Small mass, non-minimizing critical regions in the liquid Drop Model

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Workshop on Functional Inequalities

Université Paris Dauphine-PSL, June 4-6, 2025

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We continue the proof of the main step in the construction in the previous talk. We are proving:

Proposition Let $a \in (0, \frac{1}{2})$ be fixed and such that $I_a > 0$. Then for all sufficiently large integer n, there exists a number

$$0 < \gamma_n \sim \frac{1}{\ln n}$$

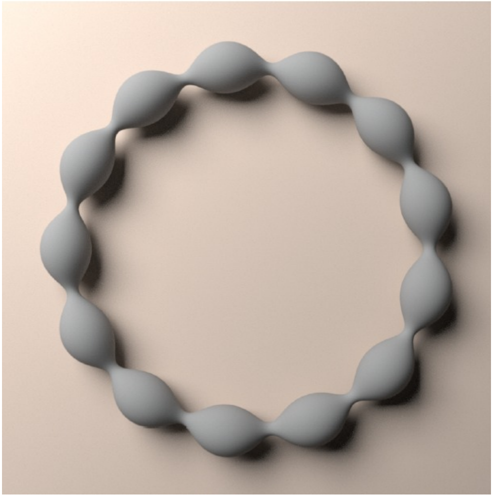
and a function h with $\|h\|_{C^{2,\alpha}} \lesssim (\log n)^{-1}$ such that

$$H_{\tilde{\Sigma}_h^n}(x) + \gamma \int_{\tilde{\Omega}_h^n} \frac{dy}{|x-y|} = \lambda, \quad \text{for all } x \in \tilde{\Sigma}_h^n \equiv \partial\tilde{\Omega}_h^n. \qquad (2)$$

$\tilde{\Sigma}_h^n$ is a "twisted n-truncated Delaunay". Let us assume

$$2\pi R = n\,T.$$

We translate Σ_h^n to the point $R\mathbf{e}_2$ and rotate it in the plane (x_2, x_3)



Recall

$$I_a = \int_0^{T/2} \frac{f(s)}{(1+f'(s)^2)^{\frac{5}{2}}} \left[ff''(2-f'(s)^2) + (1+3f'(s)^2)(1+f'(s)^2) \right] ds$$

where $r = f(x_3)$ parametrizes the Delaunay surface Σ with mean curvature $= 2$, necksize $0 < a < \frac{1}{2}$ is parametrized with

$$y(\theta, x_3) = (f(x_3)\cos\theta, f(x_3)\sin\theta, x_3)$$

where $f(y_3)$ is positive, even and T-periodic.
and the *twisted Delaunay* $\tilde{\Sigma}^n$ where $2\pi R = nT$ is given by

$$\tilde{\Sigma}^n = X(\Sigma^n), \quad X\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ (R+x_2)\cos\frac{x_3}{R} \\ (R+x_2)\sin\frac{x_3}{R} \end{pmatrix}$$

with parametrization

$$\tilde{y}(\theta, x_3) = X(y(\theta, x_3)), \quad (\theta, x_3) \in S = [0, 2\pi] \times [-\frac{nT}{2}, \frac{nT}{2}].$$

First we estimate the “error of approximation” corresponding to $h = 0$, namely

$$E_\gamma(x) = H_{\tilde{\Sigma}^n}(x) + \gamma \int_{\tilde{\Omega}^n} \frac{dy}{|y - x|}, \quad x \in \tilde{\Sigma}^n$$

We would like to estimate both mean curvature and Coulombian operator for of $\tilde{\Sigma}$ and $\tilde{\Sigma}_h^n$ regarded as perturbations of of the corresponding quantities for Σ and Σ_h.

For large n and $x \in \Omega_0$, We can approximate

$$\gamma \int_{\tilde{\Omega}^n} \frac{dy}{|y-x|} \approx \sum_{j=1}^{n} \gamma \int_{\Omega_0} \frac{dy}{|y+kTe_3-x|}$$
$$\approx |\Omega_0| \sum_{j=1}^{n} \frac{1}{kT} \approx \gamma \frac{V_a}{T} \log n.$$

We also estimate

$$H_{\tilde{\Sigma}^n}(x) \approx 2 - \frac{2\pi}{T} \tilde{f}(x_3) \sin\theta.$$

for a positive, even, T-periodic function $\tilde{f}(x_3)$.

After some computation, we find that at a point $x = X(y(\theta, x_3)) \in \tilde{\Sigma}^n$ we have

$$E_\gamma(x) \approx 2 - \gamma \frac{V_a}{T} \log n - \gamma \hat{f}(x_3) + \frac{2\pi}{n\,T} \sin\theta \left(\tilde{f}(x_3) - \gamma \log n \frac{V_a}{T} f(x_3) \right)$$

where the functions $\tilde{f}(x_3)$, $f(x_3)$ are strictly positive and T-periodic.

For $h \neq 0$, we perturb Σ^n with h in the normal direction as $\Sigma^n_h = \Sigma^n + h\nu$, so that $\tilde{\Sigma}^n_h = X(\Sigma^n_h)$.

For $h \neq 0$ the following expansions hold

$$\boxed{H_{\tilde{\Sigma}_h^n}(\tilde{y}) = H_{\Sigma_h^n}(y) + \frac{2\pi}{nT}\tilde{f}(x_3)\sin\theta + O(n^{-2}) + O(n^{-1}\|h\|_{C^2(\Sigma_n)})}$$

and

$$\boxed{H_{\Sigma_n^h} = H_{\Sigma_n} + J_{\Sigma^n}[h] + O(\|h\|^2_{C^2(\Sigma_n)}).)}$$

where $J_{\Sigma^n}[h]$ is the Jacobi operator of the surface Σ^n, defined on functions $h \in C^{2,\alpha}(\Sigma^n)$, which is expressed in local coordinates as

$$J_{\Sigma^n}[h] = \frac{1}{\sqrt{\det g}} D_j\left(g^{ij}\sqrt{\det g}\, D_i h\right) + |A|^2 h.$$

The quadratic remainder term can depend on first and second derivatives of h.

For the Coulombian nonlocal operator we get

$$\gamma \int_{\tilde{\Omega}^n_h} \frac{dy}{|x-y|} = \gamma \int_{\tilde{\Omega}^n} \frac{dy}{|x-y|} + \gamma \log n \int_{\Sigma_n} h + O(\frac{\gamma}{n}\|h\|_\infty)$$

The problem consists of finding a function h such that for some $\gamma > 0$ and some constant λ we have

$$\boxed{H_{\tilde{\Sigma}^n_h}(x) + \gamma \int_{\tilde{\Omega}^n_h} \frac{dy}{|x-y|} = \lambda \quad \text{for all} \quad x \in \tilde{\Sigma}^n_h} \qquad (2)$$

From the previous computations, Problem (2) is equivalent to finding a small function $h(x)$ and constants γ, λ such that

$$
\begin{aligned}
J_{\Sigma^n}[h] = E_\gamma(x) + \gamma \left(\frac{1}{T} \ln n + b(x) \right) \int_{\Sigma_0} h + \gamma \, \ell_1[h](x) \\
+ n^{-1} \, \ell[h, Dh, D^2 h](x) + q[h, Dh, D^2 h](x) + \lambda
\end{aligned}
\tag{3}
$$

Being more precise in the expansion of E_γ, up to an additive constant, we get

$$
E_\gamma(y) = \gamma \, F_0^n(y_3) + \frac{\gamma}{n} F_1^n(y) + \frac{y_2}{n}[F(y_3) - \gamma \ln n \, \frac{2\pi|\Omega_0|}{T^2}],
$$

$$
F(y_3) = \frac{2\pi}{Tf} \left(\frac{(2-(f')^2) \, f \, f''}{(1+(f')^2)^{\frac{5}{2}}} + \frac{1+3(f')^2}{(1+(f')^2)^{\frac{3}{2}}} \right),
$$

- The functions of y_3 involved are all even, positive and T-periodic.
- The operators ℓ_1 and ℓ have linear growth in their arguments and q is quadratic.

To keep at main order invariant the volume of the region, we work with perturbations h such that $\int_{\Sigma_n} h = 0$.

Each component of the normal vector ν, satisfies $J_{\Sigma^n}[\nu_i] = 0$ for $i = 1, 2, 3$ Actually for h T-periodic we have

$$J_{\Sigma^n}[h] = 0 \implies h = \sum_{i=1}^{3} \alpha_i \nu_i$$

This is the *non-degeneracy* property of the CMC surface Σ.

Let

$$\mathcal{C}^{m,\alpha} = \{g \in C^{m,\alpha}(\Sigma^n) \;/\; g(y) \text{ is } T\text{-periodic in } y_3 \text{ and even in } y_1 \}$$

Given $g \in \mathcal{C}^{0,\alpha}$ with $\int_{\Sigma^n} g\nu_i = 0$ for $i = 1, 2, 3$ there exists a unique $h \in \mathcal{C}^{2,\alpha}$ with $\int_{\Sigma^n} h\nu_i = 0$ for all i that solves the equation

$$J_{\Sigma^n}[h] = g \quad \text{ in } \Sigma_n$$

By variation of parameters formula one finds a explicit, positive solution $\varphi \in \mathcal{C}^{2,\alpha}$, $\varphi = \varphi(y_3)$ to the equation

$$J_{\Sigma^n}[\varphi] = 1 \quad \text{ in } \Sigma^n$$

Consider the problem of finding constants c and d and a solution h with $\int_{\Sigma_0} h = 0$ to the linear equation

$$J_{\Sigma^n}[h](y) = g(y) + c\,\nu_2(y) + d, \quad y \in \Sigma^n, \tag{4}$$

We can compute the numbers c, d as follows:

$$\int_{\Sigma_0} J_{\Sigma^n}[h]\nu_3 = \int_{\Sigma_0} J_{\Sigma^n}[\nu_3]h = 0 = \int_{\Sigma_0} g\nu_3 + c\int_{\Sigma_0} \nu_3^2 + \int_{\Sigma_0} \nu_3$$

$$\int_{\Sigma_0} J_{\Sigma^n}[h]\varphi = \int_{\Sigma_0} J_{\Sigma_0}[\varphi]h = \int_{\Sigma_0} h = 0 = \int_{\Sigma_0} g\varphi + c\int_{\Sigma_0} \nu_3\varphi + d\int_{\Sigma_0} \varphi$$

$$c = -\frac{\int_{\Sigma_0} g\nu_2}{\int_{\Sigma_0} \nu_3^2}, \quad d = -\frac{\int_{\Sigma_0} g\,\varphi}{\int_{\Sigma_0} \varphi}.$$

Here we have used $\int_{\Sigma_0} \nu_3 = \int_{\Sigma_0} \nu_3\varphi = 0$ because ν_3 is an odd function of y_3 and φ is even.

Lemma

Given $g \in \mathcal{C}^{0,\alpha}$ *there exists a unique solution* $h(y) \in \mathcal{C}^{2,\alpha}$ *to* (4) *with* $\int_{\Sigma^n} h\nu_3 = \int_{\Sigma^n} h \ = 0$ *and scalars* c, d *such that*
This solution defines a bounded linear operator $h = \mathcal{T}[g]$ *so do the scalars* c *and* d*. In fact, we have the estimate*

$$\|\mathcal{T}[g]\|_{C^{2,\alpha}(\Sigma^n)} \lesssim \|g\|_{C^{\alpha}(\Sigma^n)}$$

We want to solve Problem (3) with and h with $\int_{\Sigma_0} h = 0$, which becomes

$$\begin{aligned} J_{\Sigma^n}[h] &= E_\gamma + \gamma\,\ell_1[h] + n^{-1}\,\ell[h, Dh, D^2h] + q[h, Dh, D^2h] + d \\ &= \mathcal{E}(h,\gamma) + d. \end{aligned} \tag{3}$$

$$E_\gamma(y) = \gamma\, F_0^n(y_3) + \frac{\gamma}{n} F_1^n(y) + \frac{y_2}{n}[F(y_3) - \gamma \ln n\ \frac{2\pi|\Omega_0|}{T^2}],$$

We look for a $h \in \mathcal{C}^{2,\alpha}$ such that $\int_{\Sigma^n} h = \int_{\Sigma^n} h\nu_2 = 0$ that solves (3). We can write the problem in the form

$$\begin{cases} J_{\Sigma^n}[h] = \mathcal{E}(x,h,\gamma) + c\nu_2 + d \\ c(h,\gamma) = 0 \end{cases}$$

Thus we want to solve the system

$$\begin{cases} h = \mathcal{T}[\mathcal{E}(h,\gamma)], \\ 0 = \int_{\Sigma^n} \mathcal{E}(x,h,\gamma)\nu_2. \end{cases}$$

We solve the fixed point equation for $h = h(\gamma)$ using contraction mapping provided that $\gamma = O(\frac{1}{\log n})$. To do this we decompose $h = h_0 + h_1$ where $\|h_0\|_{C^{2,\alpha}(\Sigma_0)} = O(\gamma)$,

$$h_0 = \mathcal{T}\left[\gamma\, F_0^n(y_3) + \gamma\, \ell_1[h_0](y) + q[h_0, Dh_0, D^2 h_0](y)\right]$$

and $\|h_1\|_{C^{2,\alpha}(\Sigma_0)} \leq \frac{C}{n}$.

The following relation yields the choice of γ.

$$
\begin{aligned}
0 &= \int_{\Sigma^n} \mathcal{E}(h(\gamma),\gamma)\nu_2 \\
&= \int_{\Sigma_0} (\, E_\gamma + \gamma\, \ell_1[h(\gamma)] + n^{-1}\, \ell[h(\gamma),\ldots] + q[h(\gamma),\ldots])\nu_2 \\
&\approx \int_{\Sigma_0} (E_\gamma + \vartheta(h))\, \nu_2 \approx \frac{c_1}{n}\,(2\, I_a - c_2\gamma\, \ln n\, + O(\gamma))\,,
\end{aligned}
$$

$$
I_a = \int_{\Sigma_0} F(y_3) \sin\theta\nu_2
$$

$$
= \int_0^{T/2} \frac{f}{(1+f'(s)^2)^{\frac{5}{2}}} \Big[ff''(2-f'(s)^2) + (1+3f'(s)^2)(1+f'(s)^2) \Big]\, ds.
$$

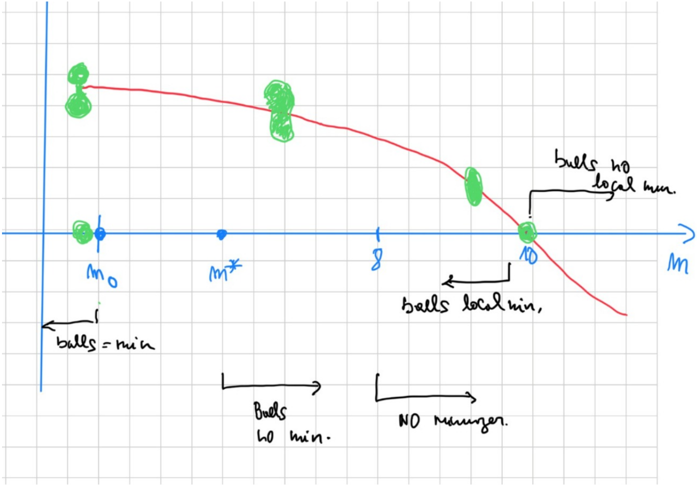
and hence at main order $\gamma \approx \gamma_n := \frac{c}{\log n}, \quad c = \frac{2}{c_2} I_a$. We indeed have that $I_a > 0$ for all sufficiently small $a > 0$ and for $\frac{1}{4} < a < \frac{1}{2}$.

We solve the full system for h and γ and get h with

$$\|h\|_{C^{2,\alpha}} \lesssim \frac{1}{\log n}$$

and the construction of a solution to Problem (2) is concluded.

The Bohr-Wheeler bifurcation branch stemming from $m = 10$, rigorously described by Frank (2019), gives rise, as pictured by Xu-Du (2023), to two balls joined by a tiny neck, whih is not a local minimizer.



Bohr-Wheeler bifurcation branch [Picture from Xu-Du 2023]

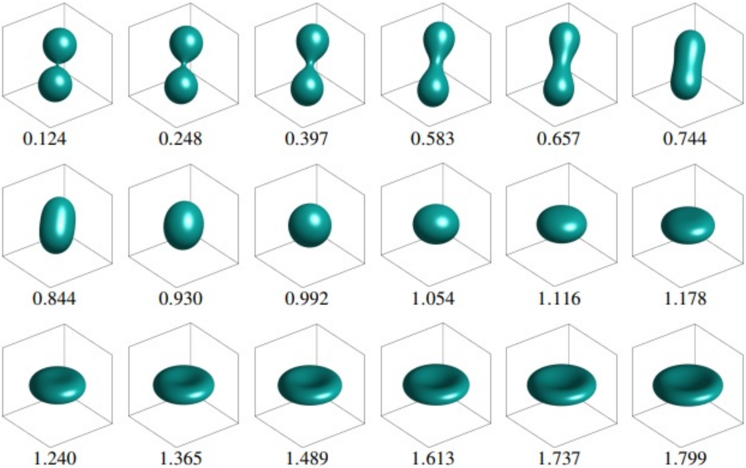
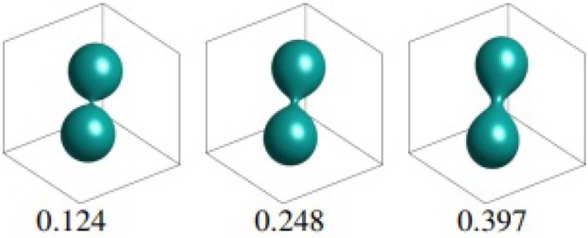


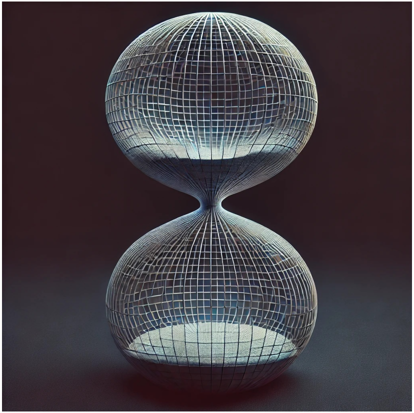
Figure 2: Computed Bohr–Wheeler bifurcation branch. Below each shape is $\widetilde{X}$.

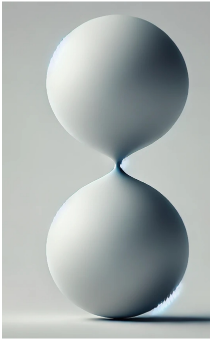


The pictures by Xu-Du suggest that the Bohr-Wheeler left branch stemming from $m = 10$ gives rise, as as $m \to 0$, to two balls joined by a tiny neck. This configuration is **not** a local minimizer for the energy.

Theorem (M.d., R. Frank, M.Musso)

For each small $m > 0$ there exists a solution of problem (P) which is approximately made out of two similar spheres with volume $\frac{m}{2}$ joined with a tiny catenoidal neck with neck size of an order ε approximately proportional to m.





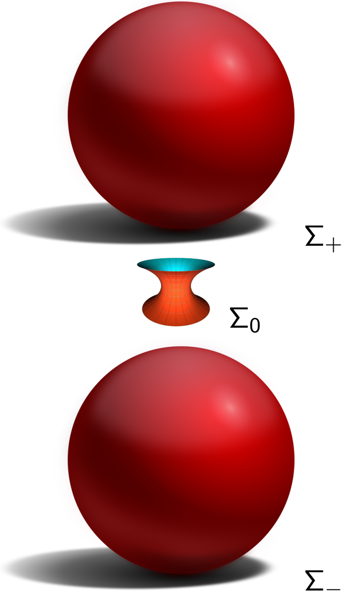


Figure: $\Sigma = \Sigma_+ \cup \Sigma_0 \cup \Sigma_-$

We want to find solutions Ω to the following problem:

$$\begin{cases} H_\Sigma(x) + \int_\Omega \frac{dy}{|x-y|} = \lambda \\ |\Omega| = \frac{8}{3}\pi m. \end{cases}$$

The normalization is made in such a way that the sought solution looks close to two tangent spheres with radius $m^{\frac{1}{3}}$ joined by a small catenoidal neck. scaling out this factor, letting $\Omega = m^{\frac{1}{3}}\tilde{\Omega}$ the problem becomes

$$\begin{cases} H_{\tilde{\Sigma}}(x) + m\int_{\tilde{\Omega}} \frac{dy}{|x-y|} = \lambda \\ |\tilde{\Omega}| = \frac{8}{3}\pi. \end{cases}$$

In this language $\tilde{\Omega}$ is approximated by the union of two tangent spheres with radius 1. Calling $\varepsilon > 0$ the neck size, let us normalize $\Omega_\varepsilon = \frac{1}{\varepsilon}\tilde{\Omega}_\varepsilon$. The problem becomes

$$\begin{cases} H_{\Sigma_\varepsilon}(x) + m\varepsilon^3 \int_{\Omega_\varepsilon} \frac{dy}{|x-y|} = \lambda \\ |\Omega_\varepsilon| = \frac{8}{3}\pi\varepsilon^{-3}. \end{cases}$$

and the domain resembles two spheres radius $\frac{1}{\varepsilon}$ joined by a large piece of a catenoid with neck size $= 1$.

In cylindrical coordinates Let $z = F_0(r)$ be the parametrization of the catenoid, as a surface of revolution. Then $F_0(r) = \log(r + \sqrt{r^2 - 1})$. Let us call $H(P)$ the mean curvature of the surface $z = P(r)$. Then $colorblue H(F_0)(r) = 0$. In general, we have

$$H(P) = \frac{P''}{(1 + |P'|^2)^{\frac{3}{2}}} + \frac{P'}{r\sqrt{1 + |P'|^2}}.$$

Let us consider a sphere with radius $R = \frac{1}{\varepsilon}$ and center $(0, 0, R + d)$. Close to the south pole, we can parametrize it with $z = G(r) = R + d - \sqrt{R^2 - r^2}$. We cansider a value R_0 such that $G'(R_0) \sim F_0'(R_0)$ Then $H(G) = 2\varepsilon$. We can find a small function $h_0(r)$ such that $F(r) = F_0(r) + h_0(r)$ satisfies approximately $H(F) = \varepsilon + O(\varepsilon^2)$ We have that for $r \gg 1$

$$F(r) = \log r + \frac{1}{2}\varepsilon r^2 + l.o.t,$$

We observe that

$$G(r) = d + \varepsilon \frac{1}{2} r^2 + \frac{1}{8}\varepsilon^2 r^4 + l.o.t$$

Matching of the derivatives takes place at $r \sim \varepsilon^{-\frac{3}{4}}$. That motivates us to choose $d = \log \varepsilon^{-\frac{3}{4}}$.

We consider a smooth cut-off function $\chi_0(s)$ with $\chi_0(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$. We let $\chi(r) = \chi_0(\varepsilon^{\frac{3}{4}}\delta^{-1}r)$

We interpolate the sphere and the catenoid in the form

$$z = P(r) = \chi(r)F(r) + (1 - \chi(r))G(r).$$

The computation of $H(P)$ leads to

$$H(P) = 2\varepsilon + \Delta\chi(F - G) + 2\nabla\chi\nabla(F - G) + l.o.t.$$

$$N(x) := m\varepsilon^3 \int_{B+} \frac{dy}{|x - y|} = N_+(x) + N_-(x) + N_0(x),$$

$$N_+(x) = m\varepsilon^3 \int_{B_R(0,0+R+d)} \frac{dy}{|x - y|}$$

$$N_0(x) = m\varepsilon^3 \int_{\Omega_0} \frac{dy}{|x - y|}$$

$$N_-(x) = N_+(\bar{x}), \quad \bar{x} = (x_1, x_2, -x_3).$$

Since $N_+(x)$ is harmonic, and invariant under rotations on $\partial B_R(0, 0 + R + d)$ we see that necessarily

$$N_+(x) = \frac{m\beta}{|x - Q|}, \quad Q = (0, 0, R + d), \quad |x - Q| > R.$$

$$N_-(x) = \frac{m\beta}{|x - \bar{Q}|}.$$

$$\Omega_0 = \{1 < r < R_0 = 2\delta\varepsilon^{-\frac{3}{4}}, F(r) < z < G(r)\},$$

$$N_0(x) \lesssim \begin{cases} m\varepsilon^3 |x| \, |\log(\frac{|x|}{R_0})| & \text{if } |x| < 5R_0 \\ \frac{m\delta^2}{|x|}\varepsilon^{\frac{3}{2}} & \text{if } |x| \geq 5R_0. \end{cases}$$

Hence, letting $\Sigma =$ the boundary of the union of the three regions. We look for a solution of the form

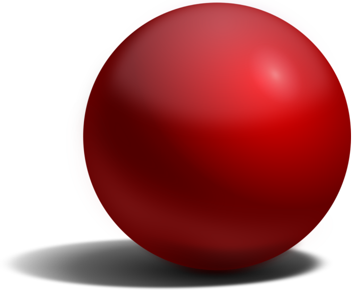
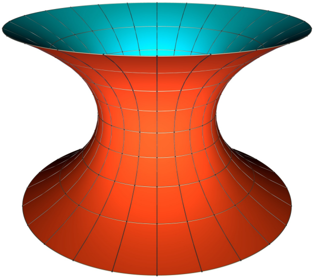
$$\Sigma_h = \{y + h(y)\nu_\Sigma(y) \ / \ y \in \Sigma_0\},$$

$$h(x) = \eta_{\Sigma_0} h_0 + \eta_{\Sigma^+} h_+(x) + \eta_{\Sigma^-} h_-(x)$$

where

$$\eta^+ = 1 \text{ on } \Sigma^+ \setminus \{\text{ small neighborhood of south pole}\},$$
$$\eta_0 = 1 \text{ on } \Sigma_0,$$

support of η_{Σ^+} compact in $\Sigma_+ \setminus \{\textit{south pole}\}$ and support of η_0 is slightly larger than Σ_0. h even with respect to the plane $z = 0$.

$$\mathcal{J}_{\Sigma^+}[\phi_{\Sigma^+}] = E_+ + A(\phi_{\Sigma^+}, \phi_{\Sigma^0}), \quad \mathcal{J}_{\Sigma_0}[\phi_{\Sigma^0}] = E_0 + B(\phi_{\Sigma^+}, \phi_{\Sigma^0})$$

.

The solvability condition for ϕ_{Σ^+} is at main order $\int_{\Sigma^+} E^+ \nu_2 = 0$. This yields a relation between the necksize of the catenoid and the mass.

We have

$$E = \frac{m\beta}{|x-Q|} + \frac{m\beta}{|x-\bar{Q}|} + \Delta\chi(F-G) + 2\nabla\chi\nabla(F-G) + \textit{smaller terms} + \varepsilon$$

Important: choice of ε in terms of m: At main order we need to solve an equation on the upper sphere σ^+ of the type

$$J_{\Sigma_+}[h_{\Sigma^+}] = E$$

We need, taking into account that $\nu_3 \approx 1$ near the south pole,

$$0 = \int_{\Sigma_+} E\nu_3 = \int_{\Sigma_+} [\frac{m\beta}{|x-Q|} + \frac{m\beta}{|x-\bar{Q}|}]\nu_3$$
$$+ \int_{\Sigma_+} (\Delta\chi(F-G) + 2\nabla\chi\nabla(F-G))\nu_3$$

Hence we need $m\varepsilon^{-1} \sim 1$ or $m \sim \varepsilon$. We solve the gluing system by fixed point. We impose $h_{\Sigma^+}(\textit{south pole}) = 0$

Thanks for your attention