The sharp σ_2-curvature inequality on the sphere in quantitative form

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1. Introduction to the stability of functional inequalities

The Sobolev inequality and its stability

The Sobolev inequality for $d > 2$

$$\|\nabla u\|^2_{L^2(\mathbb{R}^d)} \geq S^{Sob}\|u\|^2_{L^{2^*}(\mathbb{R}^d)}$$

with equality iff $u = Q$ *up to symmetries, where* $Q(x) = (1 + |x|^2)^{-\frac{d-2}{2}}$

Question of stability (qualitatively): If $\|\nabla u\|_2^2/\|u\|_{2^*}^2$ is almost S^{Sob}, is u almost equal to Q (up to symmetries)?

Question of stability (quantitatively, Brezis-Lieb 1985): Can we bound the "distance" of u to the set of optimizers by $\|\nabla u\|_2^2/\|u\|_{2^*}^2 - S^{Sob}$ from above?

Theorem (Bianchi-Egnell (1991))

$$\frac{\|\nabla u\|^2_{L^2(\mathbb{R}^d)}}{\|u\|^2_{L^{2^*}(\mathbb{R}^d)}} - S^{Sob} \gtrsim_d \inf_{a,r,\lambda} \left(\frac{\|\nabla(u - \lambda Q((\cdot - a)/r))\|_{L^2(\mathbb{R}^d)}}{\|\nabla u\|_{L^2(\mathbb{R}^d)}} \right)^2$$

Two-step method: **global-to-local** reduction (compactness!) and **local** bound

Non-quadratic stability results

- **Degenerate stability:** Zero modes of the Hessian that are not induced by symmetries lead to higher stability exponents
  Engelstein-Neumayer-Spolaor (2022), Frank (2022, displayed), Brigati-Dolbeault-Simonov (2024), Frank-P. (2024), ...

$$\frac{\|\nabla u\|_{L^2(M)}^2 + \frac{(d-2)^2}{4}\|u\|_{L^2(M)}^2}{\|u\|_{L^{2^*}(M)}^2} - Y(M) \gtrsim_d \left(\frac{\|u-\bar{u}\|_{W^{1,2}(M)}}{\|u\|_{W^{1,2}(M)}}\right)^4$$

  with $Y(M)$ Yamabe constant of $M = \mathbb{S}^1((d-2)^{-1/2}) \times \mathbb{S}^{d-1}(1)$.

- **Stability of the p-Sobolev inequality**, $1 < p < d$, $p^* = pd/(d-p)$:
  Cianchi-Fusco-Maggi-Pratelli (2009), Figalli-Neumayer (2018), Neumayer (2020), Figalli-Zhang (2022, displayed), ...

$$\frac{\|\nabla u\|_{L^p(\mathbb{R}^d)}^p}{\|u\|_{L^{p^*}(\mathbb{R}^d)}^p} - S_p^{Sob} \gtrsim_{d,p} \inf_{a,r,\lambda}\left(\frac{\|\nabla(u-\lambda Q_p((\cdot - a)/r))\|_{L^p(\mathbb{R}^d)}}{\|\nabla u\|_{L^p(\mathbb{R}^d)}}\right)^\alpha$$

  with optimizer Q_p and $\alpha = \max\{2, p\}$.

In this talk we are interested in the non-Hilbertian phenomenon!

2. The σ_2-curvature inequality and its stability

An inequality from conformal geometry

Let (M, g) be a Riemannian manifold of dimension $d > 2$. We are interested in

$$\mathcal{E}_k[g] := \int_M \sigma_k^g \, \mathrm{d}\,\mathrm{vol}(g) \qquad \textbf{total } \sigma_k\textbf{-curvature}.$$

We define

$$A^g := \frac{1}{d-2}\left(\mathrm{Ric}^g - \frac{\mathrm{R}^g}{2(d-1)} g\right) \qquad \textbf{Schouten tensor},$$

$$\sigma_k(\mu_1, \dots, \mu_d) := \sum_{i_1 < \dots < i_k} \mu_{i_1} \dots \mu_{i_k} \qquad k\textbf{-th elem. sym. polynomial}.$$

For $1 \leq k \leq d$ and (λ_i) eigenvalues of a diagonalizable matrix D, we write $\sigma_k(D) := \sigma_k(\lambda_1, \dots, \lambda_d)$, which are the coefficients of the characteristic polynomial of D.

Examples:

$$\sigma_1(D) = \mathrm{Tr}(D)\,, \quad \sigma_d(D) = \det(D)\,, \quad \sigma_2(D) = 1/2((\mathrm{Tr(D)})^2 - \mathrm{Tr}(D^2))$$

Then $\sigma_k^g := \sigma_k(g^{-1}A^g)$, in particular $\sigma_1^g \propto R^g$.

An inequality from conformal geometry

From now on, we consider $k = 2$, $d > 4$, and $M = \mathbb{S}^d$. We are interested in the optimization problem of minimizing the normalized total σ_2-curvature:

$$S := \inf \left\{ \frac{\mathcal{E}_2[g]}{\operatorname{vol}^{\frac{d-4}{d}}(g)} : \ g \sim g_{round} \, , \ \sigma_1^g > 0 \right\} ,$$

where $g \sim g_{round}$ means $g = v^2 g_{round}$ with $v > 0$ smooth function on $\mathbb{S}^d$.
The **σ_2-curvature inequality** is given by

$$\mathcal{E}_2[g] \geq S \operatorname{vol}^{\frac{d-4}{d}}(g) \, , \qquad g \sim g_{round} \, , \ \sigma_1^g > 0 \, .$$

Theorem (Viaclovsky (2000), Ge-Wang (2013), Case (2021))

The infimum S is attained precisely for $\Phi^* g_{round}$, where Φ is a Möbius transformation.

Question of stability: If $\mathcal{E}_2[g]/\operatorname{vol}^{\frac{d-4}{d}}(g)$ is almost S, is g almost g_{round} (up to Möbius transformations)?

Functional formulation

Recall: If we write $g = w^{\frac{4}{d-2}} g_{round}$, $0 < w \in C^\infty(\mathbb{S}^d)$, then

$$\begin{aligned} \mathrm{vol}(g) =& \|w\|_{L^{2^*}(\mathbb{S}^d)}^{2^*}, \\ \mathcal{E}_1[g] =& \frac{2}{d-2}\|\nabla w\|_{L^2(\mathbb{S}^d)}^2 + \frac{d}{2}\|w\|_{L^2(\mathbb{S}^d)}^2. \end{aligned}$$

If we write $g = u^{\frac{8}{d-4}} g_{round}$, $0 < u \in C^\infty(\mathbb{S}^d)$, and define

$$\sigma_1(u) := -\frac{d-4}{8}\Delta(u^2) - |\nabla u|^2 + \frac{d}{2}\left(\frac{d-4}{4}\right)^2 u^2 = \left(\frac{d-4}{4}\right)^2 u^{\frac{2d}{d-4}} \sigma_1^g,$$

$$e(u) := \left(\frac{4}{d-4}\right)^3 \left(\sigma_1(u) + \frac{1}{2}|\nabla u|^2 + \frac{d-2}{2}\left(\frac{d-4}{4}\right)^2 u^2\right)|\nabla u|^2,$$

then

$$\begin{aligned} \mathrm{vol}(g) =& \|u\|_{L^{4^*}(\mathbb{S}^d)}^{4^*}, \\ \mathcal{E}_2[g] =& \int_{\mathbb{S}^d} e(u)\,\mathrm{d}\omega + \frac{d(d-1)}{8}\|u\|_{L^4(\mathbb{S}^d)}^4 =: E_2[u]. \end{aligned}$$

Functional formulation

The σ_2-curvature inequality

$$\mathcal{E}_2[g] \geq S \operatorname{vol}^{\frac{d-4}{d}}(g)\,, \qquad g \sim g_{round}\,,\ \sigma_1^g > 0$$

becomes a **Sobolev-type** inequality

$$E_2[u] \geq S\|u\|^4_{L^{4^*}(\mathbb{S}^d)}, \qquad 0 < u \in C^\infty(\mathbb{S}^d)\,,\ \sigma_1(u) > 0\,.$$

The **optimizers** are $u \equiv$ const (up to Möbius transformation), that is,

$$u = \lambda(1)_\Psi\,, \qquad \Psi \text{ Möbius}, \lambda > 0.$$

A Möbius transformation Ψ acts on a function u via

$$(u)_\Psi := J_\Psi^{1/4^*} u \circ \Psi\,,$$

where J_Ψ denotes the Jacobian of Ψ.

Our main result

Theorem (Frank-P. (2024))

Let $d > 4$. Then for any $u \in C^\infty(\mathbb{S}^d)$ with $u > 0$ and $\sigma_1(u) > 0$ we have

$$\frac{E_2[u]}{\|u\|^4_{L^{4^*}(\mathbb{S}^d)}} - S \gtrsim_d \inf_{\lambda,\Psi} \left(\|\lambda(u)_\Psi - 1\|^2_{W^{1,2}(\mathbb{S}^d)} + \|\lambda(u)_\Psi - 1\|^4_{W^{1,4}(\mathbb{S}^d)} \right),$$

where the infimum is taken over numbers $\lambda > 0$ and Möbius transformations Ψ.

- Both sides of the inequality are **conformally invariant**.
- The exponents 2 and 4 are **optimal**, respectively.
- **Resemblance** to the 4-Sobolev inequality:

$$\frac{\|\nabla u\|^4_{L^4(\mathbb{R}^d)}}{\|u\|^4_{L^{4^*}(\mathbb{R}^d)}} - S_4^{Sob} \gtrsim_d \frac{1}{\|\nabla u\|^4_{L^4(\mathbb{R}^d)}} \inf_{a,r,\lambda} \Big(\|\nabla(u - \lambda Q_4((\cdot - a)/r))\|^4_{L^4(\mathbb{R}^d)} + \|\nabla(u - \lambda Q_4((\cdot - a)/r))\|^2_{L^2(\mathbb{R}^d, |\nabla \lambda Q_4((x-a)/r)|^2 \mathrm{d}x)} \Big).$$

- One novelty is the treatment of the 'uncontrolled' second-order term $\sigma_1(u)$.

3. Key ideas of the proof

Following the Bianchi-Egnell method

Define

$$d_p[u] := \inf_{\lambda,\Psi} \|\lambda(u)_\Psi - 1\|_{W^{1,p}(\mathbb{S}^d)} .$$

We consider sequences $(u_n) \subset C^\infty(\mathbb{S}^d)$ that satisfy $u_n, \sigma_1(u_n) > 0$.

Part 1 (Global-to-local reduction)

$$\frac{E_2[u_n]}{\|u_n\|_{4^*}^4} \to S \quad \Rightarrow \quad d_4[u_n] \to 0$$

Part 2 (Local bound)

$$d_4[u_n] \to 0 \quad \Rightarrow \quad \frac{E_2[u_n]}{\|u_n\|_{4^*}^4} - S \gtrsim_d (d_2[u_n]^2 + d_4[u_n]^4)(1 + o_{n\to\infty}(1))$$

By a standard **contradiction** argument, these two lemmas imply the stability result.

Part 1 (Global-to-local reduction)

$$\frac{E_2[u_n]}{\|u_n\|_{4^*}^4} \to S \quad \Rightarrow \quad d_4[u_n] \to 0$$

- Reduce to **Sobolev inequality** via monotonicity result by Guan-Wang (2004)

$$1 \leq \left(\frac{\mathcal{E}_1[g]}{\tilde{S}^{Sob} \, \mathrm{vol}^{\frac{d-2}{d}}(g)} \right)^{\frac{1}{d-2}} \leq \left(\frac{\mathcal{E}_2[g]}{S \, \mathrm{vol}^{\frac{d-4}{d}}(g)} \right)^{\frac{1}{d-4}}$$

- Apply **compactness** property of the Sobolev inequality (Lions)
- Caveat: Parametrization differs as

$$g = w_n^{\frac{4}{d-2}} g_{round} \quad \text{for } \mathcal{E}_1 \qquad \text{and} \qquad g = u_n^{\frac{8}{d-4}} g_{round} \quad \text{for } \mathcal{E}_2$$

- After Möbius transformation, $W^{1,2}$-conv. of w_n gives (a priori only) **pointwise a.e. conv.** of u_n
- Upgrade to $W^{1,4}$**-conv.** via positivity of $e(u_n)$

Part 2 (Local bound)

$$d_4[u_n] \to 0 \quad \Rightarrow \quad \frac{E_2[u_n]}{\|u_n\|_{4^*}^4} - S \gtrsim_d (d_2[u_n]^2 + d_4[u_n]^4)(1 + o_{n\to\infty}(1))$$

- Assume wlog $\|u_n\|_{4^*}^{4^*} = |\mathbb{S}^d|$ and after a Möbius transformation

$$u_n = 1 + r_n\,, \qquad \text{with } r_n \to 0 \text{ in } W^{1,4}\,.$$

- Expanding the σ_2-curvature inequality gives

$$E_2[u_n] - S\|u_n\|_{4^*}^4 = I_2[r_n] + I_3[r_n] + I_4[r_n] + \int_{\mathbb{S}^d} (\sigma_1(u_n) - \sigma_1(1))|\nabla r_n|^2 \, \mathrm{d}\omega + \text{error}$$

with $I_i[r]$ an i-homogeneous functional in r.

Problem: Spectral gap of $I_2[r_n] \propto \|\nabla r_n\|_2^2 - d\|r_n\|_2^2$ is **too small**!

Part 2 (Local bound)

$$d_4[u_n] \to 0 \quad \Rightarrow \quad \frac{E_2[u_n]}{\|u_n\|_{4^*}^4} - S \gtrsim_d (d_2[u_n]^2 + d_4[u_n]^4)(1 + o_{n\to\infty}(1))$$

- **Solution:** Split into **spherical harmonics** of degree $\ell \leq 1$, $1 < \ell < L$, and $\ell \geq L$ for some large but fixed $L \in \mathbb{N}$,

$$r_n = r_n^{lo} + r_n^{me} + r_n^{hi} \,.$$

- "Approximate" **orthogonality** conditions:

$$\|r_n^{lo}\|_{W^{1,2}} \lesssim |\langle r_n, 1\rangle_2| + |\langle r_n, \omega\rangle_2| \lesssim \|r_n\|_{W^{1,2}}^2 + \|r_n\|_{W^{1,4}}^4$$

- In cubic and quartic terms we have control of r_n^{med} via **uniform estimates**

$$\|r_n^{me}\|_{C^s} \lesssim_s \|r_n\|_{W^{1,2}} \,.$$

- $I_2[r_n^{hi}]$ admits a **large spectral gap**.

Thank you for your attention!