Pricing and Hedging in Financial Markets

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Outline

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Three Pillars of Financial Markets

Time	Uncertainty	Information
Money has *time value*. 1 € today $\neq$ 1 € tomorrow.	The future is *unpredictable*. Crashes, shocks, pandemics.	Uncertainty resolves *progressively* over time. We learn dynamically.

① **Time:** a finite horizon $[0, T]$.

② **Uncertainty:** a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

③ **Information:** a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ — $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s < t$.

Market Assets: Risk-Free and Risky

1. **Risk-Free Asset (B_t):**

$$\mathrm{d}B_t = r_t B_t \,\mathrm{d}t, \quad B_0 = 1$$

   Captures the *time value of money*.

2. **Risky Asset (S_t):**

$$S_t = S_0 + \underbrace{M_t}_{\text{local martingale}} + \underbrace{A_t}_{\text{predictable finite variation}}$$

   - M_t: *unpredictable* fluctuations (market noise, volatility).
   - A_t: *predictable* drift (market trend).

   This captures both *uncertainty* and *information accumulation*.

Derivatives: What Are They?

A **derivative** is a financial contract between a buyer and a seller whose payoff depends on the price of an underlying asset **at a future date**. The contract specifies terms agreed upon today (such as a strike price K and maturity T), allowing parties to manage risk or speculate on future price movements.

Examples: forwards, futures, swaps, and **options** (which we will focus on).

Definition (European Contingent Claim)

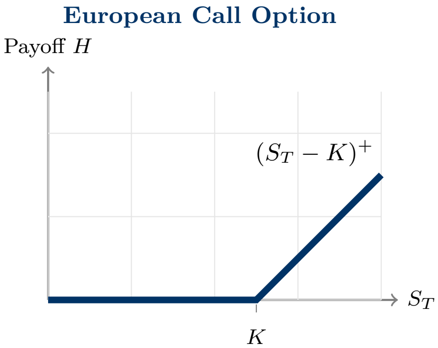
An $\mathcal{F}_T$-measurable random variable H bounded from below, modelling the payoff at T.

Definition (Options)

An ***option*** *is a derivative contract giving the holder the right, but not the obligation, to buy (call) or sell (put) the underlying asset at a strike K on (European) or before (American) maturity T. For European options, the payoff at T is:*

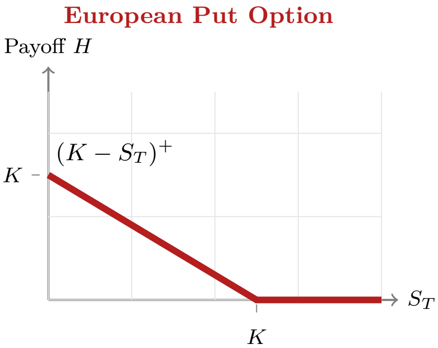
$$H_{\text{call}} = (S_T - K)^+, \qquad H_{\text{put}} = (K - S_T)^+.$$

Option Payoff Diagrams



Positive payoff if: $S_T > K$

Max payoff is unbounded as price rises.



Positive payoff if: $S_T < K$

Max payoff capped at K as price drops.

Central question: What is the fair price today for the claim H?

A Motivating Example: Air France–KLM

The situation: In January, the airline needs millions of barrels of jet fuel in July. Today's oil price is 80 €/barrel, but July's price S_T is unknown. **The risk:** A crisis could send oil to 150 €/barrel, wiping out yearly profits.

What Air France needs:

- ✓ Cap maximum cost (e.g., 90 €/barrel).
- ✓ Keep upside if prices fall.
- ✓ Eliminate catastrophic loss.

Solution: Buy a Call Option

Call option specifications:

- Underlying: Brent crude oil
- Strike $K = 90$ €/barrel, maturity $T = 6$ months
- Payoff at T: $H = (S_T - K)^+$

Result:

- If $S_T = 150$: bank pays 60, net cost = 90 ✓
- If $S_T = 70$: bank pays 0, net cost = 70 ✓

Key Insight

The call provides **downside protection** while preserving **upside participation**. It is financial insurance.

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The Bank's Dilemma and Replication

The Hedging Problem

The bank sells the call. How does it protect itself from the obligation to pay $(S_T - K)^+$?

Answer: Construct a **self-financing replicating portfolio** that exactly matches the payoff at T.

Admissible Strategies and Discounting

Definition (Trading Strategy)

A pair (ϕ_t, ψ_t) (predictable processes) holding ϕ_t shares of S and ψ_t units of B.

Why predictability? You cannot trade on today's price jump – your position at t must be based on information just before t.

Definition (Self-Financing)

$$\mathrm{d}V_t = \phi_t \mathrm{d}S_t + \psi_t \mathrm{d}B_t, \qquad V_t = \phi_t S_t + \psi_t B_t$$

No external cash flows; wealth changes only through price moves.

Definition (Admissibility)

A strategy is admissible if it is predictable, self-financing, and its discounted wealth $\tilde{V}_t = B_t^{-1} V_t$ is bounded below (prevents "doubling strategies").

Discounted Value Process

Lemma (Discounted Value)

Let $\tilde{S}_t = B_t^{-1} S_t$ and $\tilde{V}_t = B_t^{-1} V_t$. A strategy is self-financing if and only if:

$$\mathrm{d}\tilde{V}_t = \phi_t \, \mathrm{d}\tilde{S}_t$$

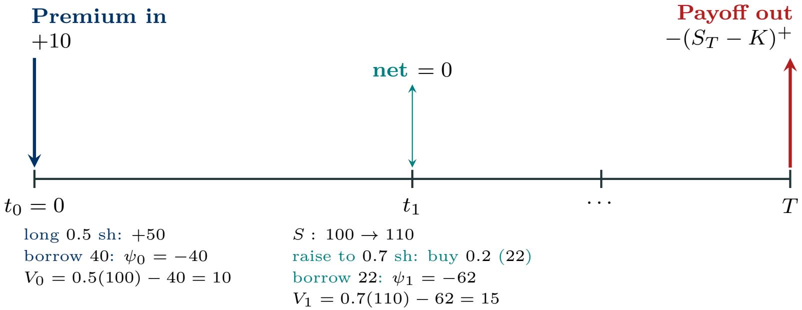
Proof. Apply Itô's product rule to $\tilde{V}_t = B_t^{-1} V_t$:

$$\begin{aligned}
\mathrm{d}\tilde{V}_t &= V_t \, \mathrm{d}(B_t^{-1}) + B_t^{-1} \, \mathrm{d}V_t + \cancel{\mathrm{d}\langle V, B^{-1} \rangle_t} && (2.1) \\
&= -r_t B_t^{-1} (\phi_t S_t + \psi_t B_t) \mathrm{d}t + B_t^{-1} (\phi_t \mathrm{d}S_t + \psi_t r_t B_t \mathrm{d}t) && (2.2) \\
&= \phi_t (B_t^{-1} \mathrm{d}S_t - r_t B_t^{-1} S_t \mathrm{d}t) = \phi_t \, \mathrm{d}\tilde{S}_t && (2.3)
\end{aligned}$$

□

Implementation: Dynamic Hedging

Bank sold the call.
Hedge: hold ϕ_t shares (long, $\phi_t > 0$),
financed by borrowing ($\psi_t < 0$, i.e. short the cash account).
The value is $V_t = \phi_t S_t + \psi_t$ ($r = 0$, $B_t = 1$).



Self-financing check.
Value before the trade: $0.5(110) - 40 = 15$.
Value after: $0.7(110) - 62 = 15$.

Wealth moved $10 \to 15$ *only* because the stock moved.

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Definition (Arbitrage Opportunity)

An admissible self-financing strategy ξ *is an* ***arbitrage*** *if:*

1. $V_0(\xi) = 0$ *(start with zero money).*
2. $\mathbb{P}(V_T(\xi) \geq 0) = 1$ *(always non-negative at end).*
3. $\mathbb{P}(V_T(\xi) > 0) > 0$ *(strictly positive with positive probability).*

No arbitrage (NA) is the economic consistency condition that ensures no risk-free profit from nothing. Without it, prices are internally inconsistent and the market cannot be used for reliable valuation.

No Free Lunch with Vanishing Risk (NFLVR)

Why NFLVR instead of NA?

In discrete time with finitely many assets, NA suffices. In continuous time, one can construct sequences of admissible strategies whose downside risk vanishes, approximating an arbitrage (a “free lunch with vanishing risk”). NA does not rule this out; NFLVR does, by requiring that no such sequence converges to a non-negative, non-zero payoff. This topological strengthening is exactly what allows the First Fundamental Theorem of AsseT Pricing (FTAP)[a].

[a] TBD :)

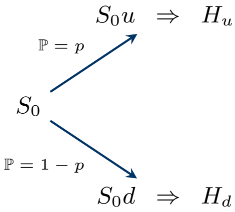
The Core Idea: The One-Period Binomial Model

1. The Physical Market

One step $[0, T]$, bank rate r.

No-arbitrage enforces:

$d < 1 + r < u$.



2. The Hedging Portfolio

Replicate H with ϕ shares $+\psi$ in bank:

$$\phi = \frac{H_u - H_d}{S_0(u - d)} \quad \text{(the Delta)}$$

The cost $V_0 = \phi S_0 + \psi$ simplifies to:

$$V_0 = \frac{1}{1 + r}\left(q\, H_u + (1 - q) H_d\right)$$

where the weight q is strictly in $(0, 1)$:

$$\boxed{q = \frac{(1 + r) - d}{u - d}}$$

The Risk-Neutral Measure $\mathbb{Q}$

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By defining a new measure $\mathbb{Q}(u) = q$, the price becomes

$$V_0 = \frac{1}{1+r}\,\mathbb{E}^{\mathbb{Q}}[H]$$

- **Martingale property:** $\mathbb{E}^{\mathbb{Q}}[S_T] = (1+r)S_0$.

The payoffs reachable from zero initial capital form the cone of **discounted gains**

$$K = \left\{ (\phi \cdot \tilde{S})_T = \int_0^T \phi_t \,\mathrm{d}\tilde{S}_t \;:\; \phi \text{ admissible} \right\}.$$

Allowing **free disposal** and keeping **bounded claims** gives the cone

$$\mathcal{C} = \left(K - L^0_+\right) \cap L^\infty.$$

Definition (Free disposal)

Subtracting L^0_+, i.e. passing from K to $K - L^0_+$, models the right to ***throw money away****: if a payoff is reachable, so is any smaller one.*

Definition (No Arbitrage)

NA holds if $\mathcal{C} \cap L^\infty_+ = \{0\}$: no nonnegative, nontrivial payoff arises from zero capital.

Definition (No Free Lunch with Vanishing Risk)

NFLVR holds if $\overline{\mathcal{C}} \cap L^\infty_+ = \{0\}$, where $\overline{\mathcal{C}}$ is the closure of $\mathcal{C}$ in $\|\cdot\|_\infty$.

FTAP

First FTAP (Delbaen–Schachermayer 1994)

No Free Lunch with Vanishing Risk (NFLVR) $\iff$ exists **Equivalent Local Martingale Measure** (ELMM) $\mathbb{Q} \sim \mathbb{P}$ with $\tilde{S}_t$ a local $\mathbb{Q}$-martingale.

Second FTAP

Market is **complete** (every bounded claim replicable) $\iff$ ELMM is **unique**.

Proof (FTAP)

Assume ELMM $\mathbb{Q} \sim \mathbb{P}$, so $\tilde{S}$ is a local $\mathbb{Q}$-martingale.

Payoff cone: $K = \{(\phi \cdot \tilde{S})_T : \phi \text{ admissible}\}, \ \mathcal{C} = (K - L^0_+) \cap L^\infty$.

Goal: **prove NFLVR:** $\overline{\mathcal{C}} \cap L^\infty_+ = \{0\}$.

- Let ϕ be admissible: $(\phi \cdot \tilde{S})_t \geq -a$ for some $a \geq 0$.

$\Rightarrow$ $\tilde{V}_t = (\phi \cdot \tilde{S})_t$ is a local $\mathbb{Q}$-martingale, *bounded below*, hence a *supermartingale.*

$\Rightarrow$ $\mathbb{E}^{\mathbb{Q}}\left[(\phi \cdot \tilde{S})_T\right] \ \leq \ (\phi \cdot \tilde{S})_0 \ = \ 0.$

- By free disposal, every $X \in \mathcal{C}$ satisfies $X \leq (\phi \cdot \tilde{S})_T$ for admissible ϕ.

$\Rightarrow$ $\mathbb{E}^{\mathbb{Q}}[X] \leq 0.$

- $X \mapsto \mathbb{E}^{\mathbb{Q}}[X]$ is $\|\cdot\|_\infty$-continuous, so this passes to the closure: $\mathbb{E}^{\mathbb{Q}}[X] \leq 0 \ \ \forall X \in \overline{\mathcal{C}}$.
- Take $f \in \overline{\mathcal{C}} \cap L^\infty_+$:
  - $\mathbb{E}^{\mathbb{Q}}[f] \leq 0, \quad f \geq 0, \quad \mathbb{Q} \sim \mathbb{P}$

  $\Rightarrow$ $f = 0.$

□

Assume NFLVR.

Goal: **find $\mathbb{Q} \sim \mathbb{P}$ making $\tilde{S}$ a local $\mathbb{Q}$-martingale.**

Theorem (Delbaen–Schachermayer 1994)

If $\tilde{S}$ is bounded and satisfies NFLVR, then $\mathcal{C}$ is weak-$*$ closed in $L^\infty = (L^1)^*$.

Lemma (Kreps–Yan separation)

Let $\mathcal{C} \subset L^\infty$ be a weak-$*$ closed convex cone with $-L^\infty_+ \subseteq \mathcal{C}$ and $\mathcal{C} \cap L^\infty_+ = \{0\}$. Then $\exists\, Z \in L^1$, $Z > 0$ a.s., with $\mathbb{E}^{\mathbb{P}}[ZX] \leq 0$ for all $X \in \mathcal{C}$.

Now apply separation.

- $\mathcal{C}$ is a weak-$*$ closed convex cone.
- Free disposal: $-L^\infty_+ \subseteq \mathcal{C}$.
- No arbitrage (weaker than NFLVR): $\mathcal{C} \cap L^\infty_+ = \{0\}$.

$\Rightarrow$ By Kreps–Yan, $\exists\, Z \in L^1$, $Z > 0$, with $\mathbb{E}^{\mathbb{P}}[ZX] \leq 0$ for all $X \in \mathcal{C}$.

$\Rightarrow$ Define the measure $\dfrac{\mathrm{d}\,\mathbb{Q}}{\mathrm{d}\,\mathbb{P}} = \dfrac{Z}{\mathbb{E}^{\mathbb{P}}[Z]}$; since $Z > 0$, $\mathbb{Q} \sim \mathbb{P}$.

- For *bounded* integrands ϕ, both $\pm(\phi \cdot \tilde{S})_T \in \mathcal{C}$, so $\mathbb{E}^{\mathbb{P}}[Z\,(\phi \cdot \tilde{S})_T] \leq 0$ both ways.

$\Rightarrow$ $\mathbb{E}^{\mathbb{Q}}\left[(\phi \cdot \tilde{S})_T\right] = 0$ for all bounded ϕ.

$\Rightarrow$ Testing $\phi = \mathbf{1}_A \mathbf{1}_{(s,t]}$, $A \in \mathcal{F}_s$, gives $\mathbb{E}^{\mathbb{Q}}[\tilde{S}_t \mid \mathcal{F}_s] = \tilde{S}_s$.

$\Rightarrow$ $\tilde{S}$ is a true $\mathbb{Q}$-martingale; for bounded $\tilde{S}$ this makes $\mathbb{Q}$ a martingale measure.

- If $\tilde{S}$ is only *locally bounded*, localize along $\tau_n \uparrow \infty$.

$\Rightarrow$ $\tilde{S}$ is a *local* $\mathbb{Q}$-martingale — an ELMM.

$\square$

Risk-Neutral Pricing Formula

Corollary (No-Arbitrage Pricing)

In a complete market, the unique no-arbitrage price of a claim H is

$$V_t = B_t \, \mathbb{E}^{\mathbb{Q}} \left[B_T^{-1} H \mid \mathcal{F}_t \right]$$

With constant r: $V_t = e^{-r(T-t)} \, \mathbb{E}^{\mathbb{Q}}[H \mid \mathcal{F}_t]$.

Outline

The Financial Market: Foundations

Hedging and Self-Financing Portfolios

Arbitrage and Fundamental Theorems

Continuous Time: From Diffusions to PDEs

Definition (Itô Diffusion)

The risky asset S follows an Itô diffusion under the physical measure $\mathbb{P}$:

$$dS_t = \mu(t, S_t)S_t\,dt + \sigma(t, S_t)S_t\,dW_t$$

where:

- W_t *is a standard Brownian motion.*
- $\mu(t, s)$ *is the drift (expected return).*
- $\sigma(t, s)$ *is the volatility (risk).*

Interpretation:

- Deterministic part: $\mu(t, S_t)S_t dt$ (predictable trend).
- Stochastic part: $\sigma(t, S_t)S_t dW_t$ (unpredictable noise).

Drift vs. Risk-Free Rate: The Risk Premium

Under the physical measure $\mathbb{P}$, the **drift** μ is the asset's real expected return.

The **risk-free rate** r is the guaranteed return on the bank account B_t.

Risk premium

$$\mu - r$$

Excess expected return demanded for bearing risk — positive when investors are risk-averse.

Market price of risk

$$\theta = \frac{\mu - r}{\sigma}$$

Premium per unit of volatility.

Girsanov's Theorem: Change of Measure

Theorem (Girsanov's Theorem)

If Novikov's condition holds:

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \theta_t^2 \mathrm{d}t\right)\right] < \infty$$

then the Radon-Nikodym derivative:

$$\left.\frac{\mathrm{d}\,\mathbb{Q}}{\mathrm{d}\,\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left(-\int_0^T \theta_t\,\mathrm{d}W_t - \frac{1}{2}\int_0^T \theta_t^2\,\mathrm{d}t\right)$$

defines $\mathbb{Q} \sim \mathbb{P}$ *under which* $\tilde{W}_t := W_t + \int_0^t \theta_s\,\mathrm{d}s$ *is a standard* $\mathbb{Q}$*-Brownian motion, and:*

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma(t, S_t)S_t\,\mathrm{d}\tilde{W}_t$$

From Expectation to PDE: Feynman–Kac

Theorem (Feynman–Kac / Pricing Dictionary)

The risk-neutral expectation $V_t = e^{-r(T-t)}\, \mathbb{E}^{\mathbb{Q}}\big[H(S_T) \mid \mathcal{F}_t\big]$ *is given by* $V_t = v(t, S_t)$*, where* $v \in C^{1,2}$ *solves the PDE:*

$$v_t + rs\, v_s + \tfrac{1}{2}\sigma^2 s^2 v_{ss} = rv, \qquad v(T,s) = H(s).$$

Derivation Sketch (The Replication Argument):

1. **Apply Itô's Lemma** to the ansatz $v(t, S_t)$:

$$\mathrm{d}V_t = \Big(v_t + \mu S\, v_s + \tfrac{1}{2}\sigma^2 S^2 v_{ss}\Big)\mathrm{d}t + \sigma S\, v_s\, \mathrm{d}W_t$$

2. **Construct a Self-Financing Portfolio** $\Pi_t = \phi_t S_t + \psi_t B_t$:

$$\mathrm{d}\Pi_t = \big(\phi_t\, \mu S + r(\Pi_t - \phi_t S)\big)\mathrm{d}t + \phi_t\, \sigma S\, \mathrm{d}W_t$$

3. **Enforce No-Arbitrage** ($V_t \equiv \Pi_t$):
   - ⇒ **Match the risk ($\mathrm{d}W_t$):** $\phi_t = v_s$ (the *Delta hedge*).
   - ⇒ **Match the drift ($\mathrm{d}t$):** The hedged portfolio is locally riskless, earning rate r. Setting the drift to rv yields the PDE.

➢ *Notice that the real-world drift* μ *cancels out completely.*

The Payoff: the Black–Scholes Formula

For a European call $H = (S_T - K)^+$ with constant r, σ, solving the PDE gives the closed form:

$$C(t,S) = S\,\Phi(d_1) - K\,e^{-r(T-t)}\,\Phi(d_2),$$

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t},$$

where Φ is the standard normal CDF.

- The replicating **hedge** is the delta: $\phi_t = \dfrac{\partial C}{\partial S} = \Phi(d_1)$.
- No drift μ appears — only the volatility σ matters for the price.

Conclusion

- **No arbitrage** is the economic axiom; FTAP turns it into the existence of an equivalent (local) martingale measure $\mathbb{Q}$.
- **Completeness** (Second FTAP) makes $\mathbb{Q}$ *unique*, so every claim has one arbitrage-free price.
- The **binomial model** and the **Black-Scholes PDE** are the discrete and continuous incarnations of the same principle: price by risk-neutral expectation, hedge by replication.

Thank you ! :))

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