Sharp Spinor Inequalities

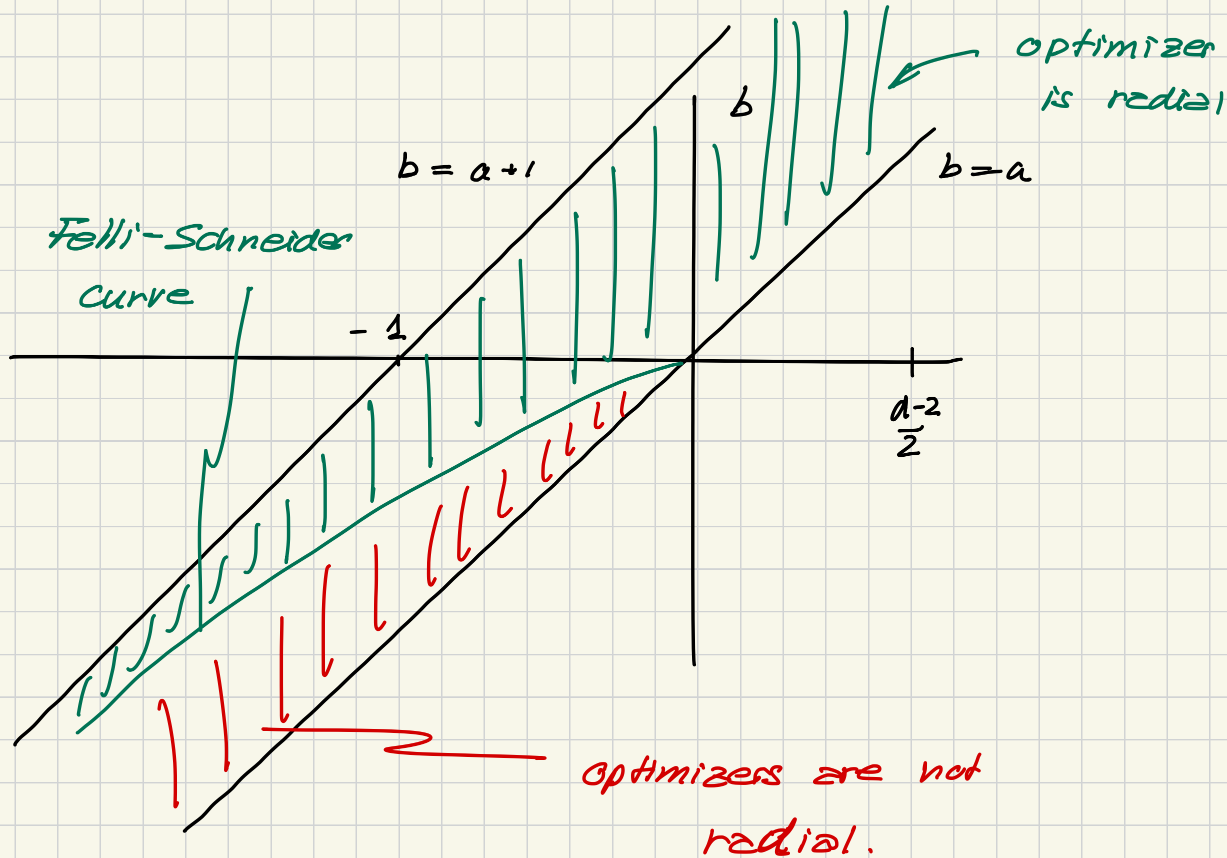
Caffarelli - Kohn - Nirenberg

$$\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\, dx \geq C_{abd} \left(\int \frac{|u|^p}{|x|^{bp}}\, dx \right)^{2/p}$$

Inequality is invariant under rotations: $u(x) \mapsto u(R^{-1}x)$

$R \in O(d)$.

$$p = \frac{2d}{d-2+2(b-a)}$$


optimizer
is radial
b
b = a+1
b = a
Felli-Schneider
curve
- 1
$\frac{d-2}{2}$
optimizers are not
radial.

CKN for Spinors

$$\int_{\mathbb{R}^3} \frac{|\sigma\cdot\nabla\Psi|^2}{|x|^{2\alpha}}\,dx \geq C_{\alpha p}\left(\int_{\mathbb{R}^3} \frac{|\Psi|^p}{|x|^{p\beta}}\,dx\right)^{2/p} \qquad p = \frac{6}{1+2(\beta-\alpha)}$$

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma\cdot\nabla = \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3$$

$$\langle \varphi, \psi \rangle = \overline{\varphi}_1\psi_1 + \overline{\varphi}_2\psi_2$$

Invariance under $SU(2)$: $\Psi(x) \mapsto A\Psi(R^{-1}(A)x) =: (U(A)\Psi)(x)$

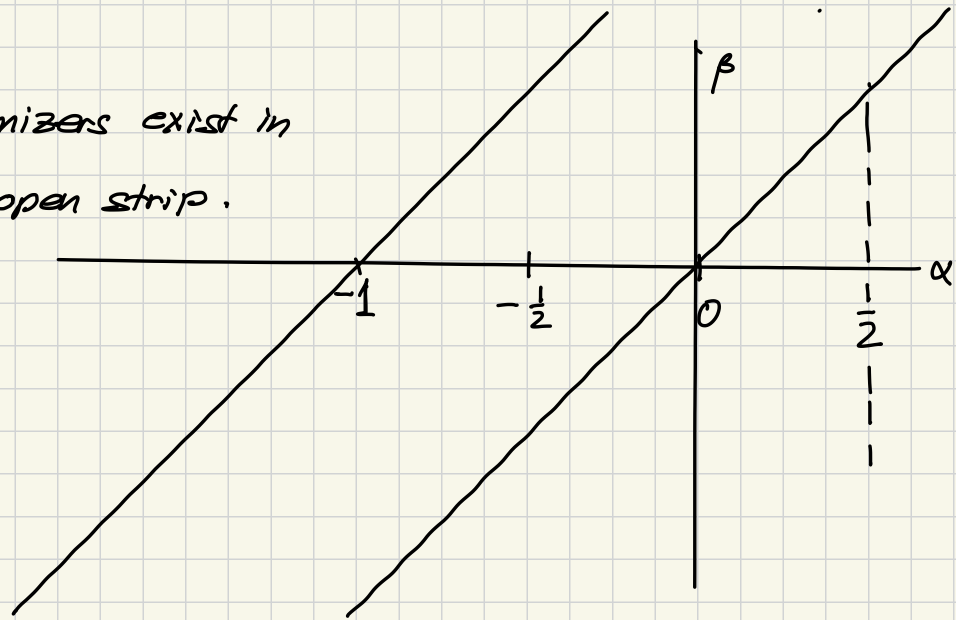
$A \in SU(2)$.

$$A^* \sigma_j A = \sum_{k=1}^{3} R_{jk}(A)\, \sigma_k \qquad R(A) \in SO(3)$$

Joint work with

$\boxed{\text{Jean Dolbeault, Maria Esteban, Rupert Frank.}}$

Optimizers exist in the open strip.



Displaying the symmetries: log. coordinates.

$$\Psi(x) = r^{\alpha-\frac{1}{2}}\,\Phi(s,\omega), \quad r = e^s, \quad \omega = \frac{x}{r}, \quad r = |x|.$$

$$\int_{-\infty}^{\infty}\int_{S^2} |\partial_s \Phi|^2 \,d\omega\, ds + \int_{-\infty}^{\infty}\int_{S^2} |(\sigma\cdot L - \alpha + \tfrac{1}{2})\Phi|^2 \,d\omega\, ds$$

$$\geq C_{\alpha,p}\left(\int_{-\infty}^{\infty}\int_{S^2} |\Phi|^p \,ds\, d\omega\right)^{2/p}$$

$L = x \wedge \frac{1}{i}\nabla$ angular momentum operator

$$L(S^2, d\omega : \mathbb{C}^2) = \bigoplus_{j=\frac{1}{2}}^{\infty} \left(\mathcal{H}_{\ell = j-\frac{1}{2}} \oplus \mathcal{H}_{\ell = j+\frac{1}{2}} \right)$$

$d\omega$ normalized

$U(A)$ acts irreducibly on $\mathcal{H}_{\ell=j-\frac{1}{2}}$ and $\mathcal{H}_{\ell=j+\frac{1}{2}}$.

Normalized eigen spinors of $\sigma\cdot L$:

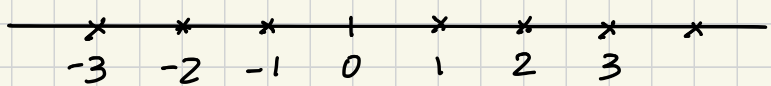
$$j=\ell+\tfrac{1}{2}: \quad \frac{1}{\sqrt{2\ell+1}}\begin{pmatrix}\sqrt{\ell+m+1/2}\; Y_{\ell,m-1/2}\\ -\sqrt{\ell-m+\frac{1}{2}}\; Y_{\ell,m+1/2}\end{pmatrix}=\chi^{+}_{\ell,m}$$

$$j=\ell-\tfrac{1}{2}: \quad \frac{1}{-\sqrt{2\ell+1}}\begin{pmatrix}\sqrt{\ell-m+1/2}\; Y_{\ell,m-1/2}\\ \sqrt{\ell+m+\frac{1}{2}}\; Y_{\ell,m+1/2}\end{pmatrix}=\chi^{-}_{\ell,m}$$

<u>Eigenvalues of $\sigma\cdot L+1$</u>

$\{\sigma\cdot L+1,\ \sigma\cdot\omega\}=0$

$j=\ell-1/2$ $\qquad\qquad$ $j=\ell+1/2$



$$\alpha = -\frac{2\ell+1}{2} \qquad \ell = 1, 2, \dots \qquad (\sigma\cdot L + 1 + \ell)\,\chi^-_{\ell,m} = 0$$

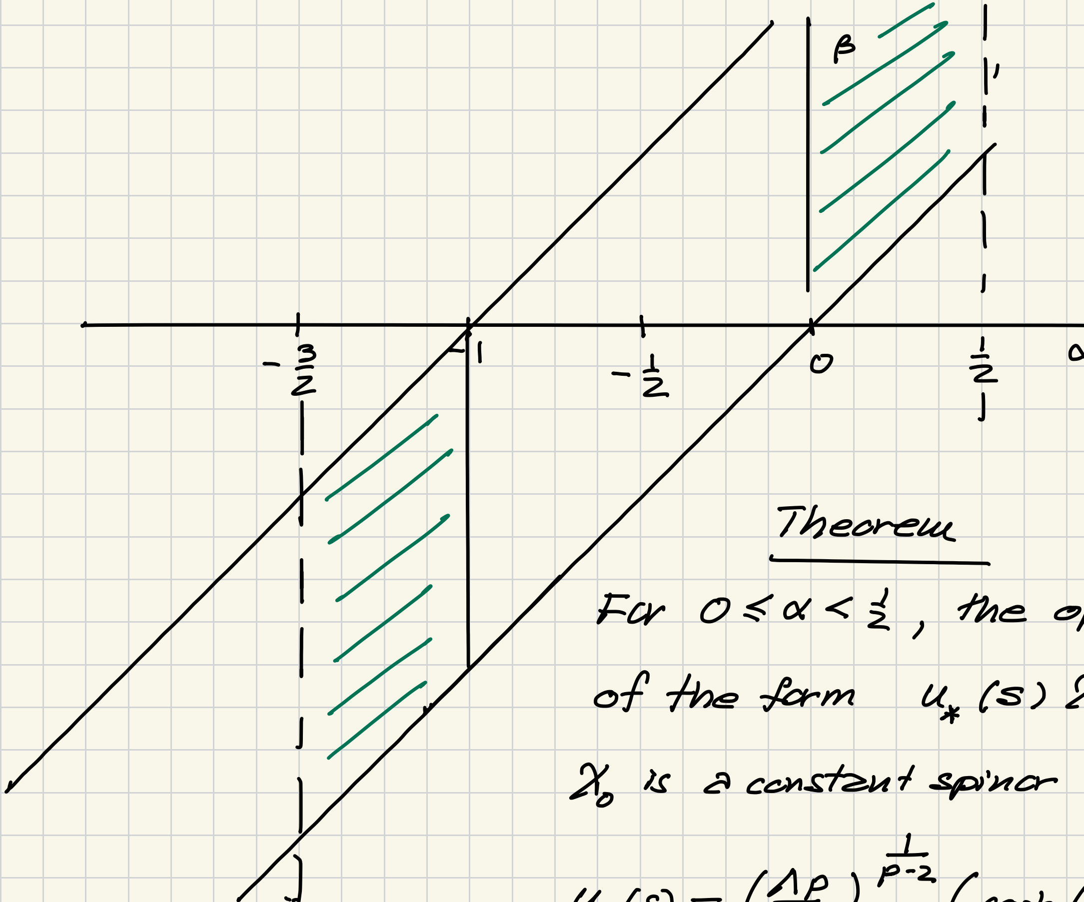
$$\inf_u \frac{\int_{-\infty}^{\infty}\int_{S^2} |\partial_s (u\,\chi^-_{\ell,m})|^2 \, d\omega\, ds}{\left(\int_{-\infty}^{\infty}\int_{S^2} |u\,\chi^-_{\ell,m}|^p \, d\omega\, ds\right)^{2/p}} = 0 .$$

Symmetry: $\displaystyle\int_{-\infty}^{\infty}\int_{S^2} |(\sigma\cdot L - \alpha + \tfrac{1}{2})\, \sigma\cdot\omega\, \Phi|^2 \, d\omega\, ds$

$$= \int_{-\infty}^{\infty}\int_{S^2} |(\sigma\cdot L + \alpha + \tfrac{3}{2})\,\Phi|^2 \, d\omega\, ds$$

using $\{\sigma\cdot L + 1, \sigma\cdot\omega\} = 0$

other terms are not affected.



Theorem

For $0 \leq \alpha < \frac{1}{2}$, the optimizer is of the form $u_*(s)\,\chi_0$ where χ_0 is a constant spinor and

$$u_*(s) = \left(\frac{\Lambda p}{2}\right)^{\frac{1}{p-2}} \left(\cosh\left(\frac{\sqrt{\Lambda}}{2}(p-2)s\right)\right)^{-\frac{1}{p-2}}$$

$$\Lambda = \left(\alpha - \tfrac{1}{2}\right)^2.$$

Call $\mathcal{H}_0$ the symmetric sector.

$C^*_{\alpha,p}$ the optimal constant in this sector.

<u>Proposition</u>

If for some pair (α, p), $-\frac{1}{2} < \alpha < \frac{1}{2}$, $p \in [2,6]$, $C_{\alpha,p} < C^*_{\alpha,p}$,

then for all $-\frac{1}{2} < \alpha' < \alpha < \frac{1}{2}$, $C_{\alpha',p} < C^*_{\alpha',p}$.

Proposition $\Longrightarrow$ Theorem, since $C_{0,p} = C^*_{0,p}$

$$\int_{\mathbb{R}^3} |\sigma \cdot \nabla \Psi|^2 dx = \int_{\mathbb{R}^3} |\nabla \Psi|^2 dx \geq \int_{\mathbb{R}^3} |\nabla |\Psi||^2 dx$$

$\rightarrow$ radial symmetry of $|\Psi|$.

Proof of Proposition:

$$G_{\frac{1}{2}-\alpha}(\Phi) = \frac{\int_{-\infty}^{\infty}\int_{S^2} |\partial_s \Phi|^2 + |(\sigma\cdot L-\alpha+\frac{1}{2})\Phi|^2 \, d\omega \, ds}{\left(\int_{-\infty}^{\infty}\int_{S^2} |\Phi|^p \, d\omega \, ds\right)^{2/p}}$$

Scaling: $\Phi(s,\omega) \longmapsto \Phi(ts,\omega) = \Phi_t$, $\frac{1}{2}-\alpha \mapsto t(\frac{1}{2}-\alpha)$

$t>1$

$$G_{t(\frac{1}{2}-\alpha)}(\Phi_t) = t^{1+\frac{2}{p}} G_{\frac{1}{2}-\alpha}(\Phi) + \underbrace{(1-t)}_{<0} t^{\frac{p}{2}} I(\Phi)$$

where $$I(\Phi) = \frac{\int_{-\infty}^{\infty}\int_{S^2} \frac{1+t}{t}|\sigma\cdot L\Phi|^2 - 2\,\mathrm{Re}\langle\sigma\cdot L\Phi, \Phi\rangle(\alpha-\frac{1}{2}) \, d\omega \, ds}{\left(\int_{-\infty}^{\infty}\int_{S^2} |\Phi|^p \, d\omega \, ds\right)^{2/p}}$$

$\frac{1+t}{t}(\sigma \cdot L)^2 + (1-2\alpha)\,\sigma L$ has eigenvalue

$$\frac{1+t}{t}\ell^2 + (1-2\alpha)\ell \qquad \ell = 0, 1, 2, \dots$$

$$\frac{1+t}{t}(\ell+1)^2 - (1-2\alpha)(\ell+1), \quad \ell = 1, 2, \dots$$

$\Longrightarrow$ $I(\Phi) > 0$ and $= 0$ only if $\Phi \in \mathcal{H}_0$

Pick $t = \frac{1-2\alpha'}{1-2\alpha} > 1$, $-\frac{1}{2} < \alpha' < \alpha < \frac{1}{2}$ and let

Φ be an optimizer of $G_{\frac{1}{2}-\alpha}(\Phi)$.

$$C_{\alpha',p} \leq G_{t(\frac{1}{2}-\alpha)}(\Phi_t) < t^{1+\frac{2}{p}} G_{\frac{1}{2}-\alpha}(\Phi) < t^{1+\frac{2}{p}} C^*_{\alpha,p}$$

$$= C^*_{\alpha',p}.$$

Theorem

Let $-\frac{1}{2} < \alpha < 0$.

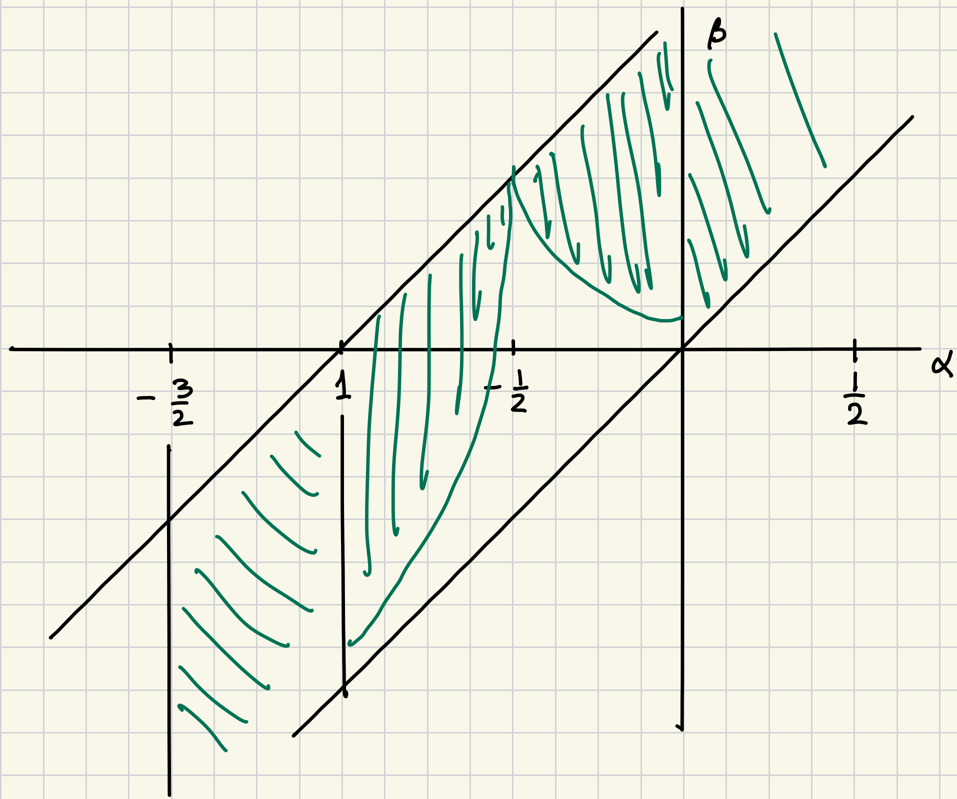
For all β with $\beta_*(\alpha) < \beta < \alpha + 1$

where

$$\beta_*(\alpha) = 1 + \alpha - \frac{6(2\alpha+1)}{4\alpha(\alpha+2)+7}$$

we have that the optimizer is of the form $u_*(s)\,\chi_{\mathrm{const}}$.

$-\frac{1}{2} < \alpha < 0$



Sobolev-type inequality

$$K_\alpha = (\sigma\cdot L)^2 + (1-2\alpha)\,\sigma\cdot L$$

Theorem For $2 < q \leq 16$,

$$\int_{\mathbb{S}^2} \langle \chi, K_\alpha \chi \rangle \, d\omega \geq 2\,\frac{1+2\alpha}{q-2}\left(\|\chi\|^2_{L^q(\mathbb{S}^2)} - \|\chi\|^2_{L^2(\mathbb{S}^2)} \right)$$

Equality only if χ is a const. spinor

$$\beta_*(\alpha) = 1 + \alpha - \frac{6(2\alpha+1)}{4\alpha(\alpha+2)+7}$$

Some ideas of the proof of symmetry

Φ optimizer

$$-\partial_s^2 \Phi + (\sigma\cdot L - \alpha + \tfrac{1}{2})^2 \Phi = |\Phi|^{p-2}\Phi .$$

$$\int_{-\infty}^{\infty}\int_{S^2} |\partial_s \Phi|^2 + |(\sigma\cdot L - \alpha + \tfrac{1}{2})\Phi|^2 \, d\omega\, ds - \int_{-\infty}^{\infty}\int_{S^2} |\Phi|^p \, d\omega\, ds = 0$$

$$\Longrightarrow \quad C_{\alpha,p} = \left(\int_{-\infty}^{\infty}\int_{S^2} |\Phi|^p \, d\omega\, ds \right)^{1-\frac{2}{p}}$$

Assume that

$$C^*_{\alpha,p} > C_{\alpha,p} .$$

$$\mathcal{F}(\phi) = \int_{-\infty}^{\infty} \int_{S^2} |\partial_s \phi|^2 - |\phi|^p + |(\sigma\cdot L - \alpha + \tfrac{1}{2})\phi|^2 \, d\omega ds = 0$$

Keller-Lieb-Thirring:

$$-\partial_s^2 u - V(s) u = -\lambda(V) u \quad \text{on } \mathbb{R}$$

$$\lambda(V)^{1/\gamma} \leq C_{LT}(\gamma) \int_{-\infty}^{\infty} V(s)^{\gamma+\frac{1}{2}} ds \, , \quad \gamma > \tfrac{1}{2}$$

Equality if $\quad V(s) = \dfrac{\gamma^2 - \frac{1}{4}}{\cosh(s)^2}$

$$u_*(s) = \pi^{-\frac{1}{4}} \left(\frac{\Gamma(\gamma)}{\Gamma(\gamma - \frac{1}{2})} \right)^{\frac{1}{2}} \cosh(s)^{\frac{1}{2} - \gamma}$$

Set $V_\omega(s) = |\phi(s,\omega)|^{p-2}$

$$\gamma = \frac{1}{2}\,\frac{p+2}{p-2}\,, \quad u(\omega) = \Big(\int_{-\infty}^{\infty} |\phi(s,\omega)|^2 ds\Big)^{1/2}$$

$$\int_{-\infty}^{\infty} |\partial_s \phi|^2 - |\phi|^{p-2}|\phi|^2 \, ds$$

$$\geq - c_{LT}(\gamma)^{\frac{1}{\gamma}} \Big(\int_{-\infty}^{\infty} |\phi(s,\omega)|^p ds\Big)^{\frac{1}{\gamma}} |u(\omega)|^2$$

$$0 = \mathcal{F}(\phi) \geq \int_{-\infty}^{\infty}\int_{S^2} |(\sigma\cdot L - \alpha + \tfrac{1}{2})\phi|^2 d\omega\, ds$$

$$- c_{LT}(\gamma)^{\frac{1}{\gamma}} \int_{S^2} \Big(\int_{-\infty}^{\infty} |\phi(s,\omega)|^p ds\Big)^{\frac{1}{\gamma}} |u(\omega)|^2 d\omega$$

$$= \int_{-\infty}^{\infty}\int_{S^2} \langle \phi, K_\alpha \phi\rangle \, d\omega\, ds + (\alpha-\tfrac{1}{2})^2 \int_{-\infty}^{\infty}\int_{S^2} |\phi|^2 \, d\omega\, ds$$

$$- C_{LT}(\gamma)^{\frac{1}{\gamma}} \int_{S^2} \left(\int_{-\infty}^{\infty} |\phi(s,\omega)|^p \, ds \right)^{\frac{1}{\gamma}} |u(\omega)|^2 \, d\omega$$

Sobolev:

$$0 = \mathcal{F}(\phi) \geq \frac{2(1+2\alpha)}{q-2} \int_{-\infty}^{\infty} \Big[\underline{\Big(\int_{S^2} |\phi|^q \, d\omega \Big)^{2/q}} - \int |\phi|^2 \, d\omega \Big] ds$$

$$+ (\alpha-\tfrac{1}{2})^2 \int_{-\infty}^{\infty}\int_{S^2} |\phi|^2 \, d\omega\, ds$$

<u>q to be determined</u>

$$\underline{- C_{LT}(\gamma)^{\frac{1}{\gamma}} \int_{S^2} \left(\int_{-\infty}^{\infty} |\phi(s,\omega)|^p \, ds \right)^{\frac{1}{\gamma}} |u(\omega)|^2 \, d\omega}$$

Minkowski's inequality

$$\left(\int_{S^2} |u(\omega)|^q \, d\omega\right)^{2/q} = \left(\int_{S^2} \left(\int_{-\infty}^{\infty} |\phi(s,\omega)|^2 \, ds\right)^{q/2} d\omega\right)^{2/q}$$

$$\leq \int_{-\infty}^{\infty} \underline{\left(\int_{S^2} |\phi(s,\omega)|^q \, d\omega\right)^{2/q}} \, ds$$

Hölder's inequality

$$\underline{\int_{S^2} \left(\int_{-\infty}^{\infty} |\phi(s,\omega)|^p \, ds\right)^{1/\gamma} |u(\omega)|^2 \, d\omega}$$

$$\leq \left(\int_{-\infty}^{\infty} \int_{S^2} |\phi(s,\omega)|^p \, d\omega \, ds\right)^{1/\gamma} \left(\int_{S^2} |u(\omega)|^{\frac{2\gamma}{\gamma-1}} \, d\omega\right)^{\frac{\gamma-1}{\gamma}}$$

$$0 = \mathcal{F}(\phi) \geq \frac{2(1+2\alpha)}{q-2}\left[\|u\|_q^2 - \|u\|_2^2\right] + (\alpha - \tfrac{1}{2})^2 \|u\|_2^2$$

$$- \underbrace{C_{LT}(\gamma)^{1/\gamma}\left(\int_{-\infty}^{\infty}\int_{S^2} |\phi|^p \, d\omega \, ds\right)^{\frac{1}{\gamma}}}_{=:D} \|u\|_{\frac{2\gamma}{\gamma-1}}^2$$

$$q = \frac{2\gamma}{\gamma-1} = 2\,\frac{p+2}{6-p}.$$

$$0 = \mathcal{F}(\phi) \geq \left[\frac{2(1+2\alpha)}{q-2} - D\right]\|u\|_q^2$$

$$+ \left[(\alpha - \tfrac{1}{2})^2 - \frac{2(1+2\alpha)}{q-2}\right]\|u\|_2^2$$

$$D = C_{LT}(\gamma)^{1/\gamma}\left(\int_{-\infty}^{\infty}\int_{S^2} |\phi|^p \, d\omega \, ds\right)^{1/\gamma}$$

$$\leq C_{LT}(\gamma)^{1/\gamma}\left(\int_{-\infty}^{\infty}\int_{S^2} |\phi_*|^p \, d\omega \, ds\right)^{1/\gamma} = (\alpha - \tfrac{1}{2})^2$$

Pick α such that $(\alpha - \frac{1}{2})^2 \leq \dfrac{2(1+2\alpha)}{q-2}$

$$\Longrightarrow \quad D \leq \frac{2(1+2\alpha)}{q-2}$$

$$\|u\|_q^2 \geq \|u\|_2^2 \qquad q > 2$$

$$0 = \mathcal{F}(\phi) \geq \left[(\alpha - \tfrac{1}{2})^2 - D \right] \|u\|_2^2 \Longrightarrow D = (\alpha - \tfrac{1}{2})^2 .$$

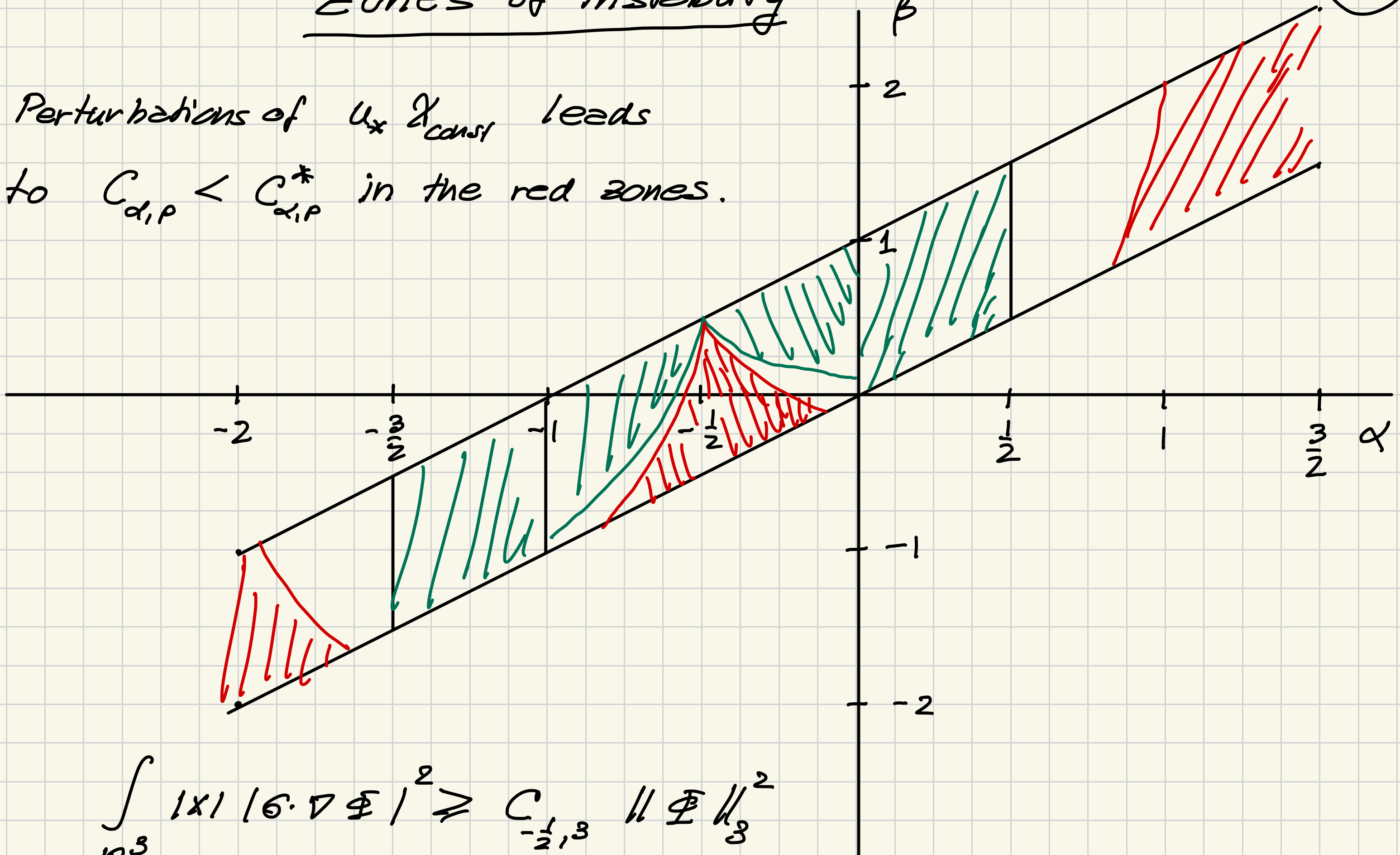
$\Longrightarrow \quad C_{\alpha,p} = C^*_{\alpha,p}$ and all inequalities must be equalities.

$\phi = u_*(s)\, \chi_{const.}$ $\quad u_*$ opt. in KLT.

$\blacksquare$

Zones of instablity

Perturbations of $u_* \chi_{const}$ leads to $C_{\alpha,\beta} < C^*_{\alpha,\beta}$ in the red zones.



$$\int_{\mathbb{R}^3} |x|\, |\sigma \cdot \nabla \Phi|^2 \geq C_{-\frac{1}{2},3} \, \|\Phi\|_3^2$$

Sobolev's inequality, Beckner's approach.

Sharp HLS on sphere:

$$\int_{\mathbb{S}^2\times\mathbb{S}^2} \frac{\langle \chi(\omega), \chi(\omega')\rangle}{|\omega-\omega'|^{\lambda}}\, d\omega\, d\omega' \le C_p \|\chi\|_p^2$$

$$\chi(\omega) = \sum_{\ell=0}^{\infty} \Phi_\ell \qquad \text{angular momentum decomp.}$$

$$\sum_{\ell=0}^{\infty} \gamma_\ell\left(\tfrac{2}{p}\right) \|\Phi_\ell\|_2^2 \le C_p \|\chi\|_p^2 . \qquad \text{Funk - Hecke}$$

$$\gamma_\ell(x) = \frac{\Gamma(x)\,\Gamma(\ell+2-x)}{\Gamma(2-x)\,\Gamma(x+\ell)}$$

Duality: $\frac{1}{p}+\frac{1}{q}=1$

$$\|x\|_q^2 \leq \sum_{\ell=0}^{\infty} \frac{1}{\gamma_\ell(\frac{2}{p})} \|\Phi_\ell\|_2^2 = \sum_{\ell=0}^{\infty} \gamma_\ell(\tfrac{2}{q}) \|\Phi_\ell\|_2^2$$

$$\frac{1}{q-2}\left(\|x\|_q^2 - \|x\|_2^2\right) \leq \sum_{\ell=1}^{\infty} \zeta_\ell(\tfrac{2}{q}) \|\Phi\|_\ell^2$$

$$\zeta_\ell(\tfrac{2}{q}) = \frac{\gamma_\ell(\frac{2}{q})-1}{q-2}$$

$$\int_{S^2} \langle x, K_\alpha x\rangle \, d\omega = \sum_{\ell=1}^{\infty} \lambda_\ell(\alpha) \|\Phi_\ell\|^2$$

show $\lambda_\ell(\alpha) \geq \zeta_\ell(\frac{2}{q})$. (need $q \leq 10$).

THANK

YOU