Exponential Estimates for Multi Type Poissonian Branching Processes

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CEREMADE

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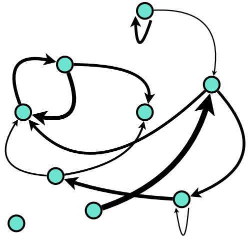

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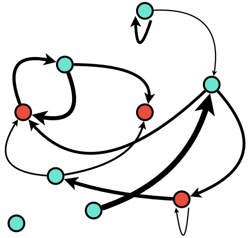

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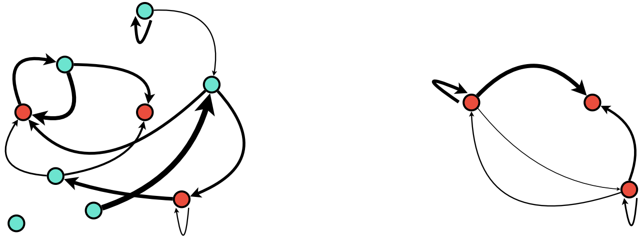

Find the best approximation of the full network interactions as a graph of interactions between the s observed nodes.

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- Given N^1, N^2, N^3, without knowing the rest of the graph of the interactions, find the best ν_i, f_j^i with $i, j = 1, 2, 3$ to approximate λ_i^i by

$$\psi_t^i(3, \nu, f) = \nu_i + \sum_{j=1}^{3} \int_{-\infty}^{t-} f_j^i(t-s)dN_s^j, \quad \text{for } i = 1, 2, 3.$$

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This statistical task requires controls on the moments of N^1, N^2, N^3: Least Square contrast

$$LS_i(\nu,f) = -2\int_0^T \psi_t^i(3,\nu,f)dN_t^i + \|\psi^i(3,\nu,f)\|_{L^2(0,T)}^2.$$

$\boldsymbol{M}$-typed tree

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- A set of all possible types $\boldsymbol{M} = [\![1, M]\!]$ with $M \in \mathbb{N}^*$.
- A $\boldsymbol{M}$-typed tree is an object $(u, \mathrm{tp}(u))_{u\in\tau}$ where τ is a tree and for any $u \in \tau$, $\mathrm{tp}(u) \in \boldsymbol{M}$ is the type of u.

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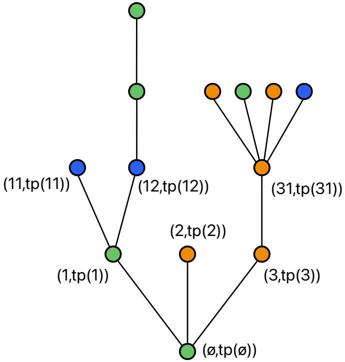
- For $\mathcal{T} = (u, \mathrm{tp}(u))_{u\in\tau}$ a $\boldsymbol{M}$-typed tree we define

$$\mathrm{Card}_{\boldsymbol{M}}(\mathcal{T}) = \Big(\mathrm{Card}(\{u \in \tau \mid \mathrm{tp}(u) = m\})\Big)_{m\in\boldsymbol{M}}.$$

Poissonian Galton Watson processes

Let $\boldsymbol{H} = (H_i^j)_{i,j\in\boldsymbol{M}}$ with $H_i^j \geq 0$. A $\mathscr{Pois}(\boldsymbol{H})$ Galton Watson tree with root of type $m_0 \in \boldsymbol{M}$ is a random $\boldsymbol{M}$-type tree $\mathcal{T}^m = (u, \mathrm{tp}(u))_{u\in\tau}$ such that

- τ is a random tree with root $\varnothing$ and $\mathrm{tp}(\varnothing) = m_0$.
- $u \in \tau$ with type $\mathrm{tp}(u) = m$ reproduces as follows: independently for each $m' \in \boldsymbol{M}$, it has $\mathscr{Pois}(H_m^{m'})$ children of type m'.



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Given $\mathcal{T}^m = (u, \mathrm{tp}(u))_{u\in\tau}$ a $\mathscr{Pois}(\boldsymbol{H})$ GW process we are interested in the moments of

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- If $u = (1, 0, \cdots, 0)$ then we only consider the first type.
- If $u = (1, 2, 1, \cdots, 1)$ then we consider all the types but type 2 counts as double.
- Depends on the type of the root.
- If $\mathrm{SpR}(\boldsymbol{H}) \geq 1$, in expectation $\mathcal{T}$ is infinite, and $\mathbb{P}\,(\text{extinction}) < 1$ if $\mathrm{SpR}(\boldsymbol{H}) > 1$.

Main result for Galton Watson trees

Definition

Let $r, K \geq 0$. A matrix $M \in \mathbf{Ge}\left(r, K\right)$ if

$$|||M^n|||_\infty \leq K r^n, \quad \forall n \geq 1.$$

$\mathrm{SpR}(\boldsymbol{H}) < 1 \iff \boldsymbol{H} \in \mathbf{Ge}\left(r, K\right)$ with $r < 1$.

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Definition

Let $u \succeq 0$, and define $\mathscr{L}(u) \in [0, \infty]^{\boldsymbol{M}}$ by

$$\mathbb{E}\left[e^{u \cdot \mathrm{Card}_{\boldsymbol{M}}(\mathcal{T}^m)}\right] = \exp\left(\boldsymbol{e}_m \cdot \mathscr{L}(u)\right)$$

where $\mathcal{T}^m$ is a $\mathscr{Pois}(\boldsymbol{H})$ GW with root of type m.

Theorem

The following holds

- *We have the following finiteness condition,*

$$|\mathscr{L}(u)|_\infty < \infty \iff \exists x \succeq 0,\ x = u + \boldsymbol{H}(e^x - \mathbf{1}). \tag{1}$$

- *If* $|\mathscr{L}(u)|_\infty < \infty$ *then* $\mathscr{L}(u)$ *is the smallest solution (for* $\preceq$*), among the solutions non negative solutions, of this equation*

$$\mathscr{L}(u) = u + \boldsymbol{H}(e^{\mathscr{L}(u)} - \mathbf{1}). \tag{2}$$

- *If* $\boldsymbol{H} \in \mathbf{Ge}\left(r, K\right)$ *with* $r < 1$*, and if we define*

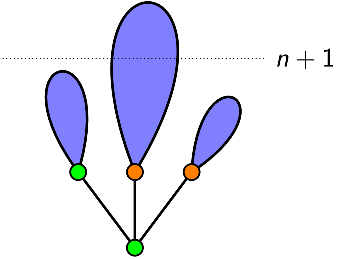
$$t_0(r, K) = \frac{\log\left(\frac{1+r}{2r}\right)}{1 + \frac{2K}{1-r}}, \tag{3}$$

for all $u \succeq 0$ *such that* $|u|_\infty \leq t_0(r, K)$ *the following holds*

$$\mathscr{L}(u) \preceq \left(\mathrm{Id} - \frac{1+r}{2r}\boldsymbol{H}\right)^{-1} u \preceq |u|_\infty \big(1 + \frac{K(1+r)}{1-r}\big)\mathbf{1}. \tag{4}$$

Sketch of proof

$$\mathrm{Card}_{\boldsymbol{M}}(\mathcal{T}^m_{\leq n+1}) \overset{d}{=} \boldsymbol{e}_m + \sum_{m' \in \boldsymbol{M}} \sum_{k=1}^{X_m^{m'}} \mathrm{Card}_{\boldsymbol{M}}(\mathcal{T}^{m',k}_{\leq n})$$



where,

- $X_m^{m'}$, $\mathcal{T}^{m',k}_{\leq n}$ for $m' \in \boldsymbol{M}$, $k \in \mathbb{N}^*$ are independent,
- $X_m^{m'} \sim \mathscr{Pois}(H_m^{m'})$,
- and $\mathcal{T}^{m',k}_{\leq n} \sim \mathcal{T}^{m'}_{\leq n}$.

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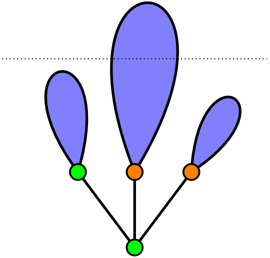

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If we denote $\mathscr{L}(u)_n = \left(\log \mathbb{E}\left[e^{u \cdot \mathrm{Card}_{\boldsymbol{M}}(\mathcal{T}^m_{\leq n})} \right] \right)_{m \in \boldsymbol{M}}$ then we have

- $\mathscr{L}(u)_0 = u = f_u(0)$,
- $\mathscr{L}(u)_{n+1} = f_u(\mathscr{L}(u)_n) = f_u^{n+1}(0)$,

with

$$f_u(x) = u + \boldsymbol{H}(e^x - \boldsymbol{1}).$$

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By Monotone Convergence Theorem

$$\mathscr{L}(u)_n \xrightarrow[n\to\infty]{} \mathscr{L}(u).$$

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Reciprocally, if $\exists y \succeq 0$ such that $y = f_u(y)$, since f_u is increasing we have

$$f_u^n(0) \leq f_u^n(y) = y, \quad \text{and thus} \quad \mathscr{L}(u) \preceq y.$$

Sketch of proof

In previous work it has been proved that if $\boldsymbol{H} \in \mathbf{Ge}\left(r, K\right)$ with $r < 1$ then we have

$$\mathscr{L}(u) \preceq \mathscr{L}(|u|_\infty \mathbf{1}) \preceq |u|_\infty \big(1 + \frac{2K}{1-r}\big)\mathbf{1} \preceq \log\big(\frac{1+r}{2r}\big)\mathbf{1}, \tag{5}$$

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The last inequality comes from

$$|||\Big(\frac{1+r}{2r}\boldsymbol{H}\Big)^{n}|||_\infty \leq K\Big(\frac{1+r}{2}\Big)^{n}, \quad n \geq 1.$$

And thus

$$|||\Big(\operatorname{Id} - \frac{1+r}{2r}\boldsymbol{H}\Big)^{-1}|||_\infty \leq 1 + K\frac{(1+r)/2}{1-(1+r)/2}.$$

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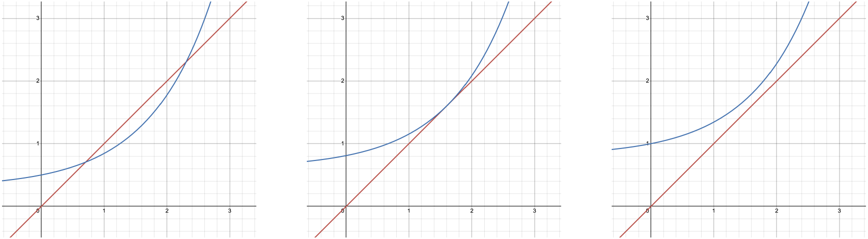

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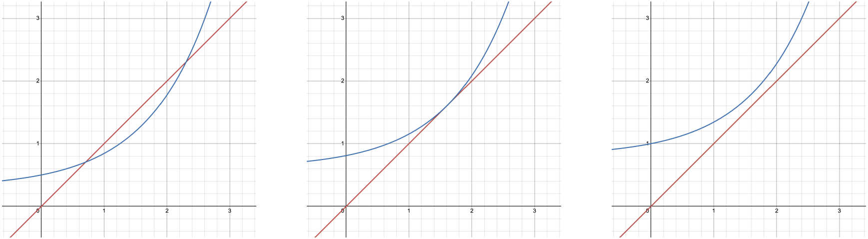

Critical case when at x_c such that $\alpha e^{x_c} = 1$ and u_c such that both curves touches, ie

$$u_c = x_c - \alpha(e^{x_c} - 1) = \log(1/\alpha) - (1 - \alpha).$$

Exact solutions

We proved the following.

Theorem

Let $\mathcal{T}$ a $\mathscr{Pois}(\alpha)$ GW tree with $\alpha < 1$. Then for any ≥ 0 we have

$$\mathbb{E}\left[e^{u\,\mathsf{Card}(\mathcal{T})}\right] < \infty \iff u \leq u_c$$

where $u_c = \log(1/\alpha) - (1-\alpha)$ and

$$\mathscr{L}(u_c) = \log\left(\mathbb{E}\left[e^{u_c\,\mathsf{Card}(\mathcal{T})}\right]\right) = \log\left(\frac{1}{\alpha}\right).$$

Moreover, for any $x \geq 0$ such that $x - \alpha(e^x - \mathbf{1}) \geq 0$ we have

$$x = \mathscr{L}(x - \alpha(e^x - 1)) \iff \alpha e^x \leq 1.$$

Is there a multitype extension of this result?

Exact solutions

Consider now $\mathscr{Pois}(\boldsymbol{H})$ Galton Watson trees with $\mathrm{SpR}(\boldsymbol{H}) < 1$.

- Let $E = \{u \succeq 0 \mid |\mathscr{L}(u)|_\infty < \infty\}$.

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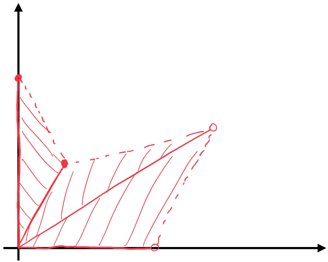
- Let $E = \{u \succeq 0 \mid |\mathscr{L}(u)|_\infty < \infty\}$.
- E is star shaped: if $u \in E$ then $tu \in E$ for all $0 \leq t \leq 1$.



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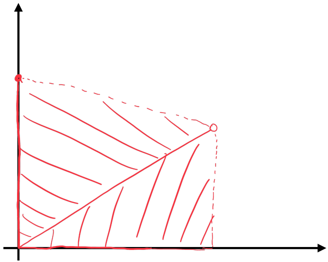

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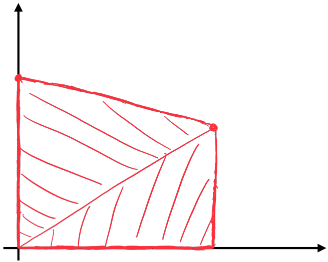
- E is star shaped: if $u \in E$ then $tu \in E$ for all $0 \leq t \leq 1$.
- E is convex (Hölder inequality).
- E is closed. Suppose $u_n = \mathscr{L}(u_n) - \boldsymbol{H}(e^{\mathscr{L}(u_n)} - \mathbf{1}) \xrightarrow[n\to\infty]{} u^*$ then $\mathscr{L}(u_n)$ must be bounded.



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Let us look at regularity properties.

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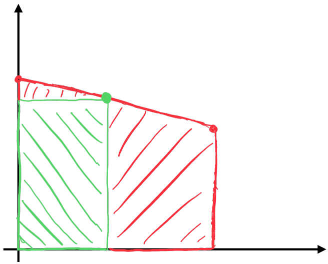

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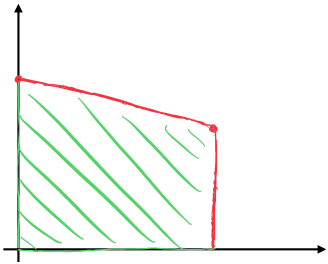

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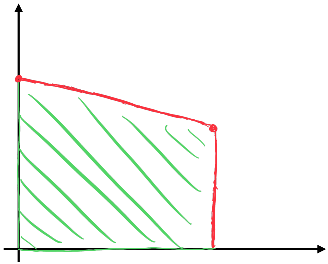

A priori no continuity on ∂E.

Exact solutions

Can we extend the characterisation $\alpha e^x \leq 1$ to the multitype case?

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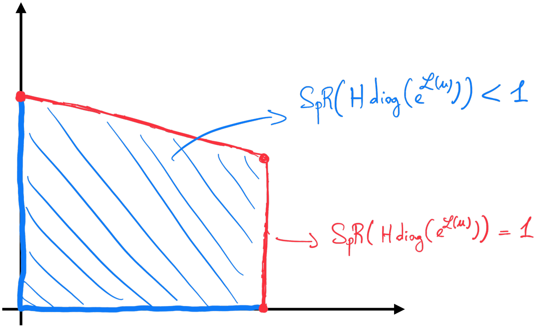
- For any $u \succeq 0$, there exists at most one solution of $y = u + \boldsymbol{H}(e^y - \mathbf{1})$ such that $y \succeq 0$ and $\mathrm{SpR}\left(\boldsymbol{H}\,\mathrm{diag}(e^y)\right) \le 1$.

Exact solutions

To sum up we have the following.



$$\forall y \geq 0, \text{ such that } y - H(e^{y} - \mathbb{1}) \geq 0,$$

$$y = \mathcal{L}(y - H(e^{y} - \mathbb{1})) \iff \mathrm{SpR}(H\,\mathrm{diag}(e^{y})) \leq 1 .$$

Poissonian Clusters

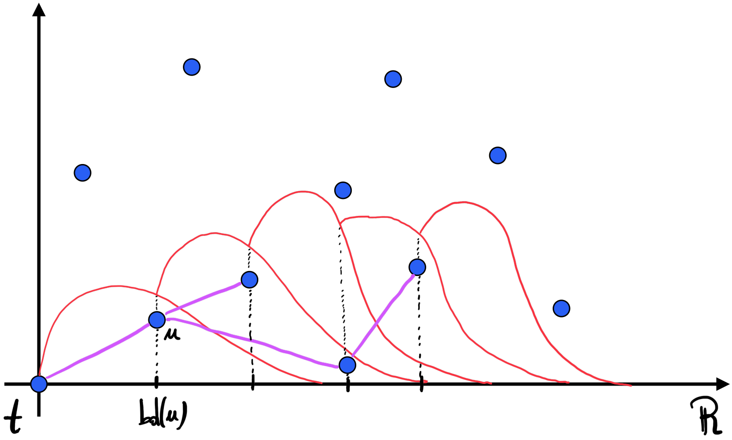
Let $\boldsymbol{h} = (h_i^j)_{i,j\in \boldsymbol{M}}$ where $h_i^j : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and denote $\boldsymbol{H} = (\|h_i^j\|_1)_{i,j}$.

A $\mathscr{Pois}(\boldsymbol{h})$ cluster with root of type $m \in \boldsymbol{M}$ and born at time $t \in \mathbb{R}$ is a random variable $G_t^m = (u, \mathrm{tp}(u), \mathrm{bd}(u))_{u\in\tau}$ such that

- $\mathrm{bd}(u) \in \mathbb{R}$ is the birth date of u,
- $(u, \mathrm{tp}(u))_{u\in\tau}$ is a $\mathscr{Pois}(\boldsymbol{H})$ Galton Watson tree with root of type m.
- Given $(u, \mathrm{tp}(u))_{u\in\tau}$, for u, v with v a child of u, the random variable $\mathrm{bd}(u) - \mathrm{bd}(v)$ are independent and

$$\mathrm{bd}(u) - \mathrm{bd}(v) \sim \frac{h_{\mathrm{tp}(u)}^{\mathrm{tp}(v)}(s)}{H_{\mathrm{tp}(u)}^{\mathrm{tp}(v)}} ds.$$

Thinning in a Poisson random measure

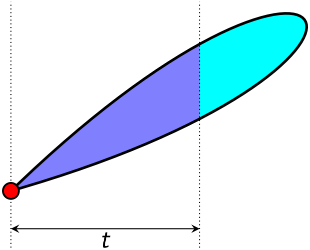


Tail of clusters

Let G_0^m a $\mathscr{Pois}(\boldsymbol{h})$ cluster. For $t \in \mathbb{R}$ we denote by $G_0^m \cap [t, \infty)$ the points u of G_0^m such that $\mathrm{bd}(u) \geq t$.

We are interested in the Laplace transform of

$$\mathrm{Card}_{\boldsymbol{M}} \left(G_m^0 \cap [t, \infty) \right).$$



Fixed point equation

Let $u \succeq 0$, $t \in \mathbb{R}$ and define

$$\boldsymbol{f}_u(t) = \left(\log \left[\mathbb{E} \left[e^{u \cdot \mathrm{Card}_{\boldsymbol{M}}(G_0^m \cap [t,\infty))} \right] \right] \right)_{m \in \boldsymbol{M}}.$$

Then the following holds.

Theorem

For all $u \in E$ and all $t \in \mathbb{R}$ we have

$$f_u(t) = \mathbb{1}_{t \leq 0} u + \left[\boldsymbol{h} \star (e^{f_u} - \mathbf{1}) \right](t)$$

and for $t \leq 0$,

$$f_u(t) = \mathscr{L}(u).$$

For all $u \in \mathring{E}$, for all $t > 0$, we have

$$\boldsymbol{f}_u(t) \preceq \int_t^\infty \sum_{n=1}^\infty \left[\boldsymbol{h}\, \mathrm{diag}(e^{\mathscr{L}(u)}) \right]^{\star n}(s) ds \times \mathscr{L}(u).$$

Sketch of proof

The fixed point equation comes from the branching property.

$$u \cdot \mathrm{Card}_{\boldsymbol{M}}(G_0^m \cap [t,\infty)) \sim \mathbb{1}_{t\leq 0} u_m + \sum_{v \in \text{first gen.}} u \cdot \mathrm{Card}_{\boldsymbol{M}}(G_{\mathrm{bd}(v)}^{\mathrm{tp}(v)} \cap [t,\infty)),$$

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Let $u \in \mathring{E}$, denote $\boldsymbol{h}_u = \boldsymbol{h}\,\mathrm{diag}(e^{\mathscr{L}(u)})$ and $g_u(t) = \mathbb{1}_{t>0} f_u(t)$. If $t > 0$ then

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$$\begin{aligned} g_u(t) &= \int_0^t \boldsymbol{h}(s)(e^{f_u(t-s)} - \mathbf{1})ds + \int_t^\infty \boldsymbol{h}(s)ds \times (e^{\mathscr{L}(u)} - \mathbf{1}) \\ &\preceq \int_0^t \boldsymbol{h}_u(s) f_u(t-s)ds + \int_t^\infty \boldsymbol{h}_u(s)ds \times \mathscr{L}(u) \\ &= \Big[\boldsymbol{h}_u \star g_u\Big](t) + \int_t^\infty \boldsymbol{h}_u(s)ds \times \mathscr{L}(u). \end{aligned}$$

Sketch of proof

Remark that for any $\psi : \mathbb{R}_+ \longrightarrow \mathcal{M}_N(\mathbb{R})$ we have

$$R(\psi)_t := \int_t^\infty \psi(s)ds = (\psi \star \mathrm{Id}_N \mathbb{1}_{\mathbb{R}_-})(t).$$

We proved that

$$g_u \preceq \boldsymbol{h}_u \star g_u + R(\boldsymbol{h}_u) \times \mathscr{L}(u).$$

Thus by iterating the previous inequality,

$$g_u \preceq \Big(\sum_{k=0}^{n-1} \boldsymbol{h}_u^{\star k}\Big) \star R(\boldsymbol{h}_u) \times \mathscr{L}(u) + \boldsymbol{h}_u^{\star n} \star g_u.$$

Since $u \in \mathring{E}$, we have $\mathrm{SpR}(\|\boldsymbol{h}_u\|_1) < 1$ and thus $\boldsymbol{h}_u^{\star n} \star g_u \xrightarrow[n\to\infty]{} 0$. Finally, we have

$$\Big(\sum_{k=0}^{n-1} \boldsymbol{h}_u^{\star k}\Big) \star R(\boldsymbol{h}_u) = R\Big(\sum_{k=1}^{n} \boldsymbol{h}_u^{\star k}\Big).$$

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Choose the right p (depends on δ) and iterate this bound to

$$\Psi(f) = f \star (\Psi(f) + \delta^0).$$

Thank you for your attention!

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