The first eigenvalue of systems

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Motivation

Minimizing or maximizing the first eigenvalue of the Laplacian with various boundary conditions is now well understood (see next slide).

But the case of systems is much less understood! Indeed, classical tools like symmetrization, maximum principle ... may not be available.

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But the case of systems is much less understood! Indeed, classical tools like symmetrization, maximum principle ... may not be available.
In this talk, we will report on partial results for

- the Stokes system
- the Elasticity (or Lamé) system
- and also a few words on the Maxwell system

The scalar case (1)

Here we consider the first eigenvalue of the Laplacian: $-\Delta u = \lambda u$

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Neumann ($\partial u / \partial n = 0$ on the boundary). The ball maximizes the first (non-trivial) eigenvalue $\mu_1(\Omega)$ with a volume constraint (**Szegö-Weinberger**), but not with a perimeter constraint ($\sup P^2 \mu_1 = +\infty$) and, among planar convex domains, it is conjectured that $\max P^2 \mu_1 = 16\pi^2$ attained by the square and the equilateral triangle, (see AH-A. Lemenant-I. Lucardesi).

The scalar case (2)

Robin $(-\Delta u = \lambda u$ and $\partial u/\partial n + \alpha u = 0$ on the boundary). The ball minimizes the first eigenvalue with a volume constraint, and with a perimeter constraint in the case $\alpha > 0$ (**Bossel-Daners**). The general result is unknown for $\alpha < 0$ (here we maximize λ_1), see **Freitas-Krejcirik** but the ball is NOT the maximizer for $|\alpha|$ large.

The scalar case (2)

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Steklov ($\Delta u = 0$ and $\partial u/\partial n = \sigma u$ on the boundary). The ball maximizes the first (non-trivial) eigenvalue $\sigma_1(\Omega)$ with a volume constraint (**Brock**), but not with a perimeter constraint (there is no maximizer, but the supremum of $P^2\sigma_1$ is 8π in the plane, see **Girouard-Karpukhin-Lagacé**).

The Stokes system

Let Ω be a bounded open set in $\mathbb{R}^N$. The first eigenvalue for Stokes (with Dirichlet boundary conditions) is defined by

$$\lambda_1(\Omega) := \min_{\substack{u \in W_0^{1,2}(\Omega;\mathbb{R}^N) \\ \nabla\cdot u = 0 \text{ in } \Omega,\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}, \tag{1}$$

where $\nabla\cdot$ stands for the divergence operator and ∇u stands for the Jacobian matrix of $u : \Omega \to \mathbb{R}^N$.

This eigenvalue is associated with the equation

$$\begin{cases} -\Delta u + \nabla p = \lambda_1(\Omega) u & \text{in } \Omega\,, \\ \nabla \cdot u = 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

The minimization problem

Here we are interested in the problem

$$\boxed{\inf_{\Omega\subset\mathbb{R}^N,|\Omega|\leq V_0} \lambda_1(\Omega)}$$

or equivalently

$$\boxed{\inf_{\Omega\subset\mathbb{R}^N} |\Omega|^{2/N}\lambda_1(\Omega).} \tag{3}$$

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The questions are:

- Existence of a minimizer?
- Is the ball the solution? (Faber-Krahn type inequality)
- If not, qualitative properties of the minimizer?

A general existence result

Theorem

The minimization problem (3) *has a solution* Ω^* *in the class of quasi-open sets.*

The method of proof relies on the concentration-compactness principle which was adapted by D. Bucur to the setting of shape optimisation. The main new difficulty here is the incompressibility condition.

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Remarks:

- according to the proof, we cannot claim that Ω^* is bounded.
- in $2-D$, in the class of simply-connected domains, we can introduce the stream function linking the Stokes problem to the buckling problem –> Ashbaugh-Bucur existence result.

The two-dimensional case

Theorem

The disk $\mathbb{B}_2$ *is a local minimizer (for smooth perturbations) for* $|\Omega|\lambda_1(\Omega)$.

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Theorem

The disk $\mathbb{B}_2$ is a local minimizer (for smooth perturbations) for $|\Omega|\lambda_1(\Omega)$.

More precisely, we can prove: The first eigenvalue $\lambda_1(\mathbb{B}_2)$ is simple. Consequently, $\mathcal{F}(\Omega) = |\Omega|\lambda_1(\Omega)$ is twice differentiable at $\mathbb{B}_2$ and:

1. For any $\Phi \in W^{3,\infty}(\mathbb{R}^2, \mathbb{R}^2)$,
$$\langle d\mathcal{F}(\mathbb{B}_2), \Phi \rangle = 0.$$
2. There exists a constant $c_0 > 0$ such that, for any $\Phi \in W^{3,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ such that $\langle \Phi, \nu \rangle \in \{1, \cos(\cdot), \sin(\cdot)\}^{\perp}$, where $\perp$ denotes the orthogonal for the $L^2(\partial\mathbb{B})$ inner product,
$$\langle d^2\mathcal{F}(\mathbb{B}_2)\Phi, \Phi \rangle \geq c_0 \|\langle \Phi, \nu \rangle\|^2_{W^{\frac{1}{2},2}(\partial\mathbb{B}_2)}$$
where ν is the normal vector on $\partial\mathbb{B}_2$.

The two-dimensional case: conjecture

Conjecture: Prove that the disk $\mathbb{B}_2$ is the (global) minimizer for $|\Omega|\lambda_1(\Omega)$.
In other words: prove that a Faber-Krahn type inequality holds true for the Stokes operator in two-dimensions.

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What support this conjecture?

- The local minimality of the disk,
- If the minimizer Ω^* is simply connected and smooth, the result will follow from Willms-Weinberger proof for the buckling problem.

The three-dimensional case

By contrast with the $2-D$ case:

Theorem

The ball $\mathbb{B}_3$ *is not the minimizer of* $\mathcal{F}(\Omega) := |\Omega|^{2/3}\lambda_1(\Omega)$.

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More precisely, we prove:
The first eigenvalue $\lambda_1(\mathbb{B}_3)$ has multiplicity 3. $\mathcal{F}$ is semi-differentiable at $\mathbb{B}_3$ but does not satisfy the first-order optimality conditions: there exists a vector field $\Phi \in W^{3,\infty}(\mathbb{R}^3;\mathbb{R}^3)$ such that

$$\langle \partial\mathcal{F}(\mathbb{B}_3)\Phi, \Phi\rangle < 0.$$

Qualitative properties of the minimizer

Assuming regularity, we can prove:

Theorem

Let Ω^* be a minimizer for $\mathcal{F}(\Omega) := |\Omega|^{2/3}\lambda_1(\Omega)$. If Ω^* has a $\mathcal{C}^{2,\alpha}$ boundary with $\alpha \in (0,1)$ then $\lambda_1(\Omega^*)$ is a simple eigenvalue. Furthermore, if u is an associated first eigenfunction, then $|(\nabla \mathrm{u})\nu|$ is constant on $\partial\Omega^*$.

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A topological consequence:

Corollary

Assume Ω^* has a $\mathcal{C}^{2,\alpha}$ boundary with $\alpha \in (0,1)$. If Σ is a connected component of $\partial\Omega^*$, Σ has Euler characteristic 0, and is thus homeomorphic to a torus.

It comes from the fact that the vector field $(\nabla \mathrm{u})\nu$ is tangential to $\partial\Omega$. Thus, the optimality condition yields the applicability of the hairy ball theorem!

A related problem: curl eigenvalue

Several recent works on the minimization of the first positive eigenvalue for the curl operator:

$$\begin{cases} \mathrm{curl}\,\mathrm{u} = \sigma \mathrm{u} & \text{in } \Omega \\ \mathrm{u}.\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

1. Enciso & Peralta-Salas : a minimiser can not have axial symmetry if it is $C^{2,\alpha}$,
2. Enciso, Gerner & Peralta-Salas : if a convex domain is optimal, it can not be analytic.
3. Gerner: if a minimizer is smooth, then the first eigenvalue is simple.

Elasticity: the Lamé system

The first eigenvalue of Ω for the Lamé system is defined by

$$\Lambda(\Omega) := \min_{u \in H_0^1(\Omega)^N \setminus \{0\}} \frac{\mu \int_\Omega |\nabla u|^2 \, dx + (\lambda + \mu) \int_\Omega (\mathrm{div}(u))^2 \, dx}{\int_\Omega |u|^2 \, dx},$$

where λ, μ are the Lamé coefficients that satisfy $\mu > 0, \lambda + \mu > 0$.

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where λ, μ are the Lamé coefficients that satisfy $\mu > 0, \lambda + \mu > 0$.
The associated vectorial PDE solved by the minimizer u is

$$\begin{cases} -\mu \Delta u - (\lambda + \mu) \nabla (\mathrm{div}(u)) = \Lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Expression with the Poisson coefficient

It is convenient to Introduce the Poisson coefficient ν related to the Lamé parameters by:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{5}$$

where E is the Young modulus and $\nu \in (-1, 0.5)$ (for many materials $\nu \in [0.2, 0.4]$).

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where E is the Young modulus and $\nu \in (-1, 0.5)$ (for many materials $\nu \in [0.2, 0.4]$). Indeed, dividing the eigenvalue Λ by μ lead us to introduce the ratio

$$\frac{\lambda+\mu}{\mu} = \frac{1}{1-2\nu}$$

and therefore, we can see that the minimization of Λ will primarily depend on the Poisson coefficient ν:

$$\frac{\Lambda(\Omega)}{\mu} := \min_{u \in H_0^1(\Omega)^N \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx + \frac{1}{1-2\nu} \int_\Omega (\operatorname{div}(u))^2 \, dx}{\int_\Omega |u|^2 \, dx}.$$

The minimization problem

We are interested in the following minimization problem:

$$\boxed{\inf\{\Lambda(\Omega), \Omega \subset \mathbb{R}^N, |\Omega| \leq V_0\}}$$

or equivalently (since $\Lambda(t\Omega) = \Lambda(\Omega)/t^2$) to the unconstrained optimization problem

$$\boxed{\inf\{|\Omega|^{2/N}\Lambda(\Omega), \Omega \subset \mathbb{R}^N\}}. \tag{6}$$

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The questions we consider are:

- Existence of a minimizer,
- In two dimensions, is the disk the minimizer?

The existence result

Our first result is:

Theorem

There exists a minimizer Ω^* *in the class of quasi-open sets Moreover in dimension* $N = 2$ *and* $N = 3$ *this set is open and any eigenfunction associated with the eigenvalue* $\Lambda(\Omega^*)$ *belongs to* $\mathcal{C}^{0,\alpha}(\mathbb{R}^N)$ *for all* $\alpha < 1$ *if* $N = 2$*, and for all* $\alpha < \frac{1}{2}$ *if* $N = 3$*.*

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For the existence, we use the standard strategy based on a concentration-compactness argument (like for Stokes). Then, to prove that the optimal domain is an open set in dimensions 2 and 3, we prove the equivalence of the minimization problem with a penalized problem and we introduce the concept of Lamé quasi-minimizers.

Is the disk the minimizer in $2-D$?

Actually, the answer depends on the Poisson coefficient: (we recall that $-1 < \nu < 0.5$)

Theorem

- *If the Poisson coefficient ν is less than 0.4, the disk* **is not** *the minimizer of Λ (among sets of given area).*

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- *if the Poisson coefficient ν is greater than ν^* defined by $\nu^* := \frac{j_{1,1}^2 - 2{j'_{1,1}}^2}{2j_{1,1}^2 - 2{j'_{1,1}}^2} \simeq 0.349895$ where $j_{1,1}$ is the first zero of the Bessel function J_1 and $j'_{1,1}$ is the first zero of its derivative J'_1, then the disk is a local minimizer.*

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Conjecture: Prove that there exists $\widehat{\nu} \in]0.4, 0.5[$ such that the the disk $\mathbb{B}_2$ is the (global) minimizer for $\nu \geq \widehat{\nu}$.

The first eigenvalue of the disk

It is not obvious to compute the first eigenvalue of the disk (transcendental equation involving Bessel functions and the Lamé coefficients), but we can prove that

- If $\nu \leq \nu^*$, the first eigenvalue $\Lambda(\mathbb{B}_2)$ is (at least) double and then, it is easy to prove that the disk is not a minimizer in that case (first order argument: we can find a perturbation that decreases the first eigenvalue).

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- If $\nu \leq \nu^*$, the first eigenvalue $\Lambda(\mathbb{B}_2)$ is (at least) double and then, it is easy to prove that the disk is not a minimizer in that case (first order argument: we can find a perturbation that decreases the first eigenvalue).
- if $\nu > \nu^*$, then $\Lambda(\mathbb{B}_2) = \mu j_{1,1}^2$ and it is simple.

When $0.349 \leq \nu \leq 0.4$

To prove that the disk is not the minimizer when $0.349 \leq \nu \leq 0.4$ we proceed in two steps. First, we exhibit some parallelograms for which we can compute the first eigenvalue:

Theorem

Let Ω be a parallelogram defined by the four lines

$$\left\{\begin{array}{l} e_1 \cdot X = \xi_1 \\ e_1 \cdot X = \hat{\xi}_1 \end{array}\right. \qquad \left\{\begin{array}{l} e_2 \cdot X = \xi_2 \\ e_2 \cdot X = \hat{\xi}_2 \end{array}\right. \qquad \textit{where } e_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad e_2 = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

and

$$\alpha = \frac{1}{\sqrt{\lambda + 2\mu}} - \frac{1}{\sqrt{\mu}}, \qquad \beta = \frac{1}{\sqrt{\lambda + 2\mu}} + \frac{1}{\sqrt{\mu}}.$$

Assume that $\hat{\xi}_1 - \xi_1 = \hat{\xi}_2 - \xi_2$. Then an eigenvalue of the parallelogram is given by

$$\Lambda(\Omega) = \frac{2\pi^2 \sqrt{\mu(\lambda + 2\mu)}}{|\Omega|} \tag{7}$$

When these parallelograms are better than the disk?

We deduce:

Corollary

Assume that the Poisson coefficient ν satisfies

$$\nu < \frac{j_{1,1}^4 - 8\pi^2}{2(j_{1,1}^4 - 4\pi^2)} \simeq 0.3879$$

then the disk is not a minimizer of Λ (among sets of given volume).

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Indeed, it suffices to compare our previous parallelogram of area π with the first eigenvalue of the disk that is $\mu j_{1,1}^2$.

How can we reach $\nu = 0.4$?

We consider a rectangle $\Omega_L = (0, L) \times (0, \ell)$ of area π. We are not able to compute the first eigenvalue of Ω_L but we can estimate from above it with enough precision, by introducing φ_1 the first (normalized) eigenfunction for the Dirichlet-Laplacian of Ω_L, defined by

$$\varphi_1(x, y) = \frac{2}{\sqrt{\pi}} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{\ell}\right),$$

and another eigenfunction

$$\varphi_2(x, y) = \frac{2}{\sqrt{\pi}} \sin\left(\frac{2\pi x}{L}\right) \sin\left(\frac{2\pi y}{\ell}\right).$$

This other eigenfunction could be the fourth one (for a rectangle not too far from the square), but can also have a larger index.

Case of the rectangle Ω_L

Now the idea is to plug in the variational formulation defining $\Lambda(\Omega_L)$ a family of vectors, for $X = (\alpha_1, \alpha_2, \beta_1, \beta_2)$:

$$U_X = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\varphi_1 + \alpha_2\varphi_2 \\ \beta_1\varphi_1 + \beta_2\varphi_2 \end{pmatrix}.$$

The ratio of two quadratic forms leads us to compute the eigenvalues of the 4×4 matrix:

$$\mathcal{M} = \begin{pmatrix} a_1 & 0 & 0 & b \\ 0 & a_2 & b & 0 \\ 0 & b & a_3 & 0 \\ b & 0 & 0 & a_4 \end{pmatrix}$$

with explicit numbers a_1, a_2, a_3, a_4, b.

Case of the rectangle Ω_L: conclusion

Now an elementary analysis allows us to prove:

Theorem

Let Ω_L be the rectangle of length $L = \sqrt{5\pi/2}$ and width $\ell = \sqrt{2\pi/5}$. Then its first eigenvalue satisfies

$$\Lambda(\Omega_L) < \mu j_{1,1}^2$$

for all values of the Poisson coefficient $\nu \in [\frac{3}{8}, \frac{2}{5}]$. Therefore the disk is not a minimizer in this range of values of ν.

Back to the conjecture

We recall that we think that the disk $\mathbb{B}_2$ could be a minimizer when the Poisson coefficient is not far from 0.5. This is supported by three properties:

- the fact that the Lamé eigenvalue Γ-converges to the Stokes eigenvalue when $\nu \to 1/2$,
- the conjecture already stated for the Stokes eigenvalue,
- the fact that the disk is a local minimizer when $\nu \geq 0.349$.

What about Maxwell ?

The first eigenvalue of the Maxwell system for $\Omega \subset \mathbb{R}^3$ is defined by

$$\lambda_1(\Omega) := \min_{\substack{\mathrm{u} \in X_N(\Omega;\mathbb{R}^3), \\ \mathrm{u} \neq 0}} \frac{\int_\Omega |\mathrm{curl}(\mathrm{u})|^2}{\int_\Omega |\mathrm{u}|^2}, \tag{8}$$

where $X_N(\Omega;\mathbb{R}^3)$, stands for the set of functions in $L^2(\Omega)^3$ with $\mathrm{curl}(\mathrm{u}) \in L^2(\Omega)^3$, $\mathrm{div}\mathrm{u} = 0$ and $\mathrm{u} \wedge \nu = 0$ on the boundary.

This eigenvalue is associated with the Maxwell system (for the electric field)

$$\begin{cases} \mathrm{curl}(\mathrm{curl}(\mathrm{u})) = \lambda_1(\Omega)\mathrm{u} & \text{in } \Omega\,, \\ \mathrm{div}\mathrm{u} = 0 & \text{in } \Omega\,, \\ \mathrm{u} \wedge \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

The first eigenvalue for a cartesian product

Let $\Omega = \omega \times (0, \ell)$ be a cartesian product. with ω simply connected. M. Costabel et M. Dauge have proved that

$$\lambda_1(\Omega) = \min\{\lambda_1^D(\omega), \mu_1(\omega) + \frac{\pi^2}{\ell^2}\}$$

where $\lambda_1^D(\omega)$ is the first Dirichlet eigenvalue of the Laplacian and $\mu_1(\omega)$ the first non-trivial Neumann eigenvalue.

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where $\lambda_1^D(\omega)$ is the first Dirichlet eigenvalue of the Laplacian and $\mu_1(\omega)$ the first non-trivial Neumann eigenvalue.
In particular for a cuboid $\Omega = (0, \ell_1) \times (0, \ell_2) \times (0, \ell_3)$ with $\ell_1 \geq \ell_2 \geq \ell_3$, we have

$$\lambda_1(\Omega) = \frac{\pi^2}{\ell_1^2} + \frac{\pi^2}{\ell_2^2}.$$

What could be the interesting problem?

The formulae $\lambda_1(\Omega) = \frac{\pi^2}{\ell_1^2} + \frac{\pi^2}{\ell_2^2}$ for cuboids immediately implies (as observed by D. Krejcirik, P.D. Lamberti, M. Zaccaron) that

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It turns out that the interesting problem is, adding a convexity assumption:

$$\inf_{\Omega \text{ convex}, \, ,P(\Omega)=P_0} \lambda_1(\Omega)$$

A new conjecture

Conjecture The "solution" of

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could be the flat disk.

We can prove

Theorem

The conjecture is true in the class of cartesian products.

The proof

Using Faber-Krahn + Payne-Weinberger leads to

$$\lambda_1(\omega \times (0,\ell)) \geq \min\{\frac{\pi^2}{diam(\omega)^2} + \frac{\pi^2}{\ell^2}, \frac{\pi j_{0,1}^2}{|\omega|}\}$$

with the perimeter constraint: $2|\omega| + P(\omega)\ell = P_0$.

The proof

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Now, since $P(\omega) \geq 2diam(\omega) := 2D$, this implies $\ell D \leq \frac{P_0}{2} - |\omega|$ and then

$$\frac{1}{D^2} + \frac{1}{\ell^2} \geq \frac{2}{D\ell} \geq \frac{2}{\frac{P_0}{2} - |\omega|}$$

and the result follows from the study of

$$A \mapsto \min\{\frac{2\pi^2}{\frac{P_0}{2} - A}, \frac{\pi j_{0,1}^2}{A}\}.$$

Thank you for your attention

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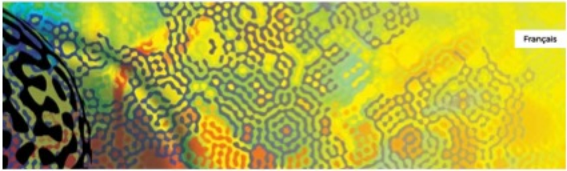

Interactions between convex geometry and spectral analysis

July 28 - August 1, 2025