex bodies.
- 14h - 16h : J. Lehec : solution to the slicing conjecture.