Super-Hedging under Transaction Costs

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Outline

Introduction

Mathematical Tools

Main steps

Choose Your Moves Wisely!

You hold a portfolio that should be worth 1000 in 3 months.

- You want to hedge the risk of market movements.
- You plan to adjust your position at several dates.
- But each trade comes with **transaction costs**.

Key question:

*What is the **minimal initial price** I need today to hedge my portfolio safely?*

Framework

- Discrete-time financial market:
  $(\Omega, (\mathcal{F}_t)_{t=0}^{T}, \mathbb{P})$
- A riskless bond with price: $S_t^0 = 1$ (zero interest rate).
- A risky asset with price process:
  $S = (S_t)_{t=0}^{T}$.
- Transaction costs represented by proportional rates:
  $(k_t)_{t=0}^{T-1}$.
- **Investment strategy:** $\theta = (\theta_t)_{t=0}^{T-1}$, representing the portfolio adjustments at each trading date.

Motivation:

- Transaction costs are considered.
- No specific assumption on how prices evolve (no fixed model).
- The portfolio can be adjusted at an arbitrary number of dates.

Self-Financing Portfolio with Transaction Costs

We define a **self-financing portfolio** $(V_t)_{t=0}^{T}$ with the following dynamics:

$$\Delta V_t = \theta_{t-1} \Delta S_t - \underbrace{k_{t-1} |\Delta \theta_{t-1}| S_{t-1}}_{\text{⚠Transaction costs}}$$

❐ V_t is interpreted as a **super-hedging price** at time t for a given payoff ξ_T if

$$V_T \geq \xi_T \quad \text{almost surely.}$$

Objective

Solve the superhedging problem:

- Define the set of admissible strategies and values:

$$\mathcal{V} = \{(V, \theta) : \ V_T \geq g(S_T)\}$$

  where g is a convex payoff function.

- Compute the minimal initial capital required to superhedge:

$$P_0^* = \text{ess inf}\{V_0(\theta) : \ (V, \theta) \in \mathcal{V}\}$$

Approach:

- Use conditional supports of the relative prices.
- Rely on conditional essential supremum / infimum.
- Apply Fenchel conjugate and biconjugate tools.

Outline

Conditional Support

Definition: Let X be an $\mathbb{R}^d$-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra. The *conditional support* of X given $\mathcal{G}$, denoted $\mathrm{supp}_{\mathcal{G}}(X)$, is the smallest $\mathcal{G}$-measurable closed set-valued map $C : \Omega \to 2^{\mathbb{R}^d}$ such that

$$\mathbb{P}(X \in C \mid \mathcal{G}) = 1 \quad \text{a.s.}$$

Conditional Essential Supremum

Definition (Cardaliaguet et al., 2003): Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra. For a non-empty family $\mathcal{A}$ of $\mathcal{F}$-measurable random variables, the *conditional essential supremum* of $\mathcal{A}$ given $\mathcal{G}$, denoted $\operatorname{ess\,sup}_{\mathcal{G}} \mathcal{A}$, is the unique (a.s.) $\mathcal{G}$-measurable random variable Z such that:

- $Z \geq X$ a.s. for all $X \in \mathcal{A}$;
- if Y is any other $\mathcal{G}$-measurable random variable with $Y \geq X$ a.s. for all $X \in \mathcal{A}$, then $Y \geq Z$ a.s.

Properties of Conditional Essential Supremum

- **Tower Property:** If $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$, then

$$\operatorname{ess\,sup}_{\mathcal{H}} \operatorname{ess\,sup}_{\mathcal{G}} \mathcal{A} = \operatorname{ess\,sup}_{\mathcal{H}} \mathcal{A} \quad \text{a.s.}$$

- **Monotonicity:** If $\mathcal{A} \subseteq \mathcal{B}$, then

$$\operatorname{ess\,sup}_{\mathcal{G}} \mathcal{A} \leq \operatorname{ess\,sup}_{\mathcal{G}} \mathcal{B} \quad \text{a.s.}$$

- **Stability under $\mathcal{G}$-measurable shifts:** For any $Z \in L^0(\Omega, \mathcal{G})$,

$$\operatorname{ess\,sup}_{\mathcal{G}}\{X + Z : X \in \mathcal{A}\} = \operatorname{ess\,sup}_{\mathcal{G}} \mathcal{A} + Z \quad \text{a.s.}$$

Reference: E. N. Barron, P. Cardaliaguet, and R. Jensen, *Conditional Essential Suprema with Applications*, Applied Mathematics and Optimization, 48(3):229–253, 2003.

Key tool

Proposition

Let $X \in L^0(\mathbb{R}, \mathcal{F})$ and $h : \Omega \times \mathbb{R} \to \mathbb{R}$ be $\mathcal{G} \otimes \mathcal{B}(\mathbb{R})$-measurable and lower semicontinuous in x. Then

$$\operatorname{ess\,sup}_{\mathcal{G}} h(X) = \sup_{x \in \operatorname{supp}_{\mathcal{G}}(X)} h(x) \quad \text{a.s.}$$

Fenchel Conjugate and Biconjugate

Definition: Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a function. Its *Fenchel (convex) conjugate* f^* is defined by:

$$f^*(y) := \sup_{x \in \mathbb{R}^d} \left\{ \langle x, y \rangle - f(x) \right\}.$$

The *biconjugate* f^{**} is the conjugate of f^*:

$$f^{**}(x) := \sup_{y \in \mathbb{R}^d} \left\{ \langle x, y \rangle - f^*(y) \right\}.$$

Fenchel Moreau Theorem: If f is proper, lower semicontinuous, and convex, then:

$$f = f^{**}.$$

Reference: Rockafellar, *Convex Analysis*, 1970.

Lemma

Let f be a function from $\mathbb{R}$ to $[-\infty, \infty]$ such that $f = \infty$ on $\mathbb{R}_-$. For every real-valued convex function φ, $f^* \circ \varphi$ is a convex function.

Let $\mathcal{A}$ denote the class of all affine functions defined on $\mathbb{R}$.

Lemma

Let $\gamma \in \mathcal{A}$, $\gamma(x) := ax + b$, and φ be a bijection. Then, $(\gamma^* \circ \varphi)^*$ is an affine function given by $(\gamma^* \circ \varphi)^*(x) := \varphi^{-1}(a)x + b$ and we have $(\gamma^* \circ \varphi)^{**} = \gamma^* \circ \varphi$.

Lemma (*)

Suppose that f is a function defined on $\mathbb{R}$ with values in $]-\infty,\infty]$ such that $f=\infty$ on $\mathbb{R}_-$. Then, for every bijection Φ such that Φ^{-1} is real-valued and convex, there exists a unique lower semi-continuous convex function h such that $h^*\circ\Phi$ is lower semi-continuous, convex and $f^{**}=(h^*\circ\Phi)^*$. Moreover,

$$\begin{aligned} h &= [(h_1^*\circ\Phi)^{**}\circ\Phi^{-1}]^*; \\ h_1(x) &:= \sup\{(\gamma^*\circ\Phi^{-1})^*(x), \gamma\in\mathcal{A} \text{ and } \gamma\leq f\}. \end{aligned}$$

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Introduction

Mathematical Tools

Main steps

Firste step

- The super-hedging problem is first solved between time $T-1$ and T for an European claim $\xi_T = g_T(S_T)$ where $g_T \geq 0$ is a convex function.

Super-Hedging Prices at Time $T-1$

We denote by:

$$\mathcal{P}_{T-1}(g_T) := \mathcal{P}_{T-1}(g_T, \theta_{T-2})$$

the set of all super-hedging prices of the contingent claim $\xi_T = g_T(S_T)$.

We show that

$$\mathcal{P}_{T-1}(g_T) = \bar{\mathcal{P}}_{T-1}(g_T) + L^0(\mathbb{R}_+, \mathcal{F}_{T-1}),$$

where:

$$\bar{\mathcal{P}}_{T-1}(g_T) = \left\{ f^*_{T-1}(-\theta_{T-1}) + \theta_{T-1} S_{T-1} + k_{T-1}|\Delta\theta_{T-1}| S_{T-1} \, : \, \theta_{T-1} \in L^0(\mathbb{R}, \mathcal{F}_{T-1}) \right\},$$

$$f_{T-1}(x) = -g_T(x) + \delta_{\mathrm{supp}_{\mathcal{F}_{T-1}}(S_T)}(x).$$

Where for a (random) set $I \subseteq \mathbb{R}$:

$$\delta_I(z) := (+\infty) \cdot 1_{\Omega \setminus I}.$$

Main idea: Use the self-financing definition.

Infimum Super-Hedging Price

The infimum price of g_T at time $T-1$ is given by:

$$p_{T-1}(g_T) = -\left(f^*_{T-1} \circ \Phi^{-1}_{\theta_{T-2}}\right)^* (S_{T-1}),$$

where the function $\Phi_{\theta_{T-2}}$ is defined as:

$$\Phi_{\theta_{T-2}}(x) := x - k_{T-1} \left|x + \theta_{T-2}\right|.$$

Key Tool:

Let f be a $\mathcal{H}$-normal integrand on $\mathbb{R}$. Then:

$$\operatorname{ess\,inf}_{\mathcal{H}} \left\{ f(A) : A \in L^0(\mathbb{R}, \mathcal{H}) \right\} = \inf_{a \in \mathbb{R}} f(a).$$

Proof Outline

$$\begin{aligned}
p_{T-1}(g_T) &= \operatorname{ess\,inf} \mathcal{P}_{T-1}(g_T) \\
&= \inf_{z \in \mathbb{R}} \left[f^*_{T-1}(-z) + z S_{T-1} + k_{T-1} |z - \theta_{T-2}| S_{T-1} \right] \\
&= -\sup_{z \in \mathbb{R}} \left[-f^*_{T-1}(-z) - z S_{T-1} - k_{T-1} |z - \theta_{T-2}| S_{T-1} \right] \\
&= -\sup_{z \in \mathbb{R}} \left[-f^*_{T-1}(z) + z S_{T-1} - k_{T-1} |z + \theta_{T-2}| S_{T-1} \right] \\
&= -\sup_{z \in \mathbb{R}} \left[S_{T-1} \Phi_{\theta_{T-2}}(z) - f^*_{T-1}(z) \right] \\
&= -\sup_{z \in \mathbb{R}} \left[S_{T-1} z - f^*_{T-1} \circ \Phi^{-1}_{\theta_{T-2}}(z) \right] \\
&= -\left(f^*_{T-1} \circ \Phi^{-1}_{\theta_{T-2}} \right)^* (S_{T-1}).
\end{aligned}$$

Dual Representation of the Infimum Price

Theorem

Let g_T be a continuous convex function. Then there exists a unique lower semi-continuous convex function h_{T-1} such that:

$$p_{T-1}(g_T) = -h_{T-1}(S_{T-1}).$$

Moreover, h_{T-1} admits the representation:

$$h_{T-1} = \left[(\bar{h}^*_{T-1} \circ \Phi_{\theta_{T-2}})^{**} \circ \Phi^{-1}_{\theta_{T-2}}\right]^*,$$

where

$$\bar{h}_{T-1}(x) := \sup\left\{(\gamma^* \circ \Phi^{-1}_{\theta_{T-2}})^*(x) \;:\; \gamma \in \mathcal{A},\ \gamma \leq f_{T-1}\right\}.$$

Since $f_{T-1} = \infty$ on $\mathbf{R}_-$ and $\Phi_{\theta_{T-2}}$ is a bijection such that $\Phi^{-1}_{\theta_{T-2}}$ is convex, Lemma (*) is in force hence there exists a unique lower semi-continuous convex function denoted h_{T-1} such that $f^*_{T-1} = h^*_{T-1} \circ \Phi_{\theta_{T-2}}$. Therefore,

$$\begin{aligned} p_{T-1}(g_T) &= -(f^*_{T-1} \circ \Phi^{-1}_{\theta_{T-2}})^*(S_{T-1}) \\ &= -(h^*_{T-1} \circ \Phi_{\theta_{T-2}} \circ \Phi^{-1}_{\theta_{T-2}})^*(S_{T-1}) \\ &= -h_{T-1}(S_{T-1}). \end{aligned}$$

Proof of Theorem

Since $f_{T-1} = \infty$ on $\mathbf{R}_-$ and $\Phi_{\theta_{T-2}}$ is a bijection such that $\Phi^{-1}_{\theta_{T-2}}$ is convex, Lemma (*) is in force hence there exists a unique lower semi-continuous convex function denoted h_{T-1} such that $f^*_{T-1} = h^*_{T-1} \circ \Phi_{\theta_{T-2}}$. Therefore,

$$\begin{aligned} p_{T-1}(g_T) &= -(f^*_{T-1} \circ \Phi^{-1}_{\theta_{T-2}})^*(S_{T-1}) \\ &= -(h^*_{T-1} \circ \Phi_{\theta_{T-2}} \circ \Phi^{-1}_{\theta_{T-2}})^*(S_{T-1}) \\ &= -h_{T-1}(S_{T-1}). \end{aligned}$$

⚠The random function $h_{T-1}(\cdot)$ may depend on S_{T-1} so that the mapping $S_{T-1} \mapsto h_{t-1}(S_{T-1})$ could be non convex.

Notations

- Denote C_{T-1} the conditional support of S_T w.r.t. the σ-algebra $\mathcal{F}_{T-1}$.
- Assume that C_{T-1} is a random compact interval of the form $[m_{T-1}, M_{T-1}]$, where $m_{T-1}, M_{T-1} \in L^0(\mathbf{R}_+, \mathcal{F}_{T-1})$ satisfy $m_{T-1} < M_{T-1}$ a.s..

We define: $y_{T-1} := g_T(m_{T-1}), \quad Y_{T-1} := g_T(M_{T-1})$

m^+_{T-1}	m^-_{T-1}	M^+_{T-1}	M^-_{T-1}
$\frac{m_{T-1}}{1+k_{T-1}}$	$\frac{m_{T-1}}{1-k_{T-1}}$	$\frac{M_{T-1}}{1+k_{T-1}}$	$\frac{M_{T-1}}{1-k_{T-1}}$

- The discrete-time delta-hedging strategy at time $T-1$ is given by:

$$a^0_{T-1} := \frac{Y_{T-1} - y_{T-1}}{M_{T-1} - m_{T-1}}.$$

Explicitly!

The infimum super-replicating price of g_T at time $T-1$ satisfies:

If $a^0_{T-1} \leq \theta_{T-2}$,

$$\begin{aligned} p_{T-1}(g_T) &= -\infty, \quad S_{T-1} < m^+_{T-1} \text{ or } S_{T-1} > M^-_{T-1}, \\ &= y_{T-1} + \theta_{T-2}(S_{T-1} - m_{T-1}), \quad m^+_{T-1} \leq S_{T-1} < m^-_{T-1}, \\ &= y_{T-1} + a^0_{T-1}(S_{T-1} - m_{T-1}) + k_{T-1}(\theta_{T-2} - a^0_{T-1})S_{T-1}, \quad m^-_{T-1} \leq S_{T-1} \leq M^-_{T-1}. \end{aligned}$$

If $a^0_{T-1} \geq \theta_{T-2}$,

$$\begin{aligned} p_{T-1}(g_T) &= -\infty, \quad S_{T-1} < m^+_{T-1} \text{ or } S_{T-1} > M^-_{T-1}, \\ &= y_{T-1} + a^0_{T-1}(S_{T-1} - m_{T-1}) + k_{T-1}(a^0_{T-1} - \theta_{T-2})S_{T-1}, \quad m^+_{T-1} \leq S_{T-1} < M^+_{T-1}, \\ &= y_{T-1} + a^0_{T-1}(S_{T-1} - m_{T-1}) + (M_{T-1} - S_{T-1})(a^0_{T-1} - \theta_{T-2}), \quad M^+_{T-1} \leq S_{T-1} \leq M \end{aligned}$$

Absence of Immediate Profit (AIP)

- An immediate profit at time $t-1$ relatively to the payoff function g_t at time t is an infimum price $p_{t-1}(g_t)$ such that we have $\mathbb{P}(p_{t-1}(g_t) = -\infty) > 0$.

Definition (AIP)

We say that the relative AIP condition holds for g_t at time $t-1$ if there is no immediate profit at time $t-1$ relatively to the payoff function g_t.

Corollary

The following statements are equivalent:

1) AIP holds at time $T-1$,

2) $m^{+}_{T-1} \le S_{T-1} \le M^{-}_{T-1}$, a.s.,

3) $p_{T-1}(\theta_{T-2} = 0, g_T) \ge 0$, a.s..

Corollary

The following statements are equivalent:

1) AIP holds at time $T-1$,
2) $m^+_{T-1} \leq S_{T-1} \leq M^-_{T-1}$, a.s.,
3) $p_{T-1}(\theta_{T-2}=0, g_T) \geq 0$, a.s..

Corollary

Suppose that $m_{T-1} = \alpha_{T-1} S_{T-1}$ and $M_{T-1} = \beta_{T-1} S_{T-1}$ where $0 \leq \alpha_{T-1} < \beta_{T-1}$ are deterministic. Then, AIP holds at time $T-1$ if and only if $\alpha_{T-1} \leq 1 + k_{T-1}$ and $\beta_{T-1} \geq 1 - k_{T-1}$.

Minimum Super-Hedging Price at Time $T-1$

Let g_T be a continuous convex function and let $\theta_{T-2} \in L^0(\mathbf{R}, \mathcal{F}_{T-2})$ be given. Suppose:

- $m_{T-1} = \alpha_{T-1} S_{T-1}$ and $M_{T-1} = \beta_{T-1} S_{T-1}$,
- where $0 \leq \alpha_{T-1} < \beta_{T-1}$ are deterministic,
- and that **AIP holds at time** $T-1$.

Then, the **infimum super-hedging price** $p_{T-1}(g_T) \in \mathcal{P}_{T-1}(g_T)$ is given by:

$$\boxed{p_{T-1}(g_T) = y_{T-1} + (1-\alpha_{T-1}) a^0_{T-1} S_{T-1} + \delta_{T-1} |a^0_{T-1} - \theta_{T-2}| S_{T-1}}$$

where:

$$\delta_{T-1} = \begin{cases} \min(k_{T-1}, 1-\alpha_{T-1}) & \text{if } a^0_{T-1} \leq \theta_{T-2}, \\ \min(k_{T-1}, \beta_{T-1}-1) & \text{if } a^0_{T-1} \geq \theta_{T-2}. \end{cases}$$

Optimal strategy

Moreover, an **optimal strategy** is:

$$\boxed{\theta_{T-1}^{\mathsf{opt}} = a_{T-1}^0 \mathbf{1}_{B_{T-1}^l} + \theta_{T-2} \mathbf{1}_{B_{T-1}^r}}$$

with:

$$\begin{aligned}
B_{T-1}^l = & \Big(\{k_{T-1} \le 1 - \alpha_{T-1}\} \cap \{a_{T-1}^0 \le \theta_{T-2}\}\Big) \\
& \cup \Big(\{k_{T-1} \le \beta_{T-1} - 1\} \cap \{a_{T-1}^0 \ge \theta_{T-2}\}\Big), \\
B_{T-1}^r = & \Big(\{k_{T-1} \ge 1 - \alpha_{T-1}\} \cap \{a_{T-1}^0 \le \theta_{T-2}\}\Big) \\
& \cup \Big(\{k_{T-1} \ge \beta_{T-1} - 1\} \cap \{a_{T-1}^0 \ge \theta_{T-2}\}\Big).
\end{aligned}$$

To summarize:

- The super-hedging problem is first solved between time $T-1$ and T for an European claim $\xi_T = g_T(S_T)$ where $g_T \geq 0$ is a convex function.

  ⚠ Absence of Immediate Profit assumption.

☞ Under the assumption that the conditional supports of the relative prices $\frac{S_{t+1}}{S_t}$ are deterministic intervals, we show that at time $T-1$,

  - There exists minimal super-hedging prices

$$P^*_{T-1} = P^*_{T-1}(\theta_{T-2})$$

  ⚠ **Depends** on the strategy θ_{T-2} chosen at time $T-2$.

We get a specified form $g_{T-1}(\theta_{T-2}, S_{T-1}) = P^*_{T-1}(\theta_{T-2})$.

☆ Note that g_{T-1} is still a convex function in S_{T-1}.

Second Step:

The dependence of g_t, $t \leq T-1$, w.r.t. θ_{t-1} is iteratively identified so that the second step consists in solving the **unusual** super-hedging price problem of the form

$$V_{t-1} + \theta_{t-1}\Delta S_t - k_{t-1}|\Delta\theta_{t-1}|S_{t-1} \geq g_t(\theta_{t-1}, S_t). \tag{3.1}$$

Our assumption is the following:

- The payoff function g_t at time t is of the form:

$$g_t(\theta_{t-1}, x) = \max_{i=1,\ldots,N} g_t^i(\theta_{t-1}, x) \qquad (\star)$$

  - The mapping $x \mapsto g_t^i(\theta_{t-1}, x)$ is convex.
  - $g_t^i(\theta_{t-1}, x) = \hat{g}_t^i(x) - \hat{\mu}_t^i\theta_{t-1}x$

    $(\hat{\mu}_t^i)_{i=1}^N$ are deterministic such that $1 + \hat{\mu}_t^i > 0$.

Third Step:

- Check the propagation property to ensure the infimum super-hedging price remains a payoff function of the underlying asset in the conjectured form.
- The AIP condition is required at each step to avoid negative infinite prices.

Main idea!

$$\begin{aligned} V_{t-1} &\geq g_t(\theta_{t-1}, S_t) - \theta_{t-1}\Delta S_t + k_{t-1}|\theta_{t-1} - \theta_{t-2}|S_{t-1}, \\ &\geq \underbrace{\text{ess sup}_{\mathcal{F}_{t-1}} \left(g_t(\theta_{t-1}, S_t) - \theta_{t-1}S_t\right)} + \theta_{t-1}S_{t-1} + k_{t-1}|\Delta\theta_{t-1}|S_{t-1}. \end{aligned}$$

$$\text{ess sup}_{\mathcal{F}_{t-1}} \left(g_t(\theta_{t-1}, S_t) - \theta_{t-1}S_t\right) = \max_{i=1,\dots,N} \underbrace{\text{ess sup}_{\mathcal{F}_{t-1}} \left(g_t^i(\theta_{t-1}, S_t) - \theta_{t-1}S_t\right)}$$

$$\begin{aligned} \text{ess sup}_{\mathcal{F}_{t-1}} \left(g_t^i(\theta_{t-1}, x) - \theta_{t-1}S_t\right) &= \sup_{z \in \text{supp}_{\mathcal{F}_{t-1}}(S_t)} \left(\hat{g}_t^i(z) - \hat{\mu}_t^i \theta_{t-1} z - \theta_{t-1} z\right) \\ &= \sup_{x \in K_{t-1}^i} \left(\bar{g}_t^i(x) - \theta_{t-1}x\right) \\ &= \sup_{x \in \mathbb{R}} \left(-\theta_{t-1}x - \bar{f}_t^i(x)\right) \\ &= (\bar{f}_t^i)^*(-\theta_{t-1}). \end{aligned}$$

- V_{t-1} is a super-hedging price if and only if

$$V_{t-1} \geq \underbrace{f_t^*(-\theta_{t-1}) - \theta_{t-1} S_t + \theta_{t-1} S_{t-1} + k_{t-1} |\Delta\theta_{t-1}| S_{t-1}}_{V_{t-1}(\theta_{t-1})}$$

- The set of all super-hedging prices at time $t-1$ is given by

$$\mathcal{V}_{t-1}(g_t) = \{V_{t-1}(\theta_{t-1}) : \theta_{t-1} \in L^0(\mathbb{R}, \mathcal{F}_{t-1})\} + L^0(\mathbb{R}_+, \mathcal{F}_{t-1})$$

- The infimum super-hedging price is defined as:

$$p_{t-1}(g_t) = \text{ess inf}\, \mathcal{V}_{t-1}(g_t)$$

Theorem

The infimum super-hedging price at time $t-1$ satisfies the following:

$$p_{t-1}(g_t) = -(f_{t-1}^{*} \circ \Phi_{\theta_{t-2}}^{-1})^{*}(S_{t-1})$$

⚠ Distorted Legendre-Fenchel biconjugate makes computations more complicated.

Proposition

The AIP condition holds at time $t-1$ relatively to the payoff function g_t if and only if

$$k_{t-1} \geq \max(\alpha^1_{t-1} - 1, 1 - \beta^N_{t-1}).$$

☞ $\text{supp}_{\mathcal{F}_{t-1}} S_t / S_{t-1} = [\alpha_{t-1}, \beta_{t-1}]$

Theorem

Suppose that AIP condition holds. The infimum super-hedging price at time $t-1$ satisfies $p_{t-1}(g_t) = g_{t-1}(\theta_{t-2}, S_{t-1})$ where g_{t-1} is a payoff function of the form $(\star)$ with t replaced by $t-1$.

Proposition

Suppose that $k_{t-1} \geq |\alpha^1_{t-1} - 1|$. Suppose that g_t is a convex payoff function satisfying the form $(\star)$. Then, the infimum superhedging price $p_{t-1}(g_t)$ at time $t-1$ satisfies [1]

$$g_{t-1}(\theta_{t-2}, S_{t-1}) = \max_{(i,j)\in\Lambda_1} g^{i,j}_{t-1}(\theta_{t-2}, S_{t-1}) \vee \max_{i\in\Lambda_2} g^{1,i}_{t-1}(\theta_{t-2}, S_{t-1}),$$

where

$$\begin{aligned}
g^{i,j}_{t-1}(\theta_{t-2}, x) &= \hat{g}^{i,j}_{t-1}(x) - \hat{\mu}^{i,j}_{t-1}\theta_{t-2}x, \quad (i,j) \in \Lambda_1, \\
\hat{g}^{i,j}_{t-1}(x) &= \lambda^{i,j}_{t-1}\tilde{g}^i_t(x) + (1 - \lambda^{i,j}_{t-1})\tilde{g}^j_t(x), \quad (i,j) \in \Lambda_1, \\
g^{1,i}_{t-1}(\theta_{t-2}, x) &= \hat{g}^{1,i}_{t-1}(x) - \hat{\mu}^{1,i}_{t-1}\theta_{t-2}x, \quad i \in \Lambda_2, \\
\hat{g}^{1,i}_{t-1}(x) &= \epsilon^i_{t-1}\tilde{g}^1_{t-1}(x) + (1 - \epsilon^i_{t-1})\tilde{g}^i_{t-1}(x), \quad i \in \Lambda_2.
\end{aligned}$$

[1] We adopt the convention $\max(\emptyset) = -\infty$.

Parameter	**Expression**
ϵ^1	1
ϵ^i_{t-1}	$1 - \dfrac{\rho_{t-1}}{\alpha_{t-1}(\hat{\mu}^i_t - \hat{\mu}^1_t)} \leq 1, \quad i = 2, \ldots, N$
ϵ^i_{t-1}	$1 - \dfrac{\rho_{t-1}}{\beta^{i-N}_{t-1} - \alpha^1_{t-1}} \leq 1, \quad i = N+1, \ldots, 2N$
ρ_{t-1}	$(1 + k_{t-1}) - (1 + \hat{\mu}^1_t)\alpha_{t-1} \geq 0$

Sets:

$$\Lambda_1 = \left\{(i,j) \in \{1, \ldots, 2N\}^2 : \epsilon^i_{t-1} \leq 0, \epsilon^j_{t-1} > 0\right\},$$

$$\Lambda_2 = \left\{j \in \{1, \ldots, 2N\} : \epsilon^j_{t-1} > 0\right\}.$$

Associated Quantities:

$$\lambda^{i,j}_{t-1} = \frac{|\epsilon^j_{t-1}|(1 - \epsilon^i_{t-1})}{|\epsilon^j_{t-1} - \epsilon^i_{t-1}|} \in [0,1], \quad (i,j) \in \Lambda_1,$$

$$\hat{\mu}^{i,j}_{t-1} = k_{t-1} + \frac{\epsilon^i_{t-1}\rho_{t-1}}{1 - \epsilon^i_{t-1}} \cdot \mathbf{1}_{\{j=1\}} > -1, \quad (i,j) \in \Lambda_1,$$

$$\hat{\mu}^{1,i}_{t-1} = (\rho_{t-1} - k_{t-1})\mathbf{1}_{\{i=1\}} + k_{t-1}\mathbf{1}_{\{i \neq 1\}} > -1, \quad i \in \Lambda_2.$$

Associated Optimal Strategy

Proposition

Let g_t be a convex payoff function of the form $(\star)$. Suppose that $k_{t-1} \geq |\alpha_{t-1}^1 - 1|$, then an optimal strategy associated to the minimal super-hedging price $p_{t-1}(g_t)$ is

$$\theta_{t-1}^{opt} = a_{t-1}(\alpha_{t-1}^*)1_{\{\theta_{t-2} < a_{t-1}(0)\}} + \theta_{t-2}1_{\{\theta_{t-2} \geq a_{t-1}(0)\}},$$

where α_{t-1}^* is solution to the following minimization problem:

$$\alpha_{t-1}^* \in \arg\min_{E \in I_{t-1}} |A_{t-1}^{(1)}(E) - p_{t-1}(g_t)|,$$

$$I_{t-1} = \{I_{i,j} : \; i \neq j, \, s_{t-1}^i \leq 0 \text{ and } s_{t-1}^j \geq 0\}.$$

Parameters:

$$
\begin{array}{ll|ll}
p^i_{t-1} & = \bar{p}^{\sigma(i)}_{t-1} & \bar{p}^j_{t-1} & = \dfrac{1}{(\alpha^j_{t-1} - \alpha^1_{t-1}) S_{t-1}}, \quad j = 2, \ldots, N \\
b^i_{t-1} & = \bar{b}^{\sigma(i)}_{t-1} & \bar{b}^j_{t-1} & = \bar{p}^j_{t-1} \left(\hat{g}^j_t(\alpha_{t-1} S_{t-1}) - \hat{g}^1_t(\alpha_{t-1} S_{t-1}) \right) \\
 & & \bar{p}^j_{t-1} & = \dfrac{1}{(\beta^{j-N}_{t-1} - \alpha^1_{t-1}) S_{t-1}}, \quad j = N+1, \ldots, 2N \\
 & & \bar{b}^j_{t-1} & = \bar{p}^j_{t-1} \left(\hat{g}^{j-N}_t(\beta_{t-1} S_{t-1}) - \hat{g}^1_t(\alpha_{t-1} S_{t-1}) \right)
\end{array}
$$

$$a_{t-1} := \max_{i=2,\ldots,2N} \left(b^i_{t-1} - p^i_{t-1} \alpha \right)$$

Auxiliary Quantities:

$$d^1_{t-1} = \theta_{t-2}(1 - \alpha^1_{t-1}) S_{t-1}, \quad s^1_{t-1} = 1,$$

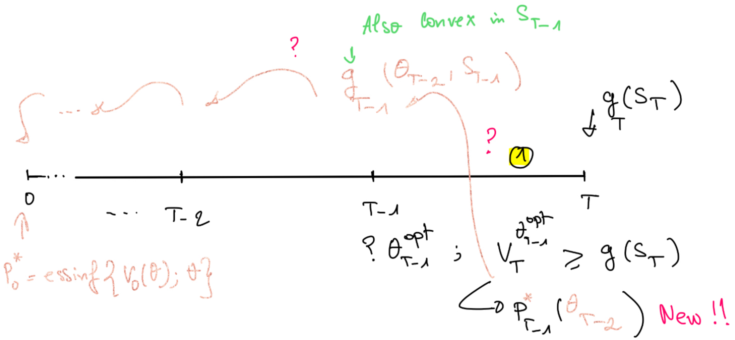
$$d^i_{t-1} = (1 - \alpha^1_{t-1}) S_{t-1} b^i_{t-1}, \quad s^i_{t-1} = 1 - (1 - \alpha^1_{t-1}) S_{t-1} p^i_{t-1}, \quad i = 2, \ldots,$$

$$A^i_{t-1}(\alpha) = d^i_{t-1} + s^i_{t-1} \alpha, \qquad i = 1, \ldots, 2N.$$

$$I_{i,j} \text{ solves } A^i_{t-1}(I_{i,j}) = A^j_{t-1}(I_{i,j}),$$

$$A^{(1)}_{t-1} = \max_{i=1,\ldots,2N} A^i_{t-1}.$$

To sum up..!



Also convex in S_{T-1}
?
$g_{T-1}(\theta_{T-2}, S_{T-1})$
$g_T(S_T)$
?
0
... T-2
T-1
T
$P_0^* = \text{essinf}\{V_0(\theta); \theta\}$
? θ_{T-1}^{opt} ; $V_T^{\theta_{T-1}^{opt}} \geq g(S_T)$
$\to P_{T-1}^*(\theta_{T-2})$ New !!

Thank you For your attention !

- Lepinette, E., Omrani, A. (2025). Beyond the Leland strategies. arXiv preprint arXiv:2503.02419.