Quantitative inequalities and convergence of thresholding schemes in optimal control theory

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Objective of the talk

1. Develop quantitative inequalities for spectral optimisation problems or more general optimal control problems.
2. Apply them to the study of numerical schemes.
3. Main reference: Chambolle, M., Privat, Math. Ann., 2025.

Reference spectral optimisation problem

Ω: bounded, smooth domain. $V \in L^\infty(\Omega)$.

$$\lambda(V) := \min_{u \in W_0^{1,2}(\Omega)\,,\int_\Omega u^2=1} \int_\Omega |\nabla u|^2 - \int_\Omega V u^2 \rightsquigarrow \begin{cases} -\Delta u = \lambda u + Vu & \text{in } \Omega\,, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

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$$\min_{0 \leq V \leq 1 \text{ a.e.},\ \int_\Omega V = V_0} \lambda(V).$$

Old and well understood problem:

1. Composite membrane problem (Cox, Lipton, MacLaughlin...)
2. Applications to mathematical biology (Cantrell, Cosner, Berestycki, Hamel, Roques, Kao, Lou, Yanagida...)

What do we want to do?

$$\min_{0\leq V\leq 1 \text{ a.e.}, \int_\Omega V = V_0} \lambda(V).$$

$$\mathscr{V}^* := \underset{0\leq V\leq 1 \text{ a.e.}, \int_\Omega V = V_0}{\operatorname{argmin}} \lambda(V)$$

Fix some $V^* \in \mathscr{V}^*$.

$$\lambda(V) - \lambda(V^*) \geq C\mathrm{dist}(V, \mathscr{V}^*)^\alpha$$

for:

1. Some distance,
2. Some exponent α,
3. And if both could be optimal...

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Spoiler:

$$\lambda(V) - \lambda(V^*) \geq C\mathrm{dist}_{L^1}(V, \mathscr{V}^*)^2.$$

Why do we want to do it?

Several applications:

1. Application to numerical schemes: allows to study the ubiquitous thresholding scheme for the numerical optimisation of potentials, see later. [Chambolle, M., Privat, Math. Ann., 2025].
2. Allows to simplify qualitative questions in optimal control: turnpike in bilinear control [M., Ruiz-Balet, SIMA, 2021], optimal placement of captors [M., Privat, Trélat, AIHP-C, 2025].
3. In general: allows to analyse perturbation problems.

Plan of the talk

1. Basic facts about the underlying optimisation problem.
2. A (short) review of the bibliography & a discussion of the coercivity norm.
3. A discussion of a part of the proof.
4. (Briefly) Application to numerical schemes.

Basic facts I: bang-bang property

Recall:

$$\min_{0\leq V\leq 1 \text{ a.e.},\ \int_\Omega V=V_0} \left(\lambda(V) := \min_{u\in W^{1,2}_0(\Omega)\,,\int_\Omega u^2=1} \int_\Omega |\nabla u|^2 - \int_\Omega Vu^2\right).$$

1. inf of linear functionals: λ is (strictly) concave.
2. Thus: any optimal V^* is an extreme point of $\{0 \leq V \leq 1\,, \int_\Omega V = V_0\}$.
3. So that: any optimal V^* is a characteristic function: $V^* = \mathbb{1}_{E^*}$.

Basic facts II: Free boundary problem

Double minimisation procedure:

$$\min_{0\leq V\leq 1\,,\int_\Omega V=V_0}\lambda(V)=\min_{u\in W^{1,2}_0(\Omega)\,,\int_\Omega u^2=1}\ \min_{0\leq V\leq 1\,,\int_\Omega V=V_0}\int_\Omega|\nabla u|^2-\int_\Omega Vu^2.$$

1. Bathtub principle (simplified and slightly wrong): $f:\Omega\to\mathbb{R}$.

$$\max_{0\leq V\leq 1\,,\int_\Omega V=V_0}\int_\Omega fV=\int_\Omega f\mathbb{1}_{\{f>c\}}$$

where c is chosen so that

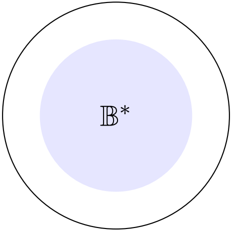
$$|\{f>c\}|=V_0.$$

2. Thus: if V^* is optimal,

$$V^*=\mathbb{1}_{\{u_{V^*}>c^*\}}\rightsquigarrow-\Delta u_{V^*}=\lambda(V^*)u_{V^*}+u_{V^*}\mathbb{1}_{\{u_{V^*}>c^*\}}.$$

Basic fact III: rearrangement inequalities

When Ω is a ball the situation is remarkably simple: $V^* = \mathbb{1}_{\mathbb{B}^*}$.

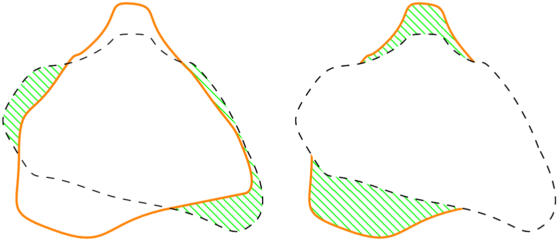


In general, the symmetries of the domain are not preserved but, if Ω is convex, so is E^* etc. We refer to [Imai, Grieser, Kurata].

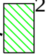
Assume: unique minimiser $V^* = \mathbb{1}_{E^*}$. We claimed:

$$\lambda(V) - \lambda(V^*) \geq C\|V - V^*\|^2_{L^1(\Omega)}.$$

For instance, if $V = \mathbb{1}_E$,



In orange, E^*, in dashed, E.

$$\lambda(\mathbb{1}_E) - \lambda(\mathbb{1}_{E^*}) \geq C$$  ^{2} (1)

Quantitative inequalities in “true” shape optimisation

Shape optimisation problem with a certain constraint:

$$\inf_{\Omega\,,|\Omega|=V_0 \text{ or } \operatorname{Per}(\Omega)=P_0} \mathcal{J}(\Omega)=\mathcal{J}(\Omega^*). \tag{2}$$

A quantitative inequality writes

$$\mathcal{J}(\Omega)-\mathcal{J}(\Omega^*)\geq C\inf_{\Omega^* \text{ optimal}} \operatorname{Vol}(\Omega\Delta\Omega^*)^2\,, \tag{3}$$

where Δ stands for the symmetric difference of sets. If $\Omega^*=\mathbb{B}$: Fraenkel asymmetry.

Some examples

- Isoperimetric inequality: $\mathcal{J} = \text{Per}$, with a volume constraint, and balls are the only solutions.
  1. Fusco, Maggi, Pratelli, Annals of Mathematics, 2008,
  2. Figalli, Maggi, Pratelli, Inventiones Mathematicae, 2010.
- Faber-Krahn inequality: $\mathcal{J} = \lambda_D$ (first eigenvalue of the Dirichlet laplacian) with a volume constraint, and balls are the only solutions.
  1. Brasco, De Philippis, Velichkov, Duke Mathematical Journal, 2015,
  2. Karpukhin, Nahon, Polterovich, Stern, 2021.

Two related contributions

To come back to our problem:

$$\min_{0\leq V\leq 1\,,\int_\Omega V=V_0} \lambda(V).$$

Related contributions for the optimisation of the Dirichlet energy/of eigenvalues w.r.t a potential with L^p $(p<\infty)$ constraints.

1. Brasco, Buttazzo, Calculus of Variations and Partial Differential Equations, 2014,
2. Carlen, Frank, Lieb, Geometric and Functional Analysis, 2014.

The norm is different and reflects the different types of constraints. The methods break down for L^∞ constraints.

Towards the optimal norm

Why choose the L^1 norm squared?
Derivative of the eigenvalue:

$$\lim_{\varepsilon\to 0}\frac{\lambda(V+\varepsilon h)-\lambda(V)}{\varepsilon}=\lambda'(V)[h]=-\int_\Omega h u_V^2.$$

At an optimal potential, with $h=V'-V^*$, the concavity of λ implies

$$\lambda(V)-\lambda(V^*)\geq \lambda'(V^*)[h]=-\int_\Omega (V-V^*)u_V^2.$$

This is another way to get $V^*=\mathbb{1}_{\{u_{V^*}>c^*\}}$.
Natural question: Can we quantify the bathtub principle and obtain something like

$$\int_\Omega V u_{V^*}^2 \leq \int_\Omega V^* u_{V^*}^2 - C\mathrm{dist}(V,V^*)^\alpha?$$

The quantitative bathtub principle

Replace $u_{V^*}^2$ by φ.

$$-\int_\Omega \varphi \mathbb{1}_{\{\varphi > c\}} + \int_\Omega V\varphi.$$

1. If $|\{\varphi = c\}| = 0$ then

$$V^* = \mathbb{1}_{\{\varphi > c\}} \text{ is the unique solution of } \max_{0 \leq V \leq 1, \int_\Omega V = V_0} T(V) = \int \varphi V.$$

2. What we have is a quantitative inequality for a linear problem:

$$T(V) - T(V^*) \leq -C\|V - V^*\|_{L^1}^2.$$

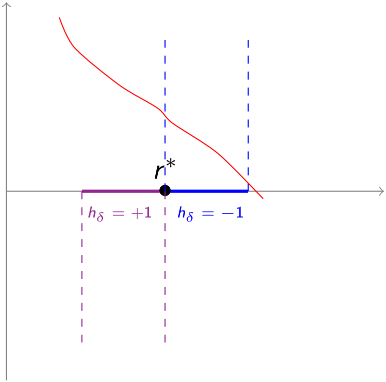
3. To obtain it: we consider

$$\max_{V, \|V - V^*\|_{L^1} = \delta} T(V) = T(V_\delta^*).$$

If we consider the one dimensional case (or radially symmetric) we have

$$T(V^*) - T(V^*_\delta) = \int h_\delta \varphi$$

with



And we can compute explicitly

$$\begin{aligned} T(V_\delta^*) - T(V^*) &= \int_{\mathbb{B}} h_\delta \varphi \\ &= -\int_{r_\delta^-}^{r^*} \varphi \, \mathrm{dr} + \int_{r^*}^{r_\delta^+} \varphi \, \mathrm{dr} \\ &\approx \varphi'(r^*) \left(\int_{r_\delta^-}^{r^*} |r - r^*| \, \mathrm{dr} + \int_{r^*}^{r_\delta^+} |r - r^*| \, \mathrm{dr} \right) \\ &\sim C\varphi'(r^*)\delta^2. \end{aligned}$$

This gives the expected norm.

Some references

The quantitative bathtub principle is found under a variety of guises:

1. Cianchi, Ferone, quantitative Hardy-Littlewood inequality,
2. Lemou, for stability issues in mathematical physics,
3. Wachsmuth, for the numerical analysis of optimal control problems,
4. and probably many other places.

Some results

Theorem (M., JDE, 2020, Chambolle, M., Privat, Math. Ann., 2025)

Let $\mathscr{V}^*$ be the set of minimisers of λ.

1. *When Ω is a ball*
2. *Or when Ω is smooth and the volume constraint V_0 is big enough*

there exists $C > 0$ such that

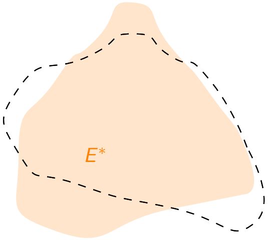
$$\lambda(V) - \lambda(V^*) \geq C\mathrm{dist}_{L^1(\Omega)}(V, \mathscr{V}^*)^2.$$

Local stability of optimal shapes

Goal:

Local stability of critical or optimal shapes

1. We fix a critical/optimal set E^* (i.e. $V^* = \mathbb{1}_{E^*}$ is optimal or critical in the sense that $E^* = \{u_{V^*} > c^*\}$).



Dashed: $(\mathrm{Id} + \Phi)(E^*) = E^*_\Phi$. We prove that if $\|\Phi\|_{W^{2,p}}$ is small enough (independently of E^*) then

$$\lambda(\mathbb{1}_{E^*_\Phi}) - \lambda(\mathbb{1}_{E^*}) \geq C \left| E^*_\Phi \triangle E^* \right|^2 \tag{4}$$

2. If this inequality is satisfied critical/optimal sets are isolated.

How to show that?

1. First and second-order shape derivatives $\rightsquigarrow$ assumes regularity of E^* $\rightsquigarrow$ uses the free boundary theory as developed by [Chanillo, Kenig & To, JEMS, 2008]. First place where V_0 large enough is used.
2. In [M., JDE, 2020] in the case of the ball, explicit computations.
3. In [Chambolle, M., Privat, Math. Ann., 2025]: indirect (non computational) argument to obtain a good enough control of the second order shape derivative. V_0 large enough also used.

Strategy of proof

We fix a minimiser V^* and V such that

$$\|V - V^*\|_{L^1(\Omega)} = \delta \ll 1.$$

To further simplify we assume that

$$V_\delta = \mathbb{1}_{E_\delta}$$

and we let u_δ be the associated eigenfunction. To improve the eigenvalue:

$$\boxed{V'_\delta \text{ solution of } \sup_{0 \leq W \leq 1\,, \int_\Omega W = V_0} \int_\Omega W u_\delta^2 \geq \int_\Omega V_\delta u_\delta^2.}$$

Indeed

$$\begin{aligned}
\lambda(V) &= \inf_{u \in W_0^{1,2}\,, \int_\Omega u^2 = 1} \left(\int_\Omega |\nabla u|^2 - \int_\Omega V_\delta u^2 \right) = \int_\Omega |\nabla u_\delta|^2 - \int_\Omega V_\delta u_\delta^2 \\
&\geq \int_\Omega |\nabla u_\delta|^2 - \int_\Omega V'_\delta u_\delta^2 \\
&\geq \lambda(V'_\delta).
\end{aligned}$$

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$$\begin{aligned}
\lambda(V) = \inf_{u \in W_0^{1,2}\,, \int_\Omega u^2 = 1} \left(\int_\Omega |\nabla u|^2 - \int_\Omega V_\delta u^2 \right) &= \int_\Omega |\nabla u_\delta|^2 - \int_\Omega V_\delta u_\delta^2 \\
&\geq \int_\Omega |\nabla u_\delta|^2 - \int_\Omega V'_\delta u_\delta^2 + \textit{remainder} \\
&\geq \lambda(V'_\delta) + \textit{remainder}.
\end{aligned}$$

The remainder

For the remainder, recall the quantitative bathtub principle; this gives

$$V'_\delta = \{u_\delta > c_\delta\}$$

and

$$\int_\Omega V'_\delta u_\delta^2 \geq \int_\Omega V_\delta u_\delta^2 + C\|V_\delta - V'_\delta\|^2_{L^1(\Omega)}$$

and, in turn,

$$\lambda(V_\delta) \geq \lambda(V'_\delta) + C\|V'_\delta - V_\delta\|^2_{L^1(\Omega)}.$$

What did we just do?

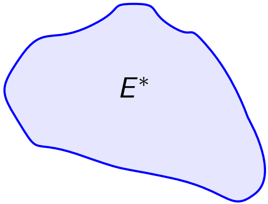

Figure: Depiction of the optimal set $E^* = \{u_{V^*} > c^*\}$.

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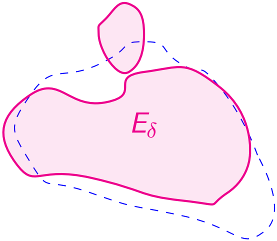

Figure: Depiction of the set E_δ s.t. $V_\delta = \mathbb{1}_{E_\delta}$; E^* is depicted in dashed blue.

What did we just do?

We replace E_δ with $E'_\delta = \{u_{E_\delta} > c_\delta\}$. We have

$$\lambda(V_\delta) - \lambda(V'_\delta) \geq C\|\mathbb{1}_{E_\delta} - \mathbb{1}_{E'_\delta}\|^2_{L^1(\Omega)}. \tag{5}$$

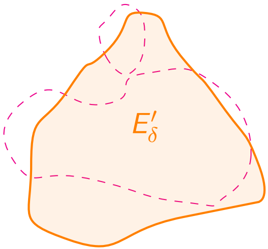

Figure: Level set E'_δ of the eigenfunction u_δ associated with E_δ and satisfying the volume constraint; E_δ is depicted in dashed magenta.

What did we just do?

But now, by elliptic regularity, $u_\delta \approx u_{V^*}$ in $W^{2,p}$.
We use the quantitative inequality for deformation of sets:

$$\lambda(V'_\delta) - \lambda(V_\delta) \geq C\|\mathbb{1}_{E'_\delta} - \mathbb{1}_{E^*}\|^2_{L^1(\Omega)}. \tag{5}$$

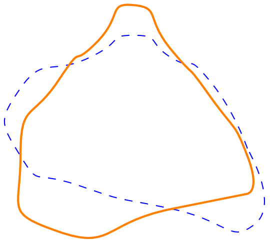

Figure: Comparison of E_δ" (in orange) and of E^* (in dashed blue).

What did we just do?

Combining these two steps we obtain

$$\lambda(V_\delta) - \lambda(V^*) \geq C \left(\|\mathbb{1}_{E'_\delta} - \mathbb{1}_{E^*}\|^2_{L^1(\Omega)} + \|\mathbb{1}_{E_\delta} - \mathbb{1}_{E'_\delta}\|^2_{L^1(\Omega)} \right). \tag{5}$$

Since

$$\|\mathbb{1}_{E^*} - \mathbb{1}_{E_\delta}\|_{L^1(\Omega)} = \delta \tag{6}$$

we have either (up to a subsequence)

$$\|\mathbb{1}_{E^*} - \mathbb{1}_{E'_\delta}\|_{L^1(\Omega)} \geq c_0 \delta$$

or

$$\|\mathbb{1}_{E_\delta} - \mathbb{1}_{E'_\delta}\|_{L^1(\Omega)} \geq c_1 \delta$$

so the contradiction follows and the inequality is proved.

Summarising

Recall that at an optimiser

$$V^* = \mathbb{1}_{\{u_{V^*} > c^*\}}.$$

We say that a set E is critical (or that $V = \mathbb{1}_E$ is critical) if

$$E = \{u_{\mathbb{1}_E} > c_E\}.$$

In fact we can show that if V_0 is large enough (or in the ball) then any critical set E is locally stable: in a small L^1 ball,

$$\lambda(V) - \lambda(\mathbb{1}_E) \geq C\|V - \mathbb{1}_E\|^2_{L^1(\Omega)}.$$

How to numerically approximate the optimal potentials?

$$\min_{0\leq V\leq 1\,,\int V=V_0} \lambda(V) \text{ first eigenvalue of } \begin{cases} -\Delta u_V = \lambda(V)u_V + Vu_V\,, \\ u_V \in W^{1,2}_0(\Omega). \end{cases}$$

Recall that at a perturbation h

$$\lambda'(V)[h] = -\int_\Omega h u_V^2.$$

This suggests a gradient descent/fixed-point algorithm

1: Initialisation at V_0
2: $k \leftarrow 0$
3: Compute u_{V_k}
4: Compute c_k such that $\mathrm{Vol}(\{u_{V_k} > c_k\}) = V_0$.
5: $V_k \leftarrow \mathbb{1}_{\{u_{V_k} > c_k\}}$
6: $k \leftarrow k+1$.

Does this algorithm converge?

The thresholding algorithm: is it successful?

The answer is **yes**. Remarkably efficient:

1. First derived by Céa, Gioan and Michel for some shape optimisation problems.
2. Optimisation of the Dirichlet energy, of eigenvalues etc...
   1. Kao, Lou, Yanagida, Mathematical biosciences and engineering, 2008
   2. Hintermüller, Kao, Laurain, Applied Mathematics & Optimization, 2012
   3. Lamboley, Laurain, Nadin, Privat, Calculus of Variations & PDEs, 2016.
3. Generalises to large classes of optimal control problems
   1. Ding, Finotti, Lenhart, Lou, Ye, Nonlinear analysis: Real world applications, 2010,
   2. M-F, Nadin, Privat, Journal de Mathématiques Pures et Appliquées, 2020,
   3. M-F, Ruiz-Balet, SIAM Journal on Applied Mathematics, 2021,
   4. Kao, Mohammadi, Journal of Mathematical Biology, 2022.
4. Topology optimisation:
   1. Amstutz, Optimization Methods and Software, 2011
   2. Amstutz, Dapogny, Ferrer, Numerische Mathematik, 2018

Main difficulties

We want to obtain convergence to a local minimiser. Let's list the difficulties:

1. **Order of the algorithm:** First-order algorithm. The most optimistic outcome is that we find a critical point. Here that means finding $V = \mathbb{1}_E$ with
$$E = \{u_V > c_E\}.$$
2. **Degenerate minimisers and regularity properties:** No unambiguous notion of "non-degenerate critical point". Quantitative inequalities will play the role of non-degeneracy conditions. They require some regularity of minimisers E^*.

What we show

With A. Chambolle, Y. Privat:

1. Convergence of the algorithm for large volume constraints ($V_0 \sim |\Omega|$)
2. Holds for the Dirichlet energy, for eigenvalue optimisation, for some classes of non-energetic optimal control problems.
3. First "general" convergence result.
    1. Kao, Mohammadi, Osting, Journal of Scientific Computing, 2021: linear convergence of rearrangement algorithms in the one-dimensional case, based on explicit computations.

A related scheme

$$V_k \Rightarrow -\Delta u_k = \Psi(V_k, u_{V_k}) \Rightarrow V_{k+1} = \mathbb{1}_{\{u_k > c_k\}}$$

Or, equivalently, $u_k = G_k \star \Psi(u_k, V_k)$, $V_{k+1} = \mathbb{1}_{\{u_k > c_k\}}$

with G_k the Green kernel of a certain operator.
Falls in the category of thresholding schemes, the main one being the Bence-Merriman-Osher approximation of the mean curvature flow.

1. Merriman, Bence, Osher. Diffusion generated motion by mean curvature, 1992,
2. Bellettini, Caselles, Chambolle, Novaga, Journal de Mathématiques Pures et Appliquées, 2009,
3. Esedoglu, Otto, Communications on Pure and Applied Mathematics, 2015,
4. Laux, Otto, Calculus of Variations and Partial Differential Equations, 2016.

Main problem here: the kernel can depend on the iteration and presence of boundary conditions.

A rough idea of the proof

Recall: λ is concave. In particular,

$$\lambda(V_{k+1}) - \lambda(V_k) \leq \lambda'(V_k)[V_{k+1} - V_k] = \int_\Omega V_k u_k^2 - \int_\Omega V_{k+1} u_k^2.$$

But we can use the quantitative bathtub principle (again...)

$$\lambda(V_{k+1}) - \lambda(V_k) \leq -C\|V_{k+1} - V_k\|_{L^1(\Omega)}^2.$$

Summing:

$$\sum_{k=0}^{\infty} \|V_{k+1} - V_k\|_{L^1(\Omega)}^2 < \infty.$$

One or infinitely many closure points

$$\sum_{k=0}^{\infty} \|V_{k+1} - V_k\|_{L^1(\Omega)}^2 < \infty \Rightarrow \|V_{k+1} - V_k\|_{L^1(\Omega)} \underset{k\to\infty}{\rightarrow} 0.$$

Conclusion: the sequence $\{V_k\}_{k\in\mathbb{N}}$ has exactly one or infinitely many closure points.

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Conclusion: the sequence $\{V_k\}_{k\in\mathbb{N}}$ has exactly one or infinitely many closure points.

But: If V_∞ is a closure point, V_∞ is critical.

One closure point

But we saw that critical points are strongly isolated in L^1. In particular, the sequence can only have one closure point, and so the algorithm converges to a stable local minimiser.

Generalisation to optimal control problems

$$\mathcal{L}u = g(u) + \Phi(u, V).$$

1. Ω: bounded smooth domain
2. $\mathcal{L}$: differential operator
3. u: state
4. g: semilinearity
5. V: control
6. Φ: state/control coupling.

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6. Φ: state/control coupling.

$$\max_{f \text{ admissible control}} / \min J(V) = \begin{cases} \text{Energy of the PDE,} \\ \int_\Omega j(x, u_V), \\ \text{Eigenvalue.} \end{cases}$$

Constraints on the controls

1. Pointwise (L^∞) constraints:

$$0 \leq V \leq 1 \text{ a.e.}.$$

2. Global (L^1) constraints:

$$\int_\Omega V = V_0.$$

3. We let $\mathcal{V}$ be the set of admissible controls.

The thresholding algorithm in the general case

$$\min_{0\leq V\leq 1\,,\int V=V_0} J(V) = \begin{cases} \text{Energy of the PDE,} \\ \int_\Omega j(x,u_V), \\ \text{Eigenvalue,}\end{cases} \qquad \begin{cases}\mathcal{L}u=\Phi(u,V)\,,\\ u_V\in W^{1,2}_0(\Omega).\end{cases}$$

We want to **compute good approximations of** $V \rightsquigarrow$ fixed-point on first order optimality conditions/gradient descent.

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Adjoint state: V admissible control. If J is smooth enough there exists p_V (the adjoint state) such that for any h such that $V+\varepsilon h \in \mathcal{F}$,

$$J'(f)[h] = -\int_\Omega p_V h.$$

Iterative scheme: V_k given, $h = V - V_k \rightsquigarrow V_{k+1}$ chosen as a solution of

$$\min_{V\in\mathcal{V}} -\int_\Omega p_{V_k}(V-V_k) \Leftrightarrow \min_{V\in\mathcal{V}} -\int_\Omega p_{V_k} V.$$

Conclusion: by the bathtub principle

$$V_{k+1} = \mathbb{1}_{\{p_{V_k} > c_k\}}$$

The thresholding algorithm in the general case

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$$J'(V)[h] = -\int_\Omega p_V h.$$

1: Initialisation at $V_0 \in \mathcal{V}$
2: $k \leftarrow 0$
3: Compute p_{V_k}
4: Compute c_k such that $\mathrm{Vol}(\{p_{V_k} > c_k\}) = V_0$.
5: $V_k \leftarrow \mathbb{1}_{\{p_{V_k} > c_k\}}$
6: $k \leftarrow k+1$.

The thresholding algorithm in the general case

$$\min_{0\leq V\leq 1, \int V=V_0} J(V) = \begin{cases} \text{Energy of the PDE}, \\ \int_\Omega j(x,u_V), \\ \text{Eigenvalue}, \end{cases} \qquad \begin{cases} \mathcal{L}u = \Phi(u,V), \\ u_V \in W_0^{1,2}(\Omega). \end{cases}$$

$$J'(V)[h] = -\int_\Omega p_V h.$$

1: Initialisation at $V_0 \in \mathcal{V}$
2: $k \leftarrow 0$
3: Compute p_{V_k} as the solution of a PDE
4: Compute c_k such that $\mathrm{Vol}(\{p_{V_k} > c_k\}) = V_0$.
5: $V_k \leftarrow \mathbb{1}_{\{p_{V_k} > c_k\}}$
6: $k \leftarrow k+1$.

Usually, p_V solves a PDE.

What we show

Essentially: if J is concave and V_0 is large enough, we have convergence to stable minimisers.

Perspectives

1. Lifting the large volume constraints for the study of regularity of optimisers?
2. General regularity theory?
3. Frank-Wolfe type methods?
4. Explicit rates of convergence?

Thank you!

1. Critical set: Lagrange multiplier c s.t. $E^* = \{u_{\mathbb{1}_{E^*}} > c\}$.
2. Shape Lagrangian:
$$L_{E^*}(F) = \mathscr{E}(F) + Vol(F).$$
3. Hadamard formula.

1. : E^* fixed critical set, $E^* = \{u_{E^*} > c\}$. Φ: smooth enough vector field. $E_t := (Id + t\Phi)E^*$.
2. Shape derivative u'_Φ: derivative at $t = 0$ of $t \mapsto u_{E_t}$:

$$\int_\Omega \langle \nabla u, \nabla v\rangle = \int_{E_t} v \Rightarrow \int_\Omega \langle \nabla u', \nabla v\rangle = \int_{\partial E^*} v\langle \Phi, \nu\rangle.$$

3. Rewrites as

$$\begin{cases} -\Delta u' = 0 \text{ in } E^* \cup (E^*)^c\,, \\ [\![\partial_\nu u']\!] = -\langle \Phi, \nu\rangle \end{cases}$$

4. First order derivative of the Lagrangian:

$$L_{E^*}(E^*)' = \int \langle \nabla u, \nabla u'\rangle - \int_{E^*} u' - \int_{\partial E^*} u\langle \Phi, \nu\rangle + c^* \int_{\partial E^*} \langle \Phi, \nu\rangle$$
$$= 0 \text{ since } u_{\mathbb{1}_{E^*}} = c^* \text{ on the boundary.}$$

1. : E^* fixed critical set, $E^* = \{u_{E^*} > c\}$. Φ: smooth enough vector field. $E_t := (Id + t\Phi)E^*$.
2. $$\begin{cases} -\Delta u' = 0 \text{ in } E^* \cup (E^*)^c\,, \\ [\![\partial_\nu u']\!] = -\langle \Phi, \nu\rangle \end{cases}$$
3. For a general E,
$$L'_{E^*}(E) = \int_{\partial E} (u_E - c^*)\langle \Phi, \nu\rangle.$$
4. Second order derivative of the Lagrangian (Hadamard second formula):
$$\boxed{L_{E^*}(E^*)'' = \int_{\partial E^*} u'\langle \Phi, \nu\rangle + \int_{\partial E^*} \frac{\partial u_{E^*}}{\partial \nu}\langle \Phi, \nu\rangle^2.}$$
5. We want to find a diagonalisation basis for the shape Hessian: the expression above is not explicit enough when $\Omega\,, E^*$ are not balls.

$$L_{E^*}(E^*)'' = \int_{\partial E^*} u'\langle \Phi, \nu\rangle + \int_{\partial E^*} \frac{\partial u_{E^*}}{\partial \nu}\langle \Phi, \nu\rangle^2 \text{ with } \begin{cases} -\Delta u' = 0 \text{ in } E^* \cup (E^*)^c, \\ [\![\partial_\nu u']\!] = -\langle \Phi, \nu\rangle \end{cases}$$

1. Diagonalisation basis:

$$\begin{cases} -\Delta \psi_k = 0 & \text{in } E^* \cup (E^*)^c, \\ [\![\partial_\nu \psi_k]\!] = -\sigma_k \frac{1}{|\partial_\nu u|}\psi_k \end{cases}$$

2. Found by looking for a suitable basis.
3. In this basis we obtain

$$L''_{E^*} = \sum \alpha_k^2 \left(1 - \frac{1}{\sigma_k}\right).$$

4. Two consequences:
    1. Best coercivity norm: $L^2(\partial E^*)$
    2. The coercivity is equivalent to requiring that

$$\sigma_1 > 1.$$

1. To obtain coercivity it suffices that

$$\sigma_1 > 1 \text{ with } \sigma_1 = \inf_{v\,,\int_{\partial E^*} v^2 > 0} \frac{\int_\Omega |\nabla v|^2}{\int_{\partial E^*} \frac{1}{|u_\nu|} v^2}.$$

2. By regularity and $u_\nu \neq 0$ this is implied by

$$\mu_1 \gg 1 \text{ with } \mu_1 = \inf_{v\,,\int_{\partial E^*} v^2 > 0} \frac{\int_\Omega |\nabla v|^2}{\int_{\partial E^*} v^2}$$

3. However, we have, for v_1 the eigenfunction: $\int_{\partial E^*} v^2 = 1$, ∂E^* close to $\partial\Omega$ (large volume), $v_1 = 0$ on $\partial\Omega$. We can then show that

$$\mu_1 \to \infty \text{ as } V_0 \to |\Omega|.$$

4. Once we have coercivity, we can adapt the tools of Dambrine & Lamboley 2019 $\rightsquigarrow$ critical points are isolated.