Relative entropy and particle systems

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Plan

1. Relative entropy

2. Particle system

Relative entropy

Definition

Let μ and ν be two probability measures with densities f and g with respect to Lebesgue measure on $\mathbb{R}$. The relative entropy of ν with respect to μ is

$$H(\nu|\mu) = \int_{\mathbb{R}} f \log\left(\frac{f}{g}\right) \mathrm{d}x$$

We can also write it, if $h = f/g$ is the density of ν with respect to μ :

$$H(\nu|\mu) = \int h \log(h) \mathrm{d}\mu$$

Proposition

Due to Jensen's inequality, the relative entropy is nonnegative.

Variational formulation

Proposition

$$H(\nu|\mu) = \sup_{\varphi} \left\{ \int \varphi \mathrm{d}\nu - \log\left(\int e^{\varphi} \mathrm{d}\mu \right) \right\}$$

Proof : take a function φ. Consider the probability measure $\tilde{\mu}$ defined as

$$\mathrm{d}\tilde{\mu}(x) = \frac{e^{\varphi(x)}}{\int e^{\varphi(y)} \mathrm{d}\mu(y)} \mathrm{d}\mu(x) = \frac{e^{\varphi(x)} g(x)}{\int e^{\varphi(y)} g(y) \mathrm{d}y} \mathrm{d}x$$

Variational formulation

$$H(\nu|\tilde{\mu}) - H(\nu|\mu) = \int f(x)\left(\log\left(\frac{f(x)\int e^{\varphi(y)}g(y)\mathrm{d}y}{e^{\varphi(x)}g(x)}\right) - \log\left(\frac{f(x)}{g(x)}\right)\right)\mathrm{d}x$$
$$= \int f(x)\log\left(\frac{\int e^{\varphi(y)}g(y)\mathrm{d}y}{e^{\varphi(x)}}\right)\mathrm{d}x$$
$$= -\int \varphi(x)f(x)\mathrm{d}x + \log\left(\int e^{\varphi(x)}g(x)\mathrm{d}x\right)$$

so

$$H(\nu|\mu) = H(\nu|\tilde{\mu}) + \int \varphi(x)\mathrm{d}\nu - \log\left(\int e^{\varphi}\mathrm{d}\mu\right)$$

and $H(\nu|\tilde{\mu}) \geq 0$ with equality if $\varphi = \log\left(\frac{f}{g}\right)$. It proves the proposition.

Entropic inequality

An important consequence of the variational formulation of the relative entropy is the following theorem, called "entropic inequality".

Theorem

For all functions φ such that the quantities are well-defined, if X is a random variable with law ν, we have

$$\forall \alpha > 0, \quad \mathbb{E}[\varphi(X)] \leq \frac{1}{\alpha}\left(H(\nu|\mu) + \log\left(\int e^{\alpha\varphi}\mathrm{d}\mu\right)\right)$$

Plan

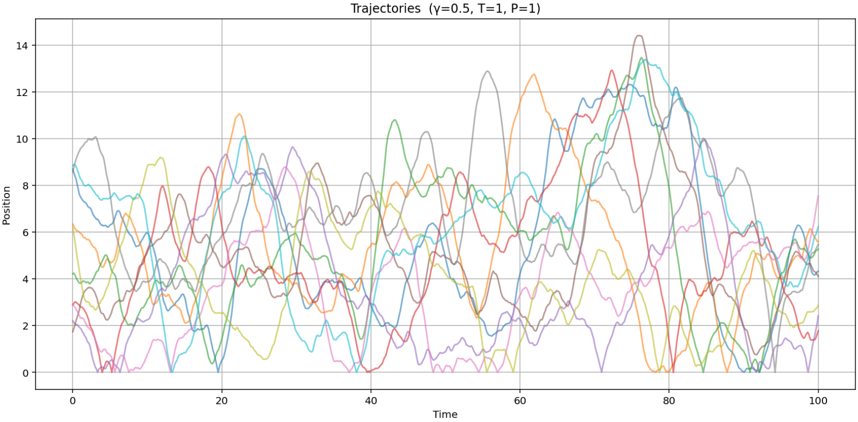
Model

We consider n particles moving on a line. q_i is the position of the particle i and p_i is its velocity. The equations of the movement are

$$\begin{aligned} \mathrm{d}q_i(t) &= p_i(t)\mathrm{d}t \\ \mathrm{d}p_i(t) &= \left(-2\delta_0(q_i(t))p_i(t^-) - P\left(\prod_{j=1}^{n} \mathbb{1}_{\{q_j(t) \leq q_i(t)\}}\right)\right)\mathrm{d}t \\ &\qquad - \gamma p_i(t)\mathrm{d}t + \sqrt{2\gamma T}\mathrm{d}W_t^i \end{aligned}$$

where the W^i are independant Brownian motions.

Simulation



Trajectories (γ=0.5, T=1, P=1)
Position
Time
0
2
4
6
8
10
12
14
0
20
40
60
80
100

Invariant measure

We say that a measure μ is an invariant measure of the system if, when the initial configuration q_0, p_0 is distributed with this measure, then it is the case at each time t.

Theorem

Let

$$g(p,q) = \frac{(\beta P)^n}{n!} \sum_{k=1}^{n} e^{-\beta P q_k} \prod_{\substack{i=1 \\ i \neq k}}^{n} \mathbb{1}_{\{q_i \leq q_k\}} \prod_{j=1}^{n} \sqrt{\frac{\beta}{2\pi}} e^{-\frac{\beta p_j^2}{2}}$$

The measure $\mu^n_{\beta,P}$ such that

$$\mathrm{d}\mu_{\beta,P} = g \mathrm{d}\lambda_{\Gamma_n}$$

(where λ_{Γ_n} is the Lebesgue measure on $\Gamma_n = (\mathbb{R}_+)^n \times \mathbb{R}^n$) is the unique invariant measure of the system.

Generator

Let $\mathcal{L} = \mathcal{A} + \gamma\mathcal{S}$ be an operator defined on

$$\mathcal{D}_n(\mathcal{L}) = \{ F \in \mathcal{C}^2\left(\mathbb{R}_+^n \times \mathbb{R}^n\right), \forall j \in [\![1, n]\!], \\ F(q, p_1, \dots, p_j, \dots, p_n)|_{q_j=0} = F(q, p_1, \dots, -p_j, \dots, p_n)|_{q_j=0} \}$$

and such that

$$\mathcal{A} = \sum_{j=1}^{n} p_j \partial_{q_j} - P \left(\prod_{i \neq j} \mathbb{1}_{\{q_j \geq q_i\}} \right) \partial_{p_j}$$

$$\mathcal{S} = \sum_{j=1}^{n} \left(T\partial_{p_j}^2 - p_j \partial_{p_j} \right)$$

Proposition

$\mathcal{S}$ is symmetric and $\mathcal{A}$ antisymmetric with respect to $\mu_{\beta,P}^n$

Generator

Proposition

The operator $\mathcal{L}$ is the generator of the system, ie

$$\forall F \in \mathcal{D}_n(\mathcal{L}), \quad \forall t \geq 0,$$
$$\mathbb{E}[F(q(t), p(t))] = \mathbb{E}[F(q_0, p_0)] + \int_0^t \mathbb{E}[\mathcal{L}F(q(s), p(s))] \mathrm{d}s$$

Application : if the law ν_t^n of $(q(t), p(t))$ is absolutely continuous with respect to $\mu_{\beta,P}^n$ with density $f_n(t, \cdot, \cdot)$ for all $t \geq 0$, then f_n satisfies the PDE

$$\partial_t f_n = \mathcal{L}^* f_n$$

(with $\mathcal{L} = -\mathcal{A} + \gamma\mathcal{S}$)

Important hypothesis

In the following we admit that such a density exists.

Relative entropy

We define

$$H_n(t) = H(\nu_t^n | \mu_{\beta,P}^n) = \int_{\Gamma_n} f_n(t,q,p) \log(f_n(t,q,p)) \mathrm{d}\mu_{\beta,P}^n.$$

Proposition

H_n is a non-increasing function.

Proof : Using the fact that $\partial_t f_n = \mathcal{L}^* f_n$, we have

$$\frac{\mathrm{d}H_n}{\mathrm{d}t}(t) = \int_{\Gamma_n} \mathcal{L}^* f_n \log(f_n) \mathrm{d}\mu_{\beta,P}^n + \int_{\Gamma_n} \partial_t f_n \mathrm{d}\mu_{\beta,P}^n$$

Relative entropy

$$\int_{\Gamma_n} \partial_t f_n \mathrm{d}\mu^n_{\beta,P} = \partial_t \int_{\Gamma_n} f_n \mathrm{d}\mu^n_{\beta,P} = \partial_t 1 = 0$$

$$\int_{\Gamma_n} f_n \mathcal{A} \log(f_n) \mathrm{d}\mu^n_{\beta,P} = \int_{\Gamma_n} \mathcal{A} f_n \mathrm{d}\mu^n_{\beta,P} = 0$$

Then

$$\begin{aligned}\frac{\mathrm{d}H_n}{\mathrm{d}t}(t) &= \gamma \int (\mathcal{S} f_n) \log(f_n) \mathrm{d}\mu_{\beta,P} \\ &= \gamma \sum_{j=1}^{n} T \int \partial^2_{p_j} f_n \log(f_n) \mathrm{d}\mu_{\beta,P} - \gamma \sum_{j=1}^{n} \int p_j \partial_{p_j} f_n \log(f_n) \mathrm{d}\mu_{\beta,P}\end{aligned}$$

Finally, by an integration by part :

$$\int p_j \partial_{p_j} f_n \log(f_n) \mathrm{d}\mu^n_{\beta,P} = T \int \partial_{p_j} \left(\partial_{p_j} f_n \log(f_n)\right) \mathrm{d}\mu^n_{\beta,P}$$

$$= T \int \left(\partial^2_{p_j} f_n \log(f_n) + \frac{\left(\partial_{p_j} f_n\right)^2}{f_n} \right) \mathrm{d}\mu^n_{\beta,P}$$

So

$$\frac{\mathrm{d}H_n}{\mathrm{d}t}(t) = -T\gamma \sum_{j=1}^{n} \frac{\left(\partial_{p_j} f_n\right)^2}{f_n} \mathrm{d}\mu^n_{\beta,P} \leq 0$$

Conjecture

For a function φ "regular enough", we set

$$\pi_n(t,\varphi) = \frac{1}{n}\sum_{i=1}^{n} \varphi\left(t, \frac{q_i(n^2 t)}{n}\right)$$

The goal is to compute the "limit" of π_n ; it is called the "hydrodynamic limit" of the particle system.

Conjecture

Conjecture

Let $\rho_0 : [0, L_0] \longrightarrow \mathbb{R}_+$ be a density function. Assume that the initial configuration ν_0^n is such that

- $\forall \varphi \in \mathcal{C}_c^2(\mathbb{R}_+), \quad \pi_n(0, \varphi) \xrightarrow[n \to +\infty]{\mathbb{P}} \int_0^{L_0} \rho_0(x)\varphi(x)\mathrm{d}x$
- $H_n = H(\nu_0^n | \mu_{\beta,P}^n) \underset{n \to +\infty}{=} O(n).$

Then, for all $\varphi \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+)$ compactly supported in space,

$$\pi_n(t, \varphi) \xrightarrow[n \to +\infty]{\mathbb{P}} \int_0^{L(t)} \rho(t, x)\varphi(t, x)\mathrm{d}x$$

where (ρ, L) is the unique weak solution of the following free boundary heat equation.

Free boundary heat equation

$$\begin{aligned} \partial_t \rho &= \frac{T}{\gamma}\partial_x^2 \rho & t \in \mathbb{R}_+, x \in [0, L(t)] \\ L'(t) &= -\frac{T^2}{\gamma P}\partial_x \rho(t, L(t)) & t \geq 0 \end{aligned}$$

$$\partial_x \rho(t,0) = 0 \qquad \rho(t, L(t)) = \frac{P}{T} \qquad \rho(0,x) = \rho_0(x)$$

Definition

(ρ, L) is a weak solution of this equation if

- $L \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, $\rho \in \mathcal{C}\left(\left\{(t,x) \in (\mathbb{R}_+)^2, x < L(t)\right\}, \mathbb{R}_+\right)$
- For all $t \geq 0$ and all test function $\varphi \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+)$ compactly supported in space such that $\partial_x \varphi(\cdot, 0) = \varphi(t, \cdot) = 0$,

$$-\int_0^t \int_0^{L(s)} \rho(s,x)\partial_t \varphi(s,x)\mathrm{d}x\mathrm{d}s - \frac{T}{\gamma}\int_0^t \int_0^{L(s)} \rho(s,x)\partial_x^2 \varphi(s,x)\mathrm{d}x\mathrm{d}s$$
$$= -\frac{P}{\gamma}\int_0^t \partial_x \varphi(s, L(s)) + \int_0^{L_0} \rho_0(x)\varphi(0,x)\mathrm{d}x.$$

Free boundary heat equation

Theorem

The equation admits at most one weak solution.

Sketch of the proof

Fix $t \geq 0$ and $\omega \in \Omega$. We can remark that $\pi_n(t,\cdot)(\omega)$ characterizes the probability measure on $\mathbb{R}_+$

$$\nu_{n,t,\omega}(A) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_A\left(\frac{q_i(n^2 t)(\omega)}{n}\right)$$

(Indeed, $\pi_n(t,\varphi) = \int_{\mathbb{R}_+} \varphi \mathrm{d}\nu_{n,t,\omega}$).
Then, we can show that $\pi_n(\omega)$ defines a continuous function from $\mathbb{R}_+$ to $\mathcal{M}_1(\mathbb{R}_+)$. Denote Q_n the probability distribution of π_n on $\mathcal{C}\left([0,t_1],\mathcal{M}_1(\mathbb{R}_+)\right)$. Q_n is a sequence in $\mathcal{M}_1\left(\mathcal{C}\left([0,t_1],\mathcal{M}_1(\mathbb{R}_+)\right)\right)$

Sketch of the proof

There are two main steps in the proof :

- Prove that the sequence $(Q_n)_{n\in\mathbb{N}}$ is tight on $\mathcal{M}_1\left(\mathcal{C}\left([0,t_1],\mathcal{M}_1(\mathbb{R}_+)\right)\right)$.
- Identify the limit point of any convergent subsequence of $(Q_n)_n$ as the weak solution of the equation.

Conclude by using the Prokhorov theorem.

Identify the limit

Suppose that for all functions $\varphi \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+)$ compactly supported in space,

$$\pi_n(t, \varphi) \xrightarrow[n \to +\infty]{\mathbb{P}} \int_0^{L(t)} \rho(t, x) \varphi(t, x) \mathrm{d}x$$

Set

$$\mathcal{M}_n(t, \partial_x \varphi) = \sqrt{2T} \sum_{i=1}^{n} \int_0^{n^2 t} \partial_x \varphi \left(s, \frac{q_i(s)}{n} \right) \mathrm{d}W_s^i$$

By a computation, we can show :

Identify the limit

$$
\begin{aligned}
\pi_n(t,\varphi) - \pi_n(0,\varphi) = &\int_0^t \pi_n(s,\partial_t\varphi)\mathrm{d}s \\
&+ \frac{1}{\gamma n^2}\left(\sum_{i=1}^n p_i(0)\partial_x\varphi\left(0, \frac{q_i(0)}{n}\right)\right. \\
&\quad \left. - \sum_{i=1}^n p_i(n^2t)\partial_x\varphi\left(t, \frac{q_i(n^2t)}{n}\right)\right) \\
&+ \frac{1}{\gamma n^2}\sum_{i=1}^n \int_0^t p_i(n^2s)\partial_{xt}\varphi\left(s, \frac{q_i(n^2s)}{n}\right)\mathrm{d}s \\
&+ \frac{1}{\gamma n}\int_0^t \sum_{i=1}^n p_i^2(n^2s)\partial_x^2\varphi\left(s, \frac{q_i(n^2s)}{n}\right)\mathrm{d}s \\
&- \frac{P}{\gamma}\int_0^t \partial_x\varphi\left(s, \frac{q_{\max}(n^2s)}{n}\right)\mathrm{d}s + \frac{1}{\sqrt{\gamma}n^2}\mathcal{M}_n(t,\partial_x\varphi)
\end{aligned}
$$

Identify the limit

Some terms are easy to identify : by hypothesis, if $\varphi(t, \cdot) = 0$, then

$$\pi_n(t, \varphi) - \pi_n(0, \varphi) \xrightarrow[n \to +\infty]{} - \int_0^{L_0} \rho_0(x)\varphi(0, x)$$

$$\int_0^t \pi_n(s, \partial_t \varphi)\mathrm{d}s \xrightarrow[n \to +\infty]{} \int_0^t \int_0^{L(s)} \rho(s, x)\partial_t \varphi(s, x)\mathrm{d}x\mathrm{d}s$$

Using some properties of the stochastic integral, we can show that

$$\frac{1}{\sqrt{\gamma} n^2} \mathcal{M}_n(t, \partial_x \varphi) \xrightarrow[n \to +\infty]{} 0$$

Identify the limit

$$
\begin{aligned}
\pi_n(t,\varphi) - \pi_n(0,\varphi) = & \int_0^t \pi_n(s,\partial_t\varphi)\mathrm{d}s \\
& + \frac{1}{\gamma n^2}\left(\sum_{i=1}^n p_i(0)\partial_x\varphi\left(0,\frac{q_i(0)}{n}\right)\right. \\
& \quad \left. - \sum_{i=1}^n p_i(n^2 t)\partial_x\varphi\left(t,\frac{q_i(n^2 t)}{n}\right)\right) \\
& + \frac{1}{\gamma n^2}\sum_{i=1}^n \int_0^t p_i(n^2 s)\partial_{xt}\varphi\left(s,\frac{q_i(n^2 s)}{n}\right)\mathrm{d}s \\
& + \frac{1}{\gamma n}\int_0^t \sum_{i=1}^n p_i^2(n^2 s)\partial_x^2\varphi\left(s,\frac{q_i(n^2 s)}{n}\right)\mathrm{d}s \\
& - \frac{P}{\gamma}\int_0^t \partial_x\varphi\left(s,\frac{q_{\max}(n^2 s)}{n}\right)\mathrm{d}s + \frac{1}{\sqrt{\gamma}n^2}\mathcal{M}_n(t,\partial_x\varphi)
\end{aligned}
$$

Identify the limit

For the term

$$\frac{1}{\gamma n^2}\sum_{i=1}^{n} p_i(n^2 t)\partial_x\varphi\left(t, \frac{q_i(n^2 t)}{n}\right),$$

we will use the entropy inequality.
Let M be a constant such that $|\partial_x\varphi| \leq M$. We have

$$\begin{aligned}\mathbb{E}\left[\left|\frac{1}{\gamma n^2}\sum_{i=1}^{n} p_i(n^2 t)\partial_x\varphi\left(t, \frac{q_i(n^2 t)}{n}\right)\right|^2\right] &\leq \frac{M^2}{\gamma^2 n^4}\mathbb{E}\left[\left(\sum_{i=1}^{n} p_i(n^2 t)\right)^2\right] \\ &\leq \frac{M^2}{\gamma^2}\sum_{i=1}^{n}\mathbb{E}\left[\frac{p_i^2(n^2 t)}{n^3}\right]\end{aligned}$$

Identify the limit

By the entropy inequality, for all $\alpha > 0$ we have

$$\mathbb{E}\left[\frac{p_i^2(n^2t)}{n^3}\right] \leq \frac{H_n(n^2t)}{\alpha} + \frac{1}{\alpha}\log\left(\int e^{\alpha\frac{p_i^2}{n^3}}\mathrm{d}\mu_{\beta,P}^n\right)$$

so

$$\sum_{i=1}^{n}\mathbb{E}\left[\frac{p_i^2(n^2t)}{n^3}\right] \leq n\frac{H_n(n^2t)}{\alpha} + \frac{n}{\alpha}\log\left(\int e^{\alpha\frac{p_1^2}{n^3}}\mathrm{d}\mu_{\beta,P}^n\right)$$

We know that the entropy is nonincreasing and $H_n(0) \underset{n\to+\infty}{=} O(n)$ so $nH_n(n^2t) \underset{n\to+\infty}{=} O(n^2)$. Then $nH_n(n^2t)/\alpha \xrightarrow[n\to+\infty]{} 0$ if $\alpha = n^\delta$ with $\delta > 2$. On the other hand,

$$\int e^{\alpha\frac{p_i^2}{n^3}}\mathrm{d}\mu_{\beta,P}^n = \int_{\mathbb{R}} \frac{e^{\alpha\frac{p^2}{n^3} - \frac{p^2}{2T}}}{\sqrt{2\pi T}}\mathrm{d}p \xrightarrow[n\to+\infty]{} 1$$

if $\alpha = n^\delta$ with $\delta < 3$.

Identify the limit

Take $\alpha = n^{2,5}$. We have finally

$$\sum_{i=1}^{n} \mathbb{E}\left[\frac{p_i^2(n^2 t)}{n^3}\right] \leq \frac{H_n(n^2 t)}{n^{1,5}} + \frac{1}{n^{1,5}} \log\left(\int e^{\frac{p_i^2}{\sqrt{n}}} \mathrm{d}\mu_{\beta,P}^n\right) \xrightarrow[n\to+\infty]{} 0$$

Conclusion :

$$\mathbb{E}\left[\left|\frac{1}{\gamma n^2} \sum_{i=1}^{n} p_i(n^2 t)\partial_x \varphi\left(t, \frac{q_i(n^2 t)}{n}\right)\right|^2\right] \xrightarrow[n\to+\infty]{} 0$$

Identify the limit

$$
\begin{aligned}
\pi_n(t,\varphi) - \pi_n(0,\varphi) &= \int_0^t \pi_n(s, \partial_t \varphi)\mathrm{d}s \\
&+ \frac{1}{\gamma n^2}\left(\sum_{i=1}^n p_i(0)\partial_x\varphi\left(0, \frac{q_i(0)}{n}\right)\right. \\
&\left. - \sum_{i=1}^n p_i(n^2 t)\partial_x\varphi\left(t, \frac{q_i(n^2 t)}{n}\right)\right) \\
&+ \frac{1}{\gamma n^2}\sum_{i=1}^n \int_0^t p_i(n^2 s)\partial_{xt}\varphi\left(s, \frac{q_i(n^2 s)}{n}\right)\mathrm{d}s \\
&+ \frac{1}{\gamma n}\int_0^t \sum_{i=1}^n p_i^2(n^2 s)\partial_x^2\varphi\left(s, \frac{q_i(n^2 s)}{n}\right)\mathrm{d}s \\
&- \frac{P}{\gamma}\int_0^t \partial_x\varphi\left(s, \frac{q_{\max}(n^2 s)}{n}\right)\mathrm{d}s + \frac{1}{\sqrt{\gamma} n^2}\mathcal{M}_n(t, \partial_x\varphi)
\end{aligned}
$$

Identify the limit

We can use again the entropy inequality to show

$$\mathbb{E}\left[\left|\frac{1}{n}\int_0^t \sum_{i=1}^{n}(p_i^2(n^2 s) - T)\partial_x^2\varphi\left(t, \frac{q_i(n^2 s)}{n}\right)\mathrm{d}s\right|\right] \xrightarrow[n\to+\infty]{} 0$$

ie

$$\frac{\gamma}{n}\int_0^t \sum_{i=1}^{n} p_i^2(n^2 s)\partial_x^2\varphi\left(t, \frac{q_i(n^2 s)}{n}\right)\mathrm{d}s \sim \frac{T}{\gamma}\int_0^t \pi_n(s, \partial_x^2\varphi)\mathrm{d}s$$

$$\xrightarrow[n\to+\infty]{} \frac{T}{\gamma}\int_0^t \int_0^{L(s)} \rho(s,x)\partial_x^2\varphi(s,x)\mathrm{d}x\mathrm{d}s$$

Identify the limit

It remains a last term :

$$\frac{P}{\gamma}\int_0^t \partial_x\varphi\left(s, \frac{q_{\max}(n^2 s)}{n}\right)\mathrm{d}s$$

It should give, in the weak formulation :

$$\frac{P}{\gamma}\int_0^t \partial_x\varphi(s, L(s))\mathrm{d}s$$

We are currently trying to show that

$$\frac{\max\left(q_1, \dots, q_n\right)\left(n^2 s\right)}{n} \xrightarrow[n\to+\infty]{} L(s)$$

but we have not finished yet.

Thank you for your attention !

All is in the title of the slide.