Mean-field limits for Coulomb-type dynamics via the modulated energy method

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The discrete coupled ODE system

Energy

$$H_N(x_1, \dots, x_N) = \frac{1}{2} \sum_{1 \le i \neq j \le N} g(x_i - x_j), \qquad x_i \in \mathbb{R}^d$$

$$\begin{cases} g(x) = \dfrac{1}{s|x|^s} & s < d \quad \textbf{Riesz case} \\ g(x) = -\log|x| & s = 0 \quad \textbf{log case} \end{cases}$$

Model case

$$\boxed{s = d - 2 \quad \textbf{Coulomb}}$$

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Evolution equation

$$\dot{x}_i=-\frac{1}{N}\nabla_i H_N(x_1,\dots,x_N) \qquad \textbf{gradient flow}$$

$$\dot{x}_i=-\frac{1}{N}\mathbb{J}\nabla_i H_N(x_1,\dots,x_N) \qquad \textbf{conservative flow} \quad (\mathbb{J}^T=-\mathbb{J})$$

$$\ddot{x}_i=-\frac{1}{N}\nabla_i H_N(x_1,\dots,x_N) \qquad \textbf{Newton's law}$$

possibly with added noise $\sqrt{\theta}dW_i^t$, N independent Brownian motions, θ=temperature

Questions

For a general system

$$\dot{x}_i = \frac{1}{N}\sum_{j\neq i} K(x_i - x_j) + \sqrt{\theta}\, dW_i^t$$

- What is the limit of the **empirical measure**? Is there μ^t such that for each t

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^t} \rightharpoonup \mu^t$$

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- if $f_N^0(x_1, \dots, x_N)$ is the probability density of position of the system at time 0, what is the limit behavior of f_N^t?
- **propagation of chaos** (Boltzmann, Kac, Dobrushin): if $f_N^0(x_1, \dots, x_N) \simeq \mu^0(x_1) \dots \mu^0(x_N)$ is it true that

$$f_N^t(x_1, \dots, x_N) \simeq \mu^t(x_1) \dots \mu^t(x_N)?$$

  in the sense of convergence of the k-point marginal $f_{N,k}$.

Formal limit

We *formally* expect $\mu_N^t := \frac{1}{N}\sum_{i=1}^N \delta_{x_i^t} \rightharpoonup \mu^t$ where μ^t solves the mean-field equation

$$\boxed{\partial_t \mu = \operatorname{div}\left((K * \mu)\mu\right) + \frac{1}{2}\theta\Delta\mu} \quad \text{(MF)}$$

So for us

$$\partial_t \mu + \operatorname{div}\left((\nabla \mathrm{g} * \mu)\mu\right) = \frac{1}{2}\theta\Delta\mu$$

How to prove propagation of chaos?

- Classical method [Mc Kean, Sznitman] for **Lipschitz** interaction kernel.
- convergence in a good metric, like Wasserstein [Braun-Hepp, Dobrushin, Neunzert-Wick,Hauray' 09, Carrillo-Choi-Hauray '14, Carrillo-Ferreira-Precioso '12, Berman-Onnheim '15]

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- **relative entropy method**: show a Gronwall relation for

$$0 \leq H_N(f_N|\mu^{\otimes N}) := \frac{1}{N}\int_{(\mathbb{R}^d)^N} f_N \log \frac{f_N}{\mu^{\otimes N}} dx_1 \dots dx_N.$$

[Jabin-Wang '16] for $\theta > 0$, kernel not too irregular.

The modulated energy method

[S, Duerinckx] Introduced for Ginzburg-Landau vortex dynamics.
Use **interaction-based metric**:

$$\|\mu-\nu\|^2 = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{g}(x-y)d(\mu-\nu)(x)d(\mu-\nu)(y).$$

Observe weak-strong uniqueness property of the solutions to (MF) for $\|\cdot\|$:

$$\|\mu_1^t-\mu_2^t\|^2 \le e^{Ct}\|\mu_1^0-\mu_2^0\|^2 \qquad C = C(\|\nabla^2(\mathrm{g}*\mu_2)\|_{L^\infty})$$

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In the discrete case, let X_N denote $(x_1,\dots,x_N)$ and take for **modulated energy**,

$$F_N(X_N,\mu) = \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} \mathrm{g}(x-y)d\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu\right)(x)d\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu\right)(y)$$

where $\triangle$ denotes the diagonal in $\mathbb{R}^d\times\mathbb{R}^d$, and $\mu=\mu^t$ solves mean-field equation.
Analogy with "relative entropy" and "modulated entropy" methods [Dafermos '79] [DiPerna '79] [Yau '91] [Brenier '00]....

The commutator estimate / main functional inequality

Computing, we find that along solutions X_N^t of the ODE system with $\theta = 0$,

$$\frac{d}{dt} F_N(X_N^t, \mu^t) \leq \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \triangle} (v(x) - v(y)) \cdot \nabla \mathrm{g}(x-y) d\Big(\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} - \mu\Big)^{\otimes 2}(x, y)$$

with $v = \nabla \mathrm{g} * \mu^t$.

The commutator estimate / main functional inequality

Theorem (S, Nguyen-Rosenzweig-S, Rosenzweig-S, Hess-Childs-Rosenzweig-S)

All Riesz cases $s < d$, v Lipschitz vector field.

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} (v(x)-v(y))\cdot\nabla \mathrm{g}(x-y)\,d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu\Big)^{\otimes 2}(x,y) \le C\|Dv\|_{L^\infty}\big(F_N(X_N,\mu)+N^{-1+\frac{s}{d}}\big).$$

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Proof by **electric formulation + stress-energy tensor structure** or by **commutator estimates**

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Proof by **electric formulation + stress-energy tensor structure** or by **commutator estimates**

Why commutator ? Let $f=\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu$

$$\int v\cdot\nabla(\mathrm{g}*f)-\mathrm{g}*(\nabla\cdot(vf)) = \langle f, \Big[v, \frac{\nabla}{(-\Delta)^{\frac{d-s}{2}}}\Big] f\rangle_{L^2}$$

Estimate used to treat the *quantum* Coulomb mean-field limit [Golse-Paul, Rosenzweig, Ben Porat], *quasi-neutral* limits [Iacobelli-Han Kwan, Rosenzweig, RS]

Theorem (S '18, H. Q. Nguyen-Rosenzweig-S '21, HCRS '25)

Case $\theta = 0$*. Assume (MF) admits a regular enough (in particular* $\mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^d))$*) solution. There exist constants* C_1, C_2 *depending on the norms of* μ^t *and* $\gamma > 0$ *depending on* d, s*, s.t.* $\forall t \in [0,T]$

$$|F_N(X_N^t, \mu^t)| \leq \left(|F_N(X_N^0, \mu^0)| + C_1 N^{-1+\frac{s}{d}} \right) e^{C_2 t}.$$

In particular, if $\mu_N^0 \rightharpoonup \mu^0$ *and is such that*

$$\lim_{N\to\infty} F_N(X_N^0, \mu^0) = 0,$$

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The electric rewriting of the energy

Set $h^f = \mathrm{g} * f$. In the Coulomb case

$$-\Delta h^f = c_d f$$

We have by IBP

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{g}(x-y) df(x) df(y) = \int_{\mathbb{R}^d} h^f df = -\frac{1}{c_d} \int_{\mathbb{R}^d} h^f \Delta h^f = \frac{1}{c_d} \int_{\mathbb{R}^d} |\nabla h^f|^2.$$

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Positivity of F_N not clear! Use suitable **truncations** obtained by replacing δ_{x_i} by $\delta_{x_i}^{(r_i)}$ with $r_i =$ nearest neighbor distance. Use almost-**monotonicity** with respect to truncation parameter.

Proof in the Coulomb case

Stress-energy tensor

$$[\nabla h^f]_{ij} = 2\partial_i h^f \partial_j h^f - |\nabla h^f|^2 \delta_{ij}.$$

For regular f,

$$\mathrm{div}\ [\nabla h^f] = 2\Delta h^f \nabla h^f = -\frac{2}{c_d} f \nabla h^f.$$

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$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(x) - v(y)) \cdot \nabla \mathrm{g}(x-y) df(x) df(y)$$
$$= 2 \int_{\mathbb{R}^d} v(x) \cdot \nabla h^f(x) df(x) = -c_d \int_{\mathbb{R}^d} v \cdot \mathrm{div}\ [\nabla h^f]$$
$$= c_d \int_{\mathbb{R}^d} Dv \cdot [\nabla h^f] \leq c_d \|Dv\|_{L^\infty} \iint \mathrm{g}(x-y) df(x) df(y)$$

Then needs to be **renormalized** by truncation procedure.

The truncation method

[Hess-Childs - Rosenzweig - S '25]

Use truncation scale $\eta = N^{-1/d}$. Find "truncation" g_η satisfying

- $\mathrm{g}_\eta \leq \mathrm{g}$
- $\hat{\mathrm{g}}_\eta \geq 0$
- $|\mathrm{g} - \mathrm{g}_\eta| \leq C_\gamma \frac{\eta^\gamma}{|x|^{s+\gamma}}$ for any $\gamma > s$.

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Set

$$\mathrm{g}_\eta(x) = c_{\phi,d,s} \int_\eta^\infty t^{d-s-1} \phi_t(x) dt$$

with $\phi_t = t^{-d}\phi(\frac{\cdot}{t})$ and ϕ is a Bessel potential, defined by $\hat{\phi}(\xi) = (1 + 4\pi^2|\xi|^2)^{-s/2}$, hence fundamental solution to $(-\Delta + I)^{s/2}$

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Handle $\iint \nabla \mathrm{g}_\eta(x-y) \cdot (v(x) - v(y)) d(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu)^{\otimes 2}(x,y)$ by stress-tensor structure +Kato-Ponce commutator estimates, using Bessel nature.

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Control $\iint \nabla(\mathrm{g} - \mathrm{g}_\eta)(x-y) \cdot (v(x) - v(y)) d(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu)^{\otimes 2}(x,y)$ more brutally by small-scale interaction control obtained by decomposing g as $\mathrm{g}_\eta + (\mathrm{g} - \mathrm{g}_\eta)$ in definition of F_N.

Modulated free energy

In the cases of gradient flow **with noise**
Modulated free energy method [Bresch-Jabin-Wang]

$$\mathcal{F}_N^\theta(f_N,\mu) := \theta H_N(f_N|\mu^{\otimes N}) + \int F(X_N,\mu)df_N(X_N).$$

Introduce **modulated Gibbs measure**

$$\mathbb{Q}_{N,\theta}(\mu) = \frac{1}{K_{N,\theta}(\mu)} e^{-\frac{1}{\theta}NF_N(X_N,\mu)} d\mu(x_1)\dots d\mu(x_N)$$

Then (remark in [Rosenzweig-S])

$$\mathcal{F}_N^\theta(f_N,\mu) = \theta H_N(f_N|\mathbb{Q}_{N,\theta}(\mu)) - \underbrace{\frac{1}{N}\log K_{N,\theta}(\mu)}_{\text{o(1) constant}}$$

Evolution of modulated free energy

When μ^t solves (MF)

$$\frac{d}{dt}\mathcal{F}_N^\theta(f_N^t,\mu^t) = -\frac{\theta^2}{N}\underbrace{\int \left|\nabla\sqrt{\frac{f_N^t}{\mathbb{Q}_{N,\theta}(\mu^t)}}\right|^2 d\mathbb{Q}_{N,\theta}(\mu^t)}_{\text{relative Fisher information}}$$

$$+\frac{1}{2}\int df_N^t \underbrace{\iint_{\triangle^c}(v^t(x)-v^t(y))\cdot\nabla \mathrm{g}(x-y)d\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu^t\right)^{\otimes 2}(x,y)dxdy}_{\text{commutator term}}$$

where $v^t = \nabla \mathrm{g} * \mu^t + \theta\nabla\log\mu^t$

Global in time convergence?

Optimized version of the functional inequality using that $v^t = \theta \nabla \log \mu^t + \nabla \mathsf{g} * \mu^t$

$$\frac{d}{dt}\mathcal{F}_N^\theta(X_N^t, \mu^t) \leq C \|\nabla v^t\|_{L^\infty} \left(\mathcal{F}_N^\theta(X_N^t, \mu^t) + N^{\frac{s}{d}-1} \|\mu^t\|_{L^\infty}^{s/d} \right),$$

$\rightsquigarrow$ prove and exploit the **decay** rate of ∇v^t as $t \to \infty$?

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$\rightsquigarrow$ prove and exploit the **decay** rate of ∇v^t as $t \to \infty$? Works in the torus setting with exponential decay rate

Theorem (Chodron de Courcel - Rosenzweig - S '23)

Riesz case $s \in [d-2, d)$, gradient flow with additive noise on the torus. We have global in time convergence:

$$\mathcal{F}_N^\theta(f_N^t, \mu^t) \leq C \left(\mathcal{F}_N^\theta(f_N^0, \mu^0) + N^{\frac{s}{d}-1} \right).$$

In whole space, problem because $\|\mu\|_{L^\infty} \to 0$ due to lack of confinement.
Replace with use of **self-similar transformation** [Rosenzweig-S '24]

$$\xi := \frac{x}{\sqrt{t+1}} \quad \tau := \log(t+1)$$

transforms the equation into

$$\partial_\tau \mu + \mathrm{div}\ \left(\left(-\nabla \mathrm{g} * \mu - \nabla(\frac{1}{4}|\xi|^2)\right)\mu\right) = \frac{1}{2}\theta\Delta\mu$$

i.e. new added quadratic confining potential, equilibrium = Gaussian.
Modulated free energy transforms well too.

Approach by functional inequalities

- [Guillin-Le Bris-Monmarché] for relative entropy method : works for 2D point vortex system on the torus, [Gong-Wang-Xie] same on the whole plane
- [Rosenzweig-S '23] for modulated free energy method
  Exploit Fisher information term to obtain global exponential convergence under some **modulated Logarithmic Sobolev Inequality** assumption:

$$N \underbrace{H_N(f_N^t|\mathbb{Q}_{N,\theta}(\mu^t))}_{\text{relative entropy}} \le C_{LS} \underbrace{\int \left|\nabla \sqrt{\frac{f_N^t}{\mathbb{Q}_{N,\theta}(\mu^t)}}\right|^2 d\mathbb{Q}_{N,\theta}(\mu^t)}_{\text{relative Fisher}}$$

  then this + commutator estimate gives

$$\frac{d}{dt}\mathcal{F}_N^\theta \le -C\mathcal{F}_N^\theta + o(1)$$

  exponential convergence to tensorized state (**generation of chaos**)

THANK YOU FOR YOUR ATTENTION!