Sharp stability for critical points of the Sobolev inequality in the absence of bubbling

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(Joint work with Yi Zhang)

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Sobolev inequality

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The Sobolev inequality states

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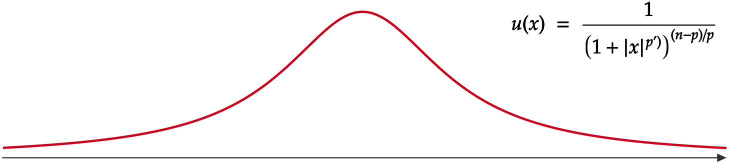

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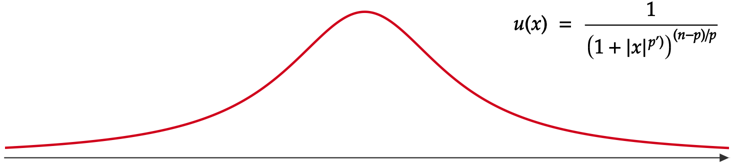

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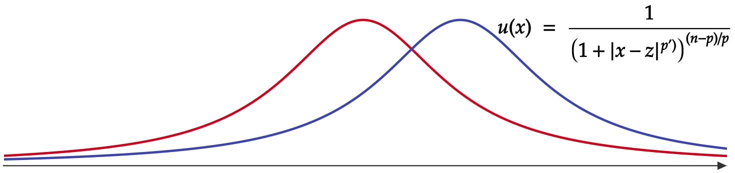

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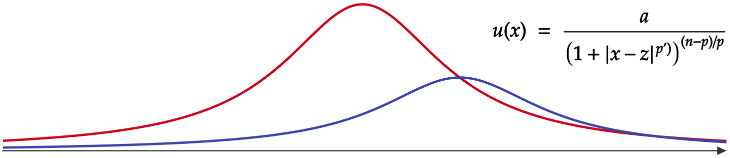

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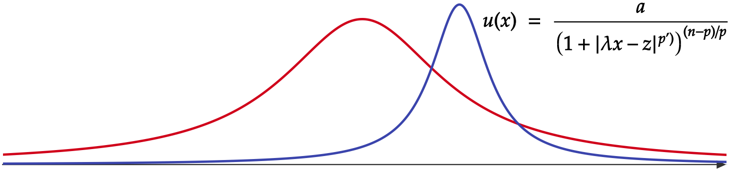

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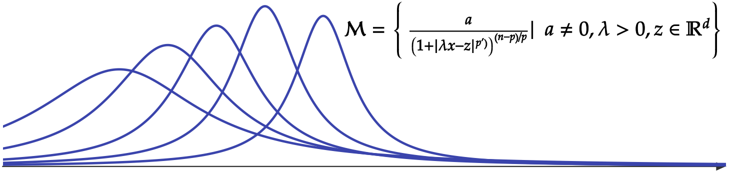

$$\mathcal{M} = \left\{ \frac{a}{\left(1+|\lambda x - z|^{p'}\right)^{(n-p)/p}} \,\middle|\, a \neq 0, \lambda > 0, z \in \mathbb{R}^d \right\}$$

- All of the functions achieving $S_{n,p}$ are in an $(n+2)$-dimensional manifold $\mathcal{M}$.

Overview of the talk

1. Stability in terms of the Sobolev energy (Minimizers)
   If $\delta(u) = \frac{\|\nabla u\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^{p^*}(\mathbb{R}^n)}} - S_{n,p} \approx 0$, then $dist(u, \mathcal{M}) \approx 0$?

2. Stability in terms of the Euler-Lagrange equations (Critical points)
   If $\Delta_p u + |u|^{p^*-2} u \approx 0$, then is u close to some $v \in \mathcal{M}$?

3. Recent result of L.–Zhang.
   Sharp stability of p-Sobolev inequality in the absence of bubbling

Stability problem of Sobolev inequality

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Stability problem in terms of the Sobolev energy ($p = 2$)

Stability problem (Brezis–Lieb, 1985, p=2)

If $\delta(u) = \frac{\|\nabla u\|_{L^2(\mathbb{R}^n)}}{\|u\|_{L^{2^*}(\mathbb{R}^n)}} - S_{n,2} \ll 1$, is u close to some $v \in \mathcal{M}$?

H. Brezis, E. Lieb, *Sobolev inequalities with remainder terms.* J. Funct. Anal. **62** (1985), 73–86.

First stability result in terms of the Sobolev energy ($p = 2$)

Theorem (Bianchi–Egnell, 1991)

It holds

$$\|\nabla u\|^2_{L^2(\mathbb{R}^n)} - S^2_{n,2}\|u\|^2_{L^{2^*}(\mathbb{R}^n)} \geq c_{BE} \inf_{v\in\mathcal{M}} \|\nabla(u-v)\|^2_{L^2(\mathbb{R}^n)}.$$

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Remark The gradient type distance and the power 2 are sharp.

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Remark This method gives no information about the size of c_{BE}.

G. Bianchi, H. Egnell, *A note on the Sobolev inequality.* J. Funct. Anal., **100** (1991), 18–24.

The optimal stability constant estimate

Theorem (Dolbeault–Esteban-Figalli–Frank–Loss, 2022)

For $n \geq 3$, there is an explicit constant $\beta > 0$, such that

$$c_{BE} \geq \frac{\beta}{n}.$$

J. Dolbeault, M. Esteban, A. Figalli, R. Frank, M. Loss, *Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence*. arXiv preprint arXiv:2209.08651 (2022).

Main idea of Bianchi–Egnell's proof

- R.H.S.:

$$\inf_{v\in\mathcal{M}} \|\nabla(u-v)\|^2_{L^2(\mathbb{R}^n)}$$

1. Compactness + Hilbert space $\Rightarrow u = v + \epsilon\varphi$, $\|\nabla\varphi\|_{L^2(\mathbb{R}^n)} = 1$ and **the orthogonality condition**

$$\langle v, \varphi \rangle_{W^{1,2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \nabla v \cdot \nabla \varphi dx = \int_{\mathbb{R}^n} v^{p^*-1}\varphi dx = 0.$$

2. Moreover,

$$\inf_{v\in\mathcal{M}} \|\nabla(u-v)\|^2_{L^2(\mathbb{R}^n)} = \epsilon^2.$$

Main idea of Bianchi–Egnell's proof

- L.H.S.:

$$\|\nabla u\|^2_{L^2(\mathbb{R}^n)} - S^2_{n,2}\|u\|^2_{L^{2^*}(\mathbb{R}^n)}.$$

The Taylor expansion $\Rightarrow$

$$\int_{\mathbb{R}^n} u^{2^*}\,dx = \int_{\mathbb{R}^n} v^{2^*}\,dx + \int_{\mathbb{R}^n} v^{p^*-1}\varphi\,dx + \epsilon^2\int_{\mathbb{R}^n} v^{p^*-2}\varphi^2\,dx + o(\epsilon^2)$$

$$\|\nabla u\|^2_{L^2(\mathbb{R}^n)} = \|\nabla v\|^2_{L^2(\mathbb{R}^n)} + \epsilon\int_{\mathbb{R}^n}\nabla v\cdot\nabla\varphi\,dx + \epsilon^2$$

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- **Spectrum analysis** of $(v^{2^*-2}\Delta)^{-1}$:

$$\int_{\mathbb{R}^n} v^{2^*-2}\varphi^2 dx \le \Lambda \|\nabla \varphi\|^2_{L^2(\mathbb{R}^n)} dx,\ \Lambda < 1.$$

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- Then,

$$\|\nabla u\|^2_{L^2(\mathbb{R}^n)} - S^2_{n,2}\|u\|^2_{L^{2^*}(\mathbb{R}^n)} \geq (1-\Lambda)\epsilon^2 + o(\epsilon^2).$$

Main idea of Bianchi–Egnell's proof

1. The orthogonality.
2. Taylor expansion.
3. Spectrum gap inequality.
4. Compactness.

Challenges of generalizing Bianchi-Egnell's result to $p \neq 2$

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  3. Does the corresponding spectrum gap inequality hold?
  4. Compactness condition and so on.

A. Figalli, R. Neumayer, *Gradient stability for the Sobolev inequality: the case $p \geq 2$*. Journal of the European Mathematical Society (EMS Publishing) **21** (2019), 319–354.

Stability in terms of the Sobolev energy ($p \neq 2$)

Theorem (Figalli–Zhang, 2022)

It holds

$$\delta(u) \geq C_{FZ} \inf_{v \in \mathcal{M}} \left(\frac{\|\nabla(u-v)\|_{L^p(\mathbb{R}^n)}}{\|\nabla u\|_{L^p(\mathbb{R}^n)}} \right)^{\max(2,p)} .$$

Remark: Here $\max\{2, p\}$ is sharp.

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- New vectorial inequalities.
- New compactness condition.
- New spectrum gap inequalities.

A. Figalli, Y. Zhang, *Sharp gradient stability for the Sobolev inequality.* Duke Math. J, **171** (2022), 2407–2459.

Stability problem of Sobolev inequality

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Stability in terms of the Euler-Lagrange equation

The Euler–Lagrange equation

$$-\Delta_p v := -\mathrm{div}(|\nabla v|^{p-2}\nabla v) = |v|^{p^*-2}v \quad \text{in } \mathbb{R}^n.$$

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Fact (Aubin, Talenti, 1976)

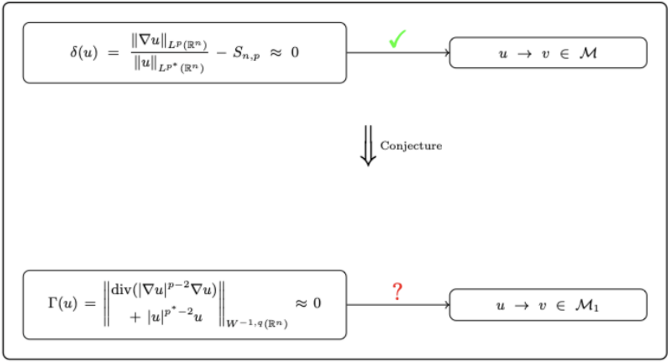
All positive critical points (Aubin-Talenti bubbles) are minimizers. ($\mathcal{M}_1$)

$$\mathcal{M}_1 = \left\{ v_i \mid v_i = v_{a,\lambda_i,z_i}(x),\ \lambda_i > 0,\ z_i \in \mathbb{R}^n, \|\nabla v_i\|^p_{L^p(\mathbb{R}^n)} = S^{np}_{n,p} \right\}.$$

T. Aubin, *Problemes isopérimétriques et espaces de Sobolev*. Journal of differential geometry **11** (1976), 573–598.

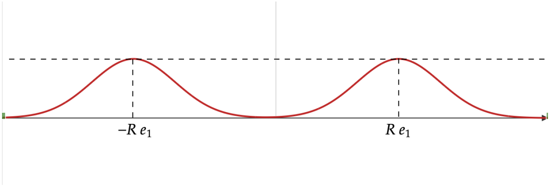
G. Talenti, *Best constant in Sobolev inequality*. Annali di Matematica pura ed Applicata **110** (1976), 353–372.

Stability Problem in terms of the Euler–Lagrange equation



Stability in terms of the Euler–Lagrange equation ($p = 2$)

No! Bubbling!



Consider $u = v_1 + v_2$ (v_1 and v_2 are supported far away from each other). Then

$$-\Delta u = -\Delta v_1 - \Delta v_2 = v_1^{2^*-1} + v_2^{2^*-1} \approx (v_1 + v_2)^{2^*-1} = u^{2^*-1}.$$

Remark: Note $\Gamma(v_1 + v_2) \approx 0$ but $\delta(v_1 + v_2) \approx S_{n,p} \gg 0$.

Stability in terms of the Euler-Lagrange equations ($p = 2$)

Refined stability problem

If $\Delta_p u + |u|^{p^*-2}u \approx 0$, then is u close to the sum of some $v_i \in \mathcal{M}_1$?

First qualitative stability: Struwe decomposition ($p = 2$)

Theorem (Struwe, 1984)

Assume

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \leq (\nu + \frac{1}{2}) S_{n,2}^{2n}.$$

Then

$$\|\nabla u - \sum_{i=1}^{\nu} \nabla v_i\|_{W^{-1,2}(\mathbb{R}^n)} = o(\Gamma(u)),$$

where

$$\Gamma(u) = \|\Delta u + |u|^{2^*-2} u\|_{W^{-1,2}(\mathbb{R}^n)}$$

and the family of Talenti-bubbles $\{v_i\}$ are supported far away from each and other.

M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities.* Math. Z., **187** (1984), 511–517.

Struwe type decomposition of p-Sobolev inequality

Theorem (Mercuri–Willem, 2010)

Let $n \geq 2$ and $1 < p < n$, assume

$$\int_{\mathbb{R}^n} |\nabla u|^p dx \leq (\nu + \frac{1}{2}) S_{n,p}^{np},$$

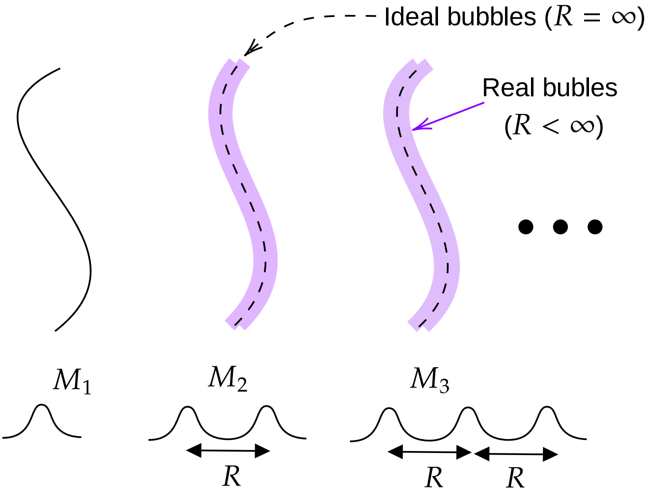
it holds

$$\left\| \nabla \left(u - \sum_{i=1}^{\nu} v_i \right) \right\|_{L^p(\mathbb{R}^n)} \leq C\omega \left(\| \Delta_p u + |u|^{p^*-2} u \|_{W^{-1,q}(\mathbb{R}^n)} \right),$$

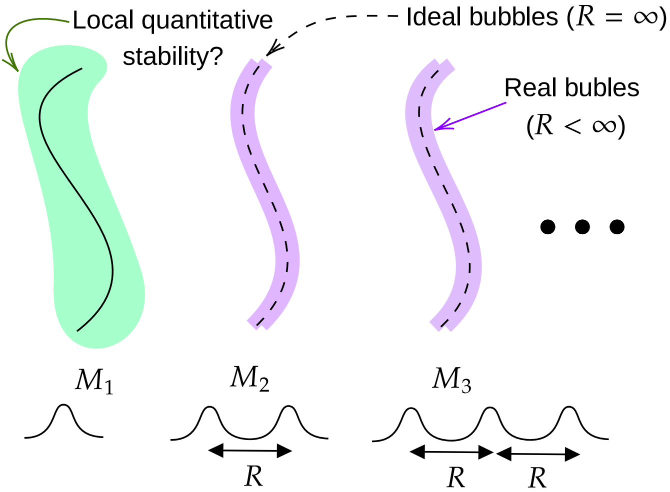
where ω is the module of continuity.

C. Mercuri, M. Willem, *A global compactness result for the p-Laplacian involving critical nonlinearities*. Discrete Contin. Dyn. Syst., **28** (2010), 469–493.

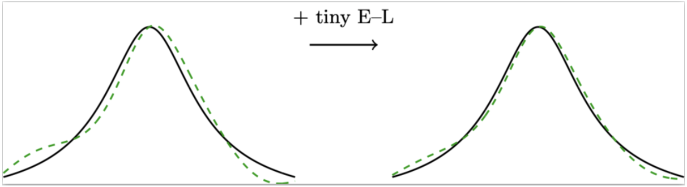
Why we need the assumption?



What quantitative stability result do we expect?



The first quantitative stability result: One bubble ($p = 2$)



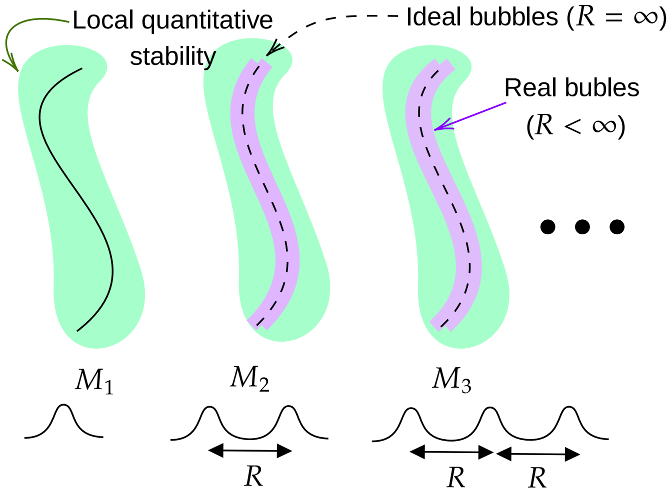
Theorem (Ciraolo–Figalli–Maggi, 2018)

For $\nu = 1$, it holds

$$\|\nabla u - \nabla v\|_{L^2(\mathbb{R}^n)} \leq C(n)\Gamma(u), \text{ where } v \in \mathcal{M}_1.$$

G. Ciraolo, A. Figalli, F. Maggi, *A quantitative analysis of metrics on $\mathbb{R}^n$ with almost constant positive scalar curvature, with applications to fast diffusion flows.* Int. Math. Res. Notices, **201** (2018), 6780–6797.

What about more than one bubble?



Quantitative stability ($p = 2$)

Theorem (Figalli–Glaudo, 2020, $3 \leq n \leq 5$.)

For $3 \leq n \leq 5$ and $\nu > 1$,

$$\|\nabla u - \sum_{i=1}^{\nu} \nabla v_i\|_{L^2(\mathbb{R}^n)} \leq C\Gamma(u)^{1}.$$

Moreover, the power 1 is sharp.

A. Figalli, F. Glaudo, *On the sharp stability of critical points of the Sobolev inequality.* Arch. Ration. Mech. Anal., **237** (2020), 201–258.

Quantitative stability ($p = 2$)

Theorem (Deng–Sun–Wei, 2025, $n \geq 6$).

For $n \geq 6$ and $\nu > 1$,

$$\|\nabla u - \sum_{i=1}^{\nu} \nabla v_i\|_{L^2(\mathbb{R}^n)} \leq \begin{cases} C(n)\Gamma(u)|\log \Gamma(u)|^{\frac{1}{2}}, & n = 6 \\ C(n)\Gamma(u)^{\frac{n+2}{2(n-2)}}, & n \geq 7 \end{cases}.$$

Moreover, the power $\frac{1}{2}$ and $\frac{n+2}{2(n-2)}$ are sharp.

B. Deng, L. Sun, J. Wei, *Sharp quantitative estimates of Struwe's decomposition.* Duke Math. J, **174** (2025), 159–228.

Why the quantitative result for multi-bubbles depends on the dimension?

- Multi-bubbles: set $u = \sum_{i=1}^{\nu} v_i + \rho, a_i > 0$, then

$$\int_{\mathbb{R}^n} u^{2^*-1}\rho dx = \int_{\mathbb{R}^n} (\sum_{i=1}^{\nu} v_i)^{2^*-1}\rho + p(\sum_{i=1}^{\nu} v_i)^{p-1}\rho^2 \; dx + o(\|\nabla\rho\|^2_{L^2(\mathbb{R}^n)})$$

$$= \int_{\mathbb{R}^n} \left((\sum_{i=1}^{\nu} v_i)^{2^*-1} - \sum_{i=1}^{\nu} v_i^{2^*-1} \right) \rho \; dx$$

$$+ (2^* - 1) \int_{\mathbb{R}^n} (\sum_{i=1}^{\nu} v_i)^{2^*-2}\rho^2 \; dx + o(\|\nabla\rho\|^2_{L^2(\mathbb{R}^n)})$$

Figalli–Glaudo's method ($p = 2$)

By the Hölder inequality, elementary estimates and computing the interactions between bubbling phenomena, they obtained

$$\int_{\mathbb{R}^n} \left((\sum_{i=1}^{\nu} v_i)^{2^*-1} - \sum_{i=1}^{\nu} v_i^{2^*-1} \right) \rho \, dx$$
$$\leq C(n) \sum_{1 \leq i \neq j \leq \nu} \int_{\mathbb{R}^n} v_i^{2^*-1} v_j \rho \, dx$$
$$\leq C(n) \sum_{1 \leq i \neq j \leq \nu} \| v_i^{2^*-2} v_j \|_{L^{(2^*)'}(\mathbb{R}^n)} \| \nabla \rho \|_{L^2(\mathbb{R}^n)}$$
$$\overset{(3 \leq n \leq 5)}{\leq} C(n) \sum_{1 \leq i \neq j \leq \nu} \int_{\mathbb{R}^n} v_i^{2^*-1} v_j \, dx \| \nabla \rho \|_{L^2(\mathbb{R}^n)}.$$

A. Figalli, F. Glaudo, *On the sharp stability of critical points of the Sobolev inequality.* Arch. Ration. Mech. Anal., **237** (2020), 201–258.

Deng–Sun–Wei's method ($p = 2$)

Let $\rho = \rho_0$ (main term) $+ \rho_1$. Using the finite-dimensional reduction method, Deng–Sun–Wei gave more precise point-wise estimates of

$$\int_{\mathbb{R}^n} \left((\sum_{i=1}^{\nu} v_i)^{2^*-1} - \sum_{i=1}^{\nu} v_i^{2^*-1} \right) \rho_0 \, dx.$$

B. Deng, L. Sun, J. Wei, *Sharp quantitative estimates of Struwe's decomposition.* Duke Math. J, **174** (2025), 159–228.

Summary

<table>
<tr><th>Stability</th><th>$p = 2$</th><th>$1 < p < n$</th></tr>
<tr><td rowspan="2">Minimizers</td><td colspan="2">Quantitative (sharp)</td></tr>
<tr><td>Bianchi–Egnell (1991)</td><td>Figalli–Zhang (2022)</td></tr>
<tr><td rowspan="6">Critical points</td><td colspan="2">Qualitative</td></tr>
<tr><td>Struwe (1984)</td><td>Mercuri–Willem (2010)</td></tr>
<tr><td colspan="2">Quantitative (sharp)</td></tr>
<tr><td>$\nu = 1, \ n \geq 3:$ Ciraolo–Figalli–Maggi (2018)</td><td rowspan="4">?</td></tr>
<tr><td>$\nu > 1, \ 3 \leq n \leq 5:$ Figalli–Glaudo (2020)</td></tr>
<tr><td>$\nu > 1, \ n \geq 6:$ Deng–Sun–Wei (2025)</td></tr>
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</table>

Stability problem of Sobolev inequality

1. Stability in terms of the Sobolev energy (Minimizers)
   If $\delta(u) = \frac{\|\nabla u\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^{p^*}(\mathbb{R}^n)}} - S_{n,p} \approx 0$, then $dist(u, \mathcal{M}) \approx 0$?

2. Stability in terms of the Euler-Lagrange equations (Critical points)
   If $\Delta_p u + |u|^{p^*-2} u \approx 0$, then is u close to some $v \in \mathcal{M}$?

3. Recent result of L.–Zhang.
   Sharp stability of p-Sobolev inequality in the absence of bubbling

Main result

Recall $q = \frac{p}{p-1}$ and $P(u) = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) - |u|^{p^*-2}u$.

Theorem (L.–Zhang, 2025)

For $1 < p < n$ and $\nu = 1$,

$$\|\nabla u - \nabla v\|_{L^p(\mathbb{R}^n)}^{\max\{1,p-1\}} \leq C\|P(u)\|_{W^{-1,q}(\mathbb{R}^n)}.$$

Moreover, the power $\max\{1, p-1\}$ is sharp.

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Remark Two explicit examples: Giving a smooth perturbation to v and setting

$$u_i = v_i + \varphi(x_i + \cdot), x_i \to \infty, \ \varphi \in C_c^\infty(B_1).$$

G. Liu, Y. Zhang, *Sharp stability for critical points of the Sobolev inequality in the absence of bubbling.* arXiv:2503.02340.

Alternative proof of Ciraolo and Gatti

Remark Ciraolo and Gatti proved at the same time a similar result but with a non-sharp exponent. Although their exponent is not sharp, their method appears to have potential for generalization to other contexts, such as the anisotropic setting.

G. Ciraolo, M. Gatti, *On the stability of the critical p-Laplace equation*. arXiv preprint arXiv:2503.01384 (2025).

Challenges for quantitative stability problem

Denote $u = v + \epsilon\varphi$. Then testing $P(u)$ against $\epsilon\varphi$ yields

$$
\begin{aligned}
&\epsilon \int_{\mathbb{R}^n} -|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi dx - \epsilon \int_{\mathbb{R}^n} |u|^{p^*-2} u\varphi dx \\
&= \langle P(u), \epsilon\varphi \rangle \leq \epsilon \|P(u)\|_{W^{-1,q}(\mathbb{R}^n)} \|\nabla \varphi\|_{L^p(\mathbb{R}^n)}.
\end{aligned}
$$

Challenges for quantitative stability problem

- $\Delta_p u$ is a non-linear operator. How to expand

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2}\nabla u \cdot \nabla \rho \, dx$$

and

$$\int_{\mathbb{R}^n} |u|^{p^*-2} u\rho \, dx?$$

Standard Taylor expansion works to get sharp result only under strong restrictions, refer to [Zhou–Zou, 2023].

- Does the corresponding spectrum gap inequality hold?

Y. Zhou, W. Zou, *Quantitative stability for the Caffarelli-Kohn-Nirenberg inequality*. arXiv preprint arXiv:2312.15735 (2023).

Step 1: Establish new vectorial inequalities

Let $x, y \in \mathbb{R}^n$ and $\kappa > 0$.

- For $1 < p < 2$, there exists a constant $c_1 = c_1(p,\kappa) > 0$ such that

$$
\begin{aligned}
|x+y|^{p-2}(x+y)\cdot y \geq |x|^{p-2}x\cdot y &+ (1-\kappa)\omega_1|y|^2 \\
&+ (p-2)(1-\kappa)\omega_2\big(|x|-|x+y|\big)^2 \\
&+ c_1 \min\{|y|^p, |x|^{p-2}|y|^2\}\big).
\end{aligned}
$$

$$
\omega_1 = \omega_1(x, x+y) := \begin{cases} |x+y|^{p-2} & \text{if } |x| \leq |x+y| \\ |x|^{p-2} & \text{if } \ |x+y| \leq |x| \end{cases}
$$

and

$$
\omega_2 = \omega_2(x, x+y) := \begin{cases} \frac{|x+y|^{p-1}}{(2-p)|x+y|+(p-1)|x|} & \text{if } |x| \leq |x+y| \\ |x|^{p-2} & \text{if } \ |x+y| \leq |x| \end{cases} .
$$

Step 1: Establish new vectorial inequalities

- For $p \geq 2$, there exist constants $c_2 = c_2(p, \kappa) > 0$ and $c_3 = c_3(p) > 0$ such that

$$|x+y|^{p-2}(x+y)\cdot y \geq |x|^{p-2}x\cdot y + (1-\kappa)\omega_3|y|^2 + (p-2)(1-\kappa)\omega_4\big(|x|-|x+y|\big)^2 + c_2|y|^p.$$

where

$$\omega_3 = \omega_3(x, x+y) := \begin{cases} |x|^{p-2} & \text{if } |x| \leq |x+y| \\ \frac{|x+y|^{p-1}}{|x|} & \text{if } \ c_3^{\frac{1}{p-1}}|x| \leq |x+y| \leq |x| \\ c_3|x|^{p-2} & \text{if } |x+y| \leq c_3^{\frac{1}{p-1}}|x| \end{cases}$$

and

$$\omega_4 = \omega_4(x, x+y) := \begin{cases} |x|^{p-2} & \text{if } |x| \leq |x+y| \\ \frac{|x+y|^{p-1}}{|x|} & \text{if } \ |x+y| \leq |x| \end{cases}.$$

Step 2: Expand $|a+b|^{p^*-2}(a+b)b$.

Let $\kappa > 0$, $C_i = C_i(p^*, \kappa) > 0$, $i = 1, 2$. Let $a, b \in \mathbb{R}$, $a \neq 0$.

1. For $1 < p \leq \frac{2n}{n+2}$, it holds
$$|a+b|^{p^*-2}(a+b)b \leq |a|^{p^*-2}ab + (p^*-1+\kappa)\frac{(|a|+C_1|b|)^{p^*}}{|a|^2+|b|^2}|b|^2.$$

2. For $\frac{2n}{n+2} < p < \infty$, it holds
$$|a+b|^{p^*-2}(a+b)b \leq |a|^{p^*-2}ab + (p^*-1+\kappa)|a|^{p^*-2}|b|^2 + C_2|b|^{p^*}.$$

Step 3: Spectrum gap inequality in three regions

For any $\gamma_0 > 0$ and $C_1 > 0$, there exists $\bar{\delta} = \bar{\delta}(n, p, \gamma_0, C_1) > 0$ such that the following statement holds:
Let $\varphi \in \dot{W}^{1,p}(\mathbb{R}^n)$ be orthogonal to $T_v\mathcal{M}$ in $L^2(\mathbb{R}^n; v^{p^*-2})$ and satisfy

$$\|\varphi\|_{\dot{W}^{1,p}(\mathbb{R}^n)} \leq \bar{\delta},$$

and define $\omega_j = \omega_j(Dv, Dv + D\varphi), j \in \{1, 2, 3, 4\}$. Then

- For $1 < p \leq \frac{2n}{n+2}$, it holds

$$\begin{aligned}
&\int_{\mathbb{R}^n} \omega_1 |D\varphi|^2 + (p-2)\omega_2(|D(v+\varphi)| - |Dv|)^2 dx \\
&\quad + \gamma_0 \int_{\mathbb{R}^n} \min\{|D\varphi|^p,\ |Dv|^{p-2}|D\varphi|^2\} dx \\
&\geq \left(p^* - 1 + \lambda S^{-p}\right) \int_{\mathbb{R}^n} \frac{(v + C_1|\varphi|)^{p^*}}{v^2 + |\varphi|^2} |\varphi|^2 dx.
\end{aligned}$$

Step 3: Spectrum gap inequality in three regions

- For $\frac{2n}{n+2} < p < 2$, it holds

$$\int_{\mathbb{R}^n} \omega_1 |D\varphi|^2 + (p-2)\omega_2 (|D(v+\varphi)| - |Dv|)^2 dx + \gamma_0 \int_{\mathbb{R}^n} \min\{|D\varphi|^p,\ |Dv|^{p-2}|D\varphi|^2\} dx \geq \left(p^* - 1 + \lambda S^{-p}\right) \int_{\mathbb{R}^n} v^{p^*-2} |\varphi|^2 dx.$$

- For $p \geq 2$, it holds

$$\int_{\mathbb{R}^n} \omega_3 |D\varphi|^2 + (p-2)\omega_4 (|D(v+\varphi)| - |Dv|)^2 dx \geq \left(p^* - 1 + \lambda S^{-p}\right) \int_{\mathbb{R}^n} v^{p^*-2} |\varphi|^2 dx.$$

Final step

Let $x = \nabla v, y = \epsilon \nabla \varphi$ and $a = v, b = \epsilon\varphi$. Combining with the above steps with

$$\epsilon \int_{\mathbb{R}^n} -|Du|^{p-2} Du \cdot D\varphi dx - \epsilon \int_{\mathbb{R}^n} |u|^{p^*-2} u\varphi dx$$
$$\leq \epsilon \|P(u)\|_{W^{-1,q}(\mathbb{R}^n)} \|D\varphi\|_{L^p(\mathbb{R}^n)},$$

we get the desired result.

Thank you for your attention!