On Gilles Pisier's approach to Gaussian concentration, isoperimetry, and Poincaré inequalities

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Politecnico di Milano

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joint project with S. G. Bobkov

0 Plan of the talk

1. Introduction: Gilles Pisier's approach to Poincaré type inequalities
2. Generalizations to different measures
3. Application with isotropic measures
4. Application: spherically invariant measures

1 Outline

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2. Generalizations to different measures
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1 Gilles Pisier's approach

Let γ_n denote the standard Gaussian measure on $\mathbb{R}^n$, thus with density

$$\frac{d\gamma_n(x)}{dx} = (2\pi)^{-\frac{n}{2}}\, e^{-|x|^2/2}, \quad x \in \mathbb{R}^n.$$

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Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a convex function. For any smooth function f on $\mathbb{R}^n$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Psi(f(y)-f(x))\, d\gamma_n(x)\, d\gamma_n(y) \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Psi\Big(\frac{\pi}{2}\langle \nabla f(x), y\rangle\Big)\, d\gamma_n(x)\, d\gamma_n(y)$$

In particular, if f has γ_n-mean zero, then

$$\int_{\mathbb{R}^n} \Psi(f)\, d\gamma_n \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Psi\Big(\frac{\pi}{2}\langle \nabla f(x), y\rangle\Big)\, d\gamma_n(x) d\gamma_n(y).$$

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For generalizations in $\{-1,1\}^n$:

P. Ivanisvili, R. van Handel, A. Volberg, *Rademacher type and Enflo type coincide*. Ann. of Math. (2020)

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$$X(t) = X \cos t + Y \sin t$$

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Since the Gaussian measure $\gamma_{2n} = \gamma_n \otimes \gamma_n$ is rotationally invariant on $\mathbb{R}^{2n}$, the couple $(X(t), X'(t))$ represents an independent copy of (X, Y): hence the second integral does not depend on t.

1 Some remarks

Choosing $\Psi(r) = |r|^p$ with $p \geq 1$, we get a Poincaré-type inequality

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x) - f(y)|^p \, d\gamma_n(x)\, d\gamma_n(x) \;\leq\; c_p \int_{\mathbb{R}^n} |\nabla f|^p \, d\gamma_n,$$

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and, by Jensen's inequality,

$$\int_{\mathbb{R}^n} \left| f - \int_{\mathbb{R}^n} f \, d\gamma_n \right|^p d\gamma_n \;\leq\; c_p \int_{\mathbb{R}^n} |\nabla f|^p \, d\gamma_n, \text{ where}$$

$$c_p = \left(\frac{\pi}{2}\right)^p \mathbb{E}\,|\xi|^p = \left(\frac{\pi}{2}\right)^p \frac{2^{\frac{p}{2}}}{\sqrt{\pi}}\,\Gamma\Big(\frac{p+1}{2}\Big),$$

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where ξ is a normal random variable with distribution γ_1. In particular, for $p = 1$ we have that $c_1 = \sqrt{\pi/2}$ is *sharp* and is the best constant in the Gaussian isoperimetric inequality

$$\gamma_n^+(\partial A) \geq 2c_1^{-1}\,\gamma_n(A)(1 - \gamma_n(A))$$

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Let us try to follow Pisier's approach to a general positive Borel measure μ on $\mathbb{R}^n \times \mathbb{R}^n$. Take a smooth function f on $\mathbb{R}^n$ and compute the fluctuations of $\Delta = f(y) - f(x)$ under μ.

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Hence, for a convex non-negative function Ψ, by Jensen's inequality

$$\Psi(\Delta) \leq \frac{2}{\pi} \int_0^{\pi/2} \Psi\Big(\frac{\pi}{2} \langle \nabla f(x(t)), x'(t) \rangle \Big)\, dt,$$

and after integration, we get

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Psi(\Delta)\, d\mu \leq \frac{2}{\pi} \int_0^{\pi/2} \int_{\mathbb{R}^{2n}} \Psi\Big(\frac{\pi}{2} \langle \nabla f(x(t)), x'(t) \rangle \Big)\, d\mu\, dt.$$

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Then if we consider the orthogonal linear transformation

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where μ_t is the push-forward measure of μ through U_t.

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$$\widehat{\mu} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\mu_t\,dt \quad \text{and inequality (1) becomes}$$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Psi(\Delta)\,d\mu \le \int_{\mathbb{R}^{2n}}\Psi\Big(\frac{\pi}{2}\,\langle\nabla f(u), v\rangle\Big)\,d\widehat{\mu}(u,v),$$

Theorem (S.Bobkov, B.V., , Elect. Journ, Prob. 2024)

If μ is any positive Borel measure μ on $\mathbb{R}^n \times \mathbb{R}^n$, for any smooth function f on $\mathbb{R}^n$ and any convex nonnegative function ψ we have

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Let $\Psi : \mathbb{R} \to [0, \infty)$ *be a convex function. Given a smooth function* f *on* $\mathbb{R}^n$*, for any non-negative Borel measurable function* $w(x, y)$ *on* $\mathbb{R}^n \times \mathbb{R}^n$*,*

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In particular, for any $p \geq 1$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(y)-f(x)|^p\, w(x,y)\, dx\, dy \leq \Big(\frac{\pi}{2}\Big)^p \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\,\langle \nabla f(u), v\rangle\,|^p\, \widehat{w}(u,v)\, du\, dv.$$

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- If μ is rotationally invariant we have $\widehat{\mu} = \mu$, *e.g.* $\widehat{\gamma}_{2n} = \gamma_{2n}$.
- For the symmetric Bernoulli measure $\nu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$, consider
$$\mu = \nu \otimes \nu = \frac{1}{4}\left(\delta_{(1,1)} + \delta_{(1,-1)} + \delta_{(-1,1)} + \delta_{(-1,-1)}\right),$$
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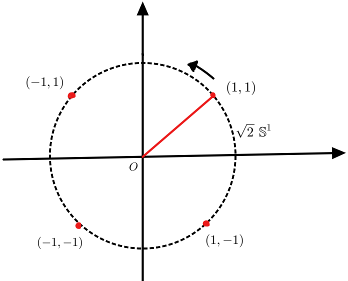

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- Let ν be the normalized Lebegue measure on $[-1,1]$, let $\mu = \nu \otimes \nu$ be the norm. Lebesgue measure on $[-1,1] \times [-1,1]$, thus with density $w(x,y) = 1/4$.

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  $$\widehat{w}(u,v) = \begin{cases} \frac{\pi}{4}, & \text{if} \quad u^2 + v^2 \le 1, \\ \arcsin \frac{1}{\sqrt{u^2+v^2}} - \frac{\pi}{4}, & \text{if} \quad 1 \le u^2 + v^2 \le 2, \\ 0, & \text{if} \quad u^2 + v^2 \ge 2. \end{cases}$$

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3. Application with isotropic measures
4. Application: spherically invariant measures

3 Disintegration of measures

As a special choice of ψ in our result, we take $\psi(t) = |t|^2$ and we obtain

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(y) - f(x)|^2 \, d\mu(x, y) \leq \frac{\pi^2}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle \nabla f(u), v \rangle^2 \, d\widehat{\mu}(u, v).$$

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Our aim is to give conditions on μ, $\widehat{\mu}$ to simplify the RHS. If ν is a finite Borel positive measure on $\mathbb{R}^n \times \mathbb{R}^n$, let π is the projection of ν on the first coordinate. By the disintegration theorem there is a (unique) family of finite measures ν_u defined fo π-almost all u, such that each ν_u is supported on $\{u\} \times \mathbb{R}^n$ and

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$$\nu = \int_{\mathbb{R}^n} \nu_u\, d\pi(u),$$

that is for each nonnegative measurable function $g(u,v)$ on $\mathbb{R}^n \times \mathbb{R}^n$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(u,v)\, d\nu(u,v) = \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} g(u,v)\, d\nu_u(v)\right] d\pi(u).$$

Thus

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\langle \nabla f(u), v\rangle^2\, d\widehat{\mu}(u,v) = \int_{\mathbb{R}^n}\underbrace{\left[\int_{\mathbb{R}^n}\langle \nabla f(u), v\rangle^2\; d\widehat{\mu}_u(v)\right]}_{\textit{how to simplify this?}} d\widehat{\pi}(u).$$

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Definition

A finite measure ν on $\mathbb{R}^n\times\mathbb{R}^n$ is *isotropic* along the first coordinate, if π-almost all conditional measures ν_u are isotropic on $\mathbb{R}^n$, i.e.

$$\int_{\mathbb{R}^n}\langle\theta,v\rangle^2\,d\nu_u(v)=\sigma^2(u)|\theta|^2\quad(\theta\in\mathbb{R}^n)$$

with some finite $\sigma^2(u)$, which we call the isotropic function of ν along the first coordinate.

Thus if we assume $\widehat{\mu}$ *isotropic along the first coordinate* we have

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} \langle \nabla f(u), v\rangle^2 \, d\widehat{\mu}(u,v) &= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \langle \nabla f(u), v\rangle^2 \, d\widehat{\mu}_u(v) \right] d\pi(u) \\ &= \int_{\mathbb{R}^n} \widehat{\sigma}^2(u) \left|\nabla f(u)\right|^2 d\pi(u),\end{aligned}$$

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therefore we have a weighted Poincaré inequality

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |f(y)-f(x)|^2 \, d\mu(x,y) \leq \left(\frac{\pi}{2}\right)^2 \int_{\mathbb{R}^n} \widehat{\sigma}^2(u)\, |\nabla f(u)|^2 \, d\pi(u).$$

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We take $\psi(t) = t^2$ and $\mu = \sigma_{2n-1}$ the uniform distribution on the sphere

$$S^{2n-1} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x|^2 + |y|^2 = 1\}.$$

We want to evaluate the conditional measures μ_x of σ_{2n-1} for $x \in B_1(0)$.

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$$S^{2n-1}_x = \sqrt{1 - |x|^2}\; S^{n-1} \quad (x \in \mathbb{R}^n,\ |x| < 1)$$

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Since μ_x is isotropic we can use

$$\int_{S_x^{2n-1}} \langle \theta, y \rangle^2 \; d\mu_x(y) = \sigma^2(x) |\theta|^2 \quad (\theta \in \mathbb{R}^n)$$

to find $\sigma^2(x)$:

$$\sigma^2(x) = \frac{1}{n} \int_{S_x^{2n-1}} |y|^2 d\nu_x(y) = \frac{1 - |x|^2}{n}, \quad |x| < 1.$$

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So we have to find the marginal measure $d\pi(x)$. An explicit computation gives

$$d\pi(x) = \frac{\Gamma(n)}{\pi^{\frac{n}{2}} \, \Gamma(\frac{n}{2})} \left(1 - |x|^2\right)^{\frac{n}{2} - 1} dx, \quad |x| < 1.$$

4 The case of the uniform distribution on the sphere

Corollary (S. G. Bobkov, B.V., 2024)

For any smooth function f on $\mathbb{R}^n$,

$$\int_{S^{2n-1}} |f(x) - f(y)|^2 \, d\sigma_{2n-1}(x, y) \leq \frac{\pi^2}{4n} \int_{B_1(0)} |\nabla f(x)|^2 \, (1 - |x|^2) \, d\pi(x).$$

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This inequality is quite similar to this Poincaré inequality

$$\int_{|x|<1} \int_{|y|<1} |f(x) - f(y)|^2 \, d\pi(x) \, d\pi(y) \le \frac{c}{n} \int_{|u|<1} |\nabla f(u)|^2 \, d\pi(u),$$

which is derived for more general measures π in

S. G. Bobkov, *Spectral gap and concentration for some spherically symmetric probability measures. Geometric aspects of functional analysis*, 2003

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If $|\nabla f| \le 1$, we have

$$\int_{S^{2n-1}} |f(x) - f(y)|^2 \, d\sigma_{2n-1}(x,y) \le \frac{\pi^2}{8n}.$$

4 The case of Cauchy measures

The n-dimensional probability Cauchy measure $\mathfrak{m}_{n,\alpha}$ on $\mathbb{R}^n$ of order $\alpha > \frac{n}{2}$ has density

$$w_{n,\alpha}(x) = \frac{1}{c_{n,\alpha}} (1 + |x|^2)^{-\alpha}, \quad x \in \mathbb{R}^n,$$

where

$$c_{n,\alpha} = \frac{\Gamma(\alpha - \frac{n}{2})\,\Gamma(\frac{n}{2})}{\Gamma(\alpha)} = \pi^{\frac{n}{2}} \frac{\Gamma(\alpha - \frac{n}{2})}{\Gamma(\alpha)}.$$

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$$\widetilde{w}_{n,\alpha}(x) = (2\alpha)^{-n/2}\, w_{n,\alpha}\Big(\frac{1}{\sqrt{2\alpha}}\, x\Big) = \frac{1}{c'_{n,\alpha}(1 + \frac{1}{2\alpha}\, |x|^2)^{\alpha}}, \quad x \in \mathbb{R}^n,$$

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thus

$$\widetilde{w}_{n,\alpha}(x) \to (2\pi)^{-n/2}\,e^{-|x|^2/2} \quad \text{as} \quad \alpha \to \infty,$$

$$\widetilde{\mathfrak{m}}_{n,\alpha} \to \gamma_n \quad \text{as} \quad \alpha \to \infty :$$

the class of Cauchy measures might serve as pre-Gaussian model.

4 Poincaré type inequalities with the Cauchy measures

The following inequality is well-known

$$\int_{\mathbb{R}^n} \left| f - \int_{\mathbb{R}^n} f \, d\mathfrak{m}_{n,\alpha} \right|^2 d\mathfrak{m}_{n,\alpha} \leq c \int_{\mathbb{R}^n} |\nabla f|^2 \left(1 + |x|^2\right) d\mathfrak{m}_{n,\alpha},$$

where the value of the best constant $c = c(\alpha, n)$ is known for any $\alpha > n/2$.

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where the value of the best constant $c = c(\alpha, n)$ is known for any $\alpha > n/2$. Such Poincaré inequality are fundamental in the study of the rates of convergence for the solutions to the *fast diffusion equation*

$$u_t = \Delta u^m, \quad m \in (0,1) :$$

one of the main connection is due to the form of the self-similar profile

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Some references:

- A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo and J. L. Vázquez, ARMA-PNAS, 2009
- S.G. Bobkov and M. Ledoux, Ann. Prob., 2009

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We consider the $2n$-dimensional Cauchy measures $\mathfrak{m}_{2n,\alpha}$ on $\mathbb{R}^n \times \mathbb{R}^n$, having the densities

$$w_{2n,\alpha}(x,y) = \frac{1}{c_{2n,\alpha}} \left(1 + |x|^2 + |y|^2\right)^{-\alpha}, \quad x, y \in \mathbb{R}^n, \tag{1}$$

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Let us try to simplify the RHS.

4 Poincaré type inequalities with the Cauchy measures

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |\langle \nabla f(x), y\rangle|^p \, d\mathfrak{m}_{2n,\alpha}(x,y) = \frac{1}{c_{2n,\alpha}} \int_{\mathbb{R}^n} I_p(x, \nabla f(x))\, dx,$$

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where, for $v \in \mathbb{R}^n$, $v = |v|\theta$,

$$I_p(x,v) = \int_{\mathbb{R}^n} \frac{|\langle v, y\rangle|^p}{(1+|x|^2+|y|^2)^\alpha}\, dy = |v|^p \underbrace{\int_{\mathbb{R}^n} \frac{|\langle \theta, y\rangle|^p}{(1+|x|^2+|y|^2)^\alpha}\, dy}_{I_p(x,\theta)}.$$

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now

$$\mathbb{E}_\theta |\langle \theta, y\rangle|^p = |y|^p G(n,p)$$

for some constant $G(n,p)$:

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$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |\langle \nabla f(x), y\rangle|^p \, d\mathfrak{m}_{2n,\alpha}(x,y) = \frac{1}{c_{2n,\alpha}} \int_{\mathbb{R}^n} I_p(x, \nabla f(x))\, dx,$$

where, for $v \in \mathbb{R}^n$, $v = |v|\theta$,

$$I_p(x,v) = \int_{\mathbb{R}^n} \frac{|\langle v, y\rangle|^p}{(1+|x|^2+|y|^2)^\alpha}\, dy = |v|^p \underbrace{\int_{\mathbb{R}^n} \frac{|\langle \theta, y\rangle|^p}{(1+|x|^2+|y|^2)^\alpha}\, dy}_{I_p(x,\theta)}.$$

But $I_p(x,\theta)$ does not depend on θ thus

$$I_p(x,\theta) = \mathbb{E}_\theta I_p(x,\theta) = \int_{\mathbb{R}^n} \frac{\mathbb{E}_\theta |\langle \theta, y\rangle|^p}{(1+|x|^2+|y|^2)^\alpha}\, dy;$$

now

$$\mathbb{E}_\theta |\langle \theta, y\rangle|^p = |y|^p G(n,p)$$

for some constant $G(n,p)$:taking $y = e_1$ we have $G(n,p) = \mathbb{E}_\theta |\theta_1|^p$.

One can show that

$$G(n,p) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{p+1}{2})}{\Gamma(\frac{n+p}{2})}.$$

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Therefore

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this last integral is finite if and only if

$$\alpha > \frac{n+p}{2}.$$

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An explicit computation gives

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$$I_p(x,v) = A\,(1+|x|^2)^{-\beta}\,|v|^p, \quad \text{where } \beta = \alpha - \frac{n+p}{2},$$

and

$$A = G(n,p)\,\frac{n\omega_n}{2}\,B\Big(\alpha - \frac{n+p}{2}, \frac{n+p}{2}\Big)$$

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Therefore:

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |\,\langle \nabla f(x), y\rangle\,|^p\,d\mathfrak{m}_{2n,\alpha}(x,y) = \frac{1}{c_{2n,\alpha}}\int_{\mathbb{R}^n} I_p(x,\nabla f(x))\,dx$$

$$= \frac{1}{\sqrt{\pi}}\,\frac{\Gamma(\frac{p+1}{2})\Gamma(\alpha-\frac{2n+p}{2})}{\Gamma(\alpha-n)}\int_{\mathbb{R}^n} |\nabla f(x))|^p\,d\mathfrak{m}_{n,\beta}\,:$$

in order to have $\mathfrak{m}_{n,\beta}$ well defined we must impose

$$\beta = \alpha - \frac{n+p}{2} > \frac{n}{2}.$$

Theorem (S.G. Bobkov, B.V, 2024)

Let $\alpha > n + \frac{1}{2}$ *and* $1 \le p < 2(\alpha - n)$. *For any smooth function* f *on* $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)|^p \, d\mathfrak{m}_{2n,\alpha}(x,y) \le C\left(\frac{\pi}{2}\right)^p \int_{\mathbb{R}^n} |\nabla f(x)|^p \, d\mathfrak{m}_{n,\beta}(x),$$

where $\beta = \alpha - \frac{n+p}{2}$,*where*

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If we use the scaling $\tilde{f}(x) := f(\sqrt{\alpha}\, x)$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)|^p \, d\widetilde{\mathfrak{m}}_{2n,\alpha}(x,y) \le C\,(2\alpha)^{\frac{p}{2}} \left(\frac{\pi}{2}\right)^p \int_{\mathbb{R}^n} |\nabla f|^p \, d\widetilde{\mathfrak{m}}_{n,\beta} :$$

letting $\alpha \to \infty$ we find

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$$C = \frac{1}{\sqrt{\pi}} \, \frac{\Gamma(\frac{p+1}{2})\Gamma(\alpha - n - \frac{p}{2})}{\Gamma(\alpha - n)}.$$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)-f(y)|^p \, d\gamma_n(x)\, d\gamma_n(x) \;\le\; \left(\frac{\pi}{2}\right)^p \frac{2^{\frac{p}{2}}}{\sqrt{\pi}} \, \Gamma\left(\frac{p+1}{2}\right) \int_{\mathbb{R}^n} |\nabla f|^p \, d\gamma_n.$$

4 Isoperimetric inequalities for Cauchy measures

Writing the inequality for $p = 1$ we have for $\alpha > n + \frac{1}{2}$ and $\beta = \alpha - \frac{n+1}{2}$, for any smooth function f on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)| \, d\mathfrak{m}_{2n,\alpha}(x, y) \leq \frac{\sqrt{\pi}}{2} \frac{\Gamma(\alpha - n - \frac{1}{2})}{\Gamma(\alpha - n)} \int_{\mathbb{R}^n} |\nabla f| \, d\mathfrak{m}_{n,\beta}. \tag{2}$$

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Scaling and passing to the limit in (2) gives the L^1-Poncaré

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(y)| \, d\gamma_n(x) \, d\gamma_n(y) \; \leq \; \underbrace{\sqrt{\frac{\pi}{2}}}_{c_1} \int_{\mathbb{R}^n} |\nabla f| \, d\gamma_n.$$

$$\gamma_n^+(\partial A) \geq 2c_1^{-1} \, \gamma_n(A)(1 - \gamma_n(A))$$

4 Isoperimetric inequalities for Cauchy measures

Let $\alpha \geq n+1$, we have the following

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)| \, d\mathfrak{m}_{2n,\alpha}(x,y) \leq \sqrt{\pi} \, \frac{1}{\sqrt{\alpha - n}} \int_{\mathbb{R}^n} |\nabla f| \, d\mathfrak{m}_{n,\beta}.$$

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$$\mathfrak{m}^{+}_{n,\beta}(\partial A) \geq \frac{2\sqrt{\alpha - n}}{\sqrt{\pi}} \, \mathfrak{m}_{2n,\alpha}(A \times A^c).$$

valid for any closed set A in $\mathbb{R}^n$.

Let $\alpha \geq n+1$, we have the following

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x)-f(y)|\, d\mathfrak{m}_{2n,\alpha}(x,y) \leq \sqrt{\pi}\,\frac{1}{\sqrt{\alpha-n}}\int_{\mathbb{R}^n} |\nabla f|\, d\mathfrak{m}_{n,\beta}.$$

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$$\mathfrak{m}_{2n,\alpha} \geq d\,\mathfrak{m}_{n,\alpha}\otimes\mathfrak{m}_{n,\alpha}, \quad d = d_{n,\alpha} = \frac{\Gamma(\alpha-\frac{n}{2})^2}{\Gamma(\alpha-n)\,\Gamma(\alpha)}.$$

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thus finally we find the isoperimetric inequality for the Cauchy measures

$$\mathfrak{m}^+_{n,\beta}(\partial A) \geq \frac{2}{\sqrt{\pi}} d\sqrt{\alpha - n} \, \mathfrak{m}_{n,\alpha}(A) \, (1 - \mathfrak{m}_{n,\alpha}(A))$$

Recall the Poincaré inequality

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Let $\alpha > n + \frac{1}{2}$ *and* $1 \le p < 2\,(\alpha - n)$. *For any smooth function* f *on* $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)|^p \, d\mathfrak{m}_{2n,\alpha}(x,y) \;\le\; C\Big(\frac{\pi}{2}\Big)^p \int_{\mathbb{R}^n} |\nabla f(x)|^p \, d\mathfrak{m}_{n,\beta}(x),$$

where $\beta = \alpha - \frac{n+p}{2}$, *and where the constant depends on* (n, p, α) *and is given by*

$$C = \frac{1}{\sqrt{\pi}}\,\frac{\Gamma(\frac{p+1}{2})\Gamma(\alpha - n - \frac{p}{2})}{\Gamma(\alpha - n)}.$$

It follows that, for any function f on $\mathbb{R}^n$ with Lipschitz semi-norm $\|f\|_{\mathrm{Lip}} \leq 1$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(y)|^p \, d\mathfrak{m}_{2n,\alpha}(x, y) \;\leq\; \frac{1}{\sqrt{\pi}} \left(\frac{\pi}{2}\right)^p \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\alpha - n - \frac{p}{2}\right)}{\Gamma(\alpha - n)}.$$

4 Moderate and large deviations

It follows that, for any function f on $\mathbb{R}^n$ with Lipschitz semi-norm $\|f\|_{\mathrm{Lip}} \leq 1$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x) - f(y)|^p \, d\mathfrak{m}_{2n,\alpha}(x,y) \;\leq\; \frac{1}{\sqrt{\pi}} \left(\frac{\pi}{2}\right)^p \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\alpha - n - \frac{p}{2}\right)}{\Gamma(\alpha - n)}.$$

We wish to explore probabilities of moderate and large deviations of $f(x) - f(y)$ under the Cauchy measure $\mathfrak{m}_{2n,\alpha}$.

It follows that, for any function f on $\mathbb{R}^n$ with Lipschitz semi-norm $\|f\|_{\mathrm{Lip}} \leq 1$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)-f(y)|^p \, d\mathfrak{m}_{2n,\alpha}(x,y) \leq \frac{1}{\sqrt{\pi}} \left(\frac{\pi}{2}\right)^p \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\alpha - n - \frac{p}{2}\right)}{\Gamma(\alpha - n)}.$$

We wish to explore probabilities of moderate and large deviations of $f(x)-f(y)$ under the Cauchy measure $\mathfrak{m}_{2n,\alpha}$.Using suitable estimates from above and below for the Gamma functions, for $\alpha \geq n+1$and $1 \leq p \leq 2(\alpha - n) - 1$ we have

Corollary

For any function f on $\mathbb{R}^n$ with $\|f\|_{\mathrm{Lip}} \leq 1$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)-f(y)|^p \, d\mathfrak{m}_{2n,\alpha}(x,y) \leq 2\left(\frac{cp}{\alpha - n}\right)^{p/2}$$

with $c = \pi^2/4$.

4 Moderate and large deviations

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Now consider

$$p_{n,\alpha}(t) = \mathfrak{m}_{2n,\alpha}\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \sqrt{\alpha - n}\,|f(x) - f(y)| \geq t\}.$$

We have the following result

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We have the following result

Corollary

If $\alpha \geq n+1$, for any function f on $\mathbb{R}^n$ with $\|f\|_{\mathrm{Lip}} \leq 1$,

$$p_{n,\alpha}(t) \leq \begin{cases} 2\,\exp\{-t^2/14\}, & 0 \leq t \leq t_0, \\ 2\,\exp\{-(t\log t)/5\}, & t_0 \leq t \leq t_1, \\ 2\left(\frac{2t_0}{t}\right)^{t_1}, & t \geq t_1, \end{cases} \tag{3}$$

where $t_0 = \sqrt{\alpha - n}$ and $t_1 = \alpha - n$.

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We have the following result

Corollary

If $\alpha \geq n+1$, for any function f on $\mathbb{R}^n$ with $\|f\|_{\mathrm{Lip}} \leq 1$,

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Analogous relations for similar regions for the product measures $\mathfrak{m}_{n,\alpha} \otimes \mathfrak{m}_{n,\alpha}$ have been explored in

S.G. Bobkov and M. Ledoux *Weighted Poincaré-type inequalities for Cauchy and other convex measures*, Ann. Prob., 2009

4 Some open problems

- Study more properties about the spherical cap measure $\widehat{\mu}$ and relevant implications.

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- Generalize Pisier's approach in the discrete setting.

Thank you for your attention!

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