Hydrodynamic limits of particle systems

Guillaume Soenen

CEREMADE - Université Paris-Dauphine PSL

09/06/2026

Plan

Simple symmetric Random Walk on $\mathbb{Z}$

Definition

We consider a sequence $(z_i)_{i\in\mathbb{N}^*}$ of independent random variables such that

$$\mathbb{P}(z_i = -1) = \mathbb{P}(z_i = 1) = \frac{1}{2}.$$

We set, for $n \geq 1$,

$$X_n = \sum_{i=1}^{n} z_i.$$

The sequence $(X_n)_n$ is called a simple symmetric random walk on $\mathbb{Z}$.

Donsker's theorem

Theorem

We set

$$Y_n(t) = \frac{X_{[n^2 t]}}{n}.$$

Then, the process (Y_n) converges in distribution to the standard brownian motion.

Rescaling : the space becomes continuous ($1/n$ factor), the time is accelerated ($n^2 t$). It is an example of hydrodynamic limit (at diffusive scale).
Idea : we describe a behaviour at the microscopic scale and we deduce the behaviour of the system at the macroscopic scale.

A weaker result

Theorem

For $f \in C_b^2(\mathbb{R})$, $n \geq 1$, $t \geq 0$ and $x \in \mathbb{R}$, we consider

$$u_n(t,x) = \mathbb{E}^x[f(Y_n(t))].$$

Then,

$$u_n \xrightarrow[n \to +\infty]{} u$$

where u is the unique solution of the heat equation

$$\begin{cases} \partial_t u &= \frac{1}{2}\partial_x^2 u \\ u(0,\cdot) &= f \end{cases}$$

It is a consequence of Donsker's theorem since the density $p(t,\cdot)$ of the brownian motion B_t satisfies the heat equation.

Sketch of the proof

If we show that

- The sequence $(u_n)_n$ is relatively compact ;
- It has only one subsequential limit.

Then it proves that the sequence converges to this limit.

Identification of the limit

We assume that $u_n \xrightarrow[n\to+\infty]{} u$. By definition of the system, we have

$$\begin{aligned}
u_n\left(t+\frac{1}{n^2},x\right)-u_n(t,x) &= \mathbb{E}^x\left[f\left(Y_n\left(t+\frac{1}{n}\right)\right)-f(Y_n(t))\right]\\
&= \mathbb{E}^x\left[f\left(Y_n(t)+\frac{Z_{[n^2t]}}{n}\right)-f(Y_n(t))\right]\\
&= \mathbb{E}^x\left[\frac{1}{2}\left(f\left(Y_n(t)+\frac{1}{n}\right)+f\left(Y_n(t)-\frac{1}{n}\right)\right)\right.\\
&\qquad\left.-f(Y_n(t))\right]\\
&= \frac{1}{n^2}\mathbb{E}^x[\mathcal{A}_n f(Y_n(t))]
\end{aligned}$$

with $\mathcal{A}_n f(x) = \frac{n^2}{2}\left(f\left(x+\frac{1}{n}\right)+f\left(x-\frac{1}{n}\right)-2f(x)\right)$.

Definition

The operator $\mathcal{A}_n$ is called the generator of the dynamic.

Identification of the limit

Then, we have

$$u_n(t,x) = u_n(0,x) + \sum_{k=0}^{[n^2 t]-1} \frac{1}{n^2}\mathbb{E}^x\left[\mathcal{A}_n f\left(Y_n\left(\frac{k}{n^2}\right)\right)\right].$$

and

- $\mathcal{A}_n \xrightarrow[n\to+\infty]{} \frac{1}{2}\partial_x^2$
- $\mathbb{E}^x\left[Y_n(t)\right] \xrightarrow[n\to+\infty]{} u(t,x)$.

Heuristically, we recognize a Riemann sum and we obtain, taking the limit when $n \to +\infty$:

$$u(t,x) - u(0,x) = \frac{1}{2}\int_0^t \partial_x^2 u(t,x)\mathrm{d}x.$$

We "differentiate" this expression : we obtain

$$\partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x)$$

and we obviously have $u(0,x) = f(x)$. Since this equation admits a unique solution, the sequence $(u_n)_n$ has at most one subsequential limit.

Compactness

Since f is bounded, the sets

$$A(t,x) = \{u_n(t,x), \quad n \in \mathbb{N}\}$$

are relatively compact for every t, x.
Moreover, $(u_n)_n$ is uniformly Lipschitz. Indeed,

$$\begin{aligned} \left|u_n(t,x) - u_n(t,y)\right| &\leq \mathbb{E}\left[\left|f\left(x + \frac{X_{[n^2 t]}}{n}\right) - f\left(y + \frac{X_{[n^2 t]}}{n}\right)\right|\right] \\ &\leq \|f'\|_\infty \, |x - y| \end{aligned}$$

and

$$\left|u_n(t,x) - u_n(s,x)\right| \leq \|\mathcal{A}_n f\|_\infty \, |t - s| \leq 2\|f\|_\infty \, |t - s|$$

In particular, it is equicontinuous. Then, due to Ascoli theorem, we know that $(u_n)_n$ is relatively compact.

Plan

Model

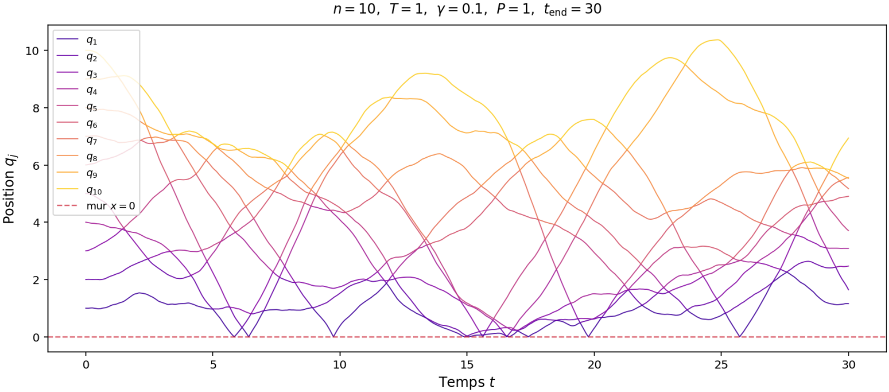
n particles ; q_j is the position of the j^{th} particle, p_j its velocity.

$$
\begin{aligned}
\mathrm{d}q_j(t) &= p_j(t)\,\mathrm{d}t \\
\mathrm{d}p_1(t) &= -2\delta_0(q_1(t))p_1(t)\,\mathrm{d}t \\
&\qquad + (p_2(t^-) - p_1(t^-))\delta_0(q_2(t) - q_1(t))\,\mathrm{d}t \\
&\qquad - \gamma p_1(t)\,\mathrm{d}t + \sqrt{2\gamma T}\,\mathrm{d}W_t^1 \\
\mathrm{d}p_j(t) &= (p_{j-1}(t^-) - p_j(t^-))\delta_0(q_j(t) - q_{j-1}(t))\,\mathrm{d}t \\
&\qquad + (p_{j+1}(t^-) - p_j(t^-))\delta_0(q_{j+1}(t) - q_j(t))\,\mathrm{d}t \\
&\qquad - \gamma p_j(t)\,\mathrm{d}t + \sqrt{2\gamma T}\,\mathrm{d}W_t^j \\
\mathrm{d}p_n(t) &= (p_{n-1}(t^-) - p_n(t^-))\delta_0(q_n(t) - q_{n-1}(t))\,\mathrm{d}t \\
&\qquad - \gamma p_n(t)\,\mathrm{d}t + \sqrt{2\gamma T}\,\mathrm{d}W_t^n \qquad - P\,\mathrm{d}t.
\end{aligned}
$$

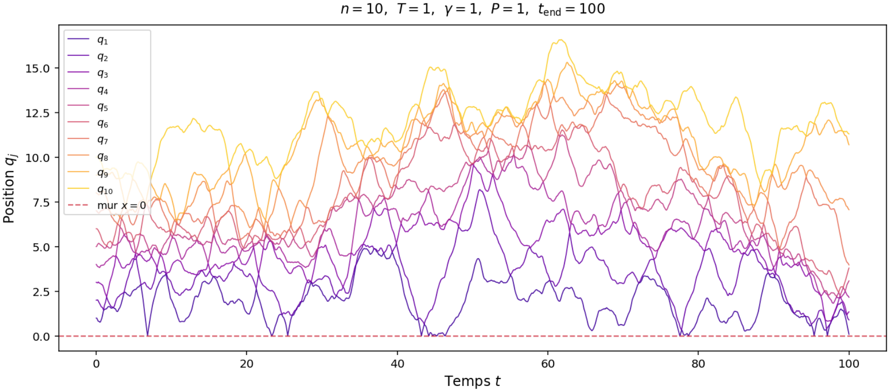
T is the temperature, P the pressure, γ a parameter.

$$\Gamma_n = \left\{(q,p) \in \mathbb{R}_+^n \times \mathbb{R}^n, \quad q_1 \leq \cdots \leq q_n\right\}$$

Simulation


$n = 10,\ T = 1,\ \gamma = 0.1,\ P = 1,\ t_{\text{end}} = 30$
Position q_j
Temps t
q_1
q_2
q_3
q_4
q_5
q_6
q_7
q_8
q_9
q_{10}
mur $x = 0$
0
2
4
6
8
10
0
5
10
15
20
25
30

Simulation



$n = 10,\ T = 1,\ \gamma = 1,\ P = 1,\ t_{\text{end}} = 100$
q_1
q_2
q_3
q_4
q_5
q_6
q_7
q_8
q_9
q_{10}
mur $x = 0$
Position q_j
Temps t
15.0
12.5
10.0
7.5
5.0
2.5
0.0
0
20
40
60
80
100

Generator

The generator of this system is $\mathcal{L} = \mathcal{A} + \gamma\mathcal{S}$, with

$$\mathcal{A} = \sum_{j=1}^{n} p_j \partial_{q_j} - P\partial_{p_n};$$

$$\mathcal{S} = \sum_{j=1}^{n} \left(T\partial_{p_j}^2 - p_j \partial_{p_j} \right).$$

A function φ is in the domain $\mathcal{D}_n(\mathcal{L})$ of the generator $\mathcal{L}$ if

(i) Regularity : φ is of class $\mathcal{C}^2$ on Γ_n with compact support ;

(ii) Collision rule : for any $k \in [\![1, n-1]\!]$,

$$\varphi(\ldots, q_k, q_k, \ldots, p_k, p_{k+1}, \ldots) = \varphi(\ldots, q_k, q_k, \ldots, p_{k+1}, p_k, \ldots);$$

(iii) Reflexion at 0 :

$$\varphi(0, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n) = \varphi(0, q_2, \ldots, q_n, -p_1, p_2, \ldots, p_n).$$

Invariant measure

An invariant measure of this dynamic is

$$\mathrm{d}\mu^n_{P,T} = \mathbb{1}_{\{q_1 \leq \cdots \leq q_n\}} e^{-\frac{P}{T} q_n} \prod_{j=1}^{n} \frac{e^{-p_j^2/2T}}{\sqrt{2\pi T}} \mathrm{d}q \mathrm{d}p.$$

It means that if the initial configuration $(q_1(0), \ldots, q_n(0), p_1(0), \ldots, p_n(0))$ is distributed by $\mu^n_{P,T}$, then at each time $t \geq 0$, we have

$$(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t)) \sim \mu^n_{P,T}$$

Empirical measure

Definition

We set

$$\pi_t^n = \frac{1}{n}\sum_{j=1}^{n} \delta_{\frac{q_j(n^2 t)}{n}}(\mathrm{d}x)$$

the empirical measure in the diffusive scale.

We define also

$$\langle \pi_t^n, \varphi \rangle = \int \varphi \mathrm{d}\pi_t^n = \frac{1}{n}\sum_{j=1}^{n} \varphi\left(\frac{q_j(n^2 t)}{n}\right).$$

Identification of the limit

$$
\begin{aligned}
-\left\langle \pi_0^n, \psi \right\rangle = & \int_0^t \langle \pi_s^n, \partial_t \psi \rangle \,\mathrm{d}s \\
& + \frac{1}{n^2\gamma} \sum_{j=1}^n p_j(0) \partial_x \psi \left(0, \frac{q_j(0)}{n} \right) \\
& + \frac{1}{n\gamma} \sum_{j=1}^n \int_0^t p_j^2(n^2 s) \partial_x^2 \psi \left(s, \frac{q_j(n^2 s)}{n} \right) \mathrm{d}s \\
& - \frac{P}{\gamma} \int_0^t \partial_x \psi \left(s, \frac{q_n(n^2 s)}{n} \right) \mathrm{d}s \\
& + \frac{1}{n^2\gamma} \sum_{j=1}^n \int_0^t p_j(n^2 s) \partial_{xt} \psi \left(s, \frac{q_j(n^2 s)}{n} \right) \mathrm{d}s \\
& + \frac{1}{n^2\gamma} \mathcal{M}_{n,t}.
\end{aligned}
$$

Identification of the limit

We can show that

$$\frac{1}{n\gamma}\sum_{j=1}^{n} p_j^2(n^2 s)\partial_x^2\psi\left(s, \frac{q_j(n^2 s)}{n}\right)\mathrm{d}s \underset{n\to+\infty}{\approx} \frac{T}{\gamma}\int_0^t \left\langle \pi_s^n, \partial_x^2\psi\right\rangle \mathrm{d}s$$

Then if we know that $\pi^n \xrightarrow[n\to+\infty]{d} \pi$ and if $\frac{q_j(n^2 s)}{n} \xrightarrow[n\to+\infty]{} L(s)$, then, considering R the cdf of π, we have the expression

$$-\int_0^t\int_0^{+\infty} R(s,x)\partial_t\varphi(s,x)\mathrm{d}x\mathrm{d}s - \frac{T}{\gamma}\int_0^t\int_0^{+\infty} R(s,x)\partial_x^2\varphi(s,x)\mathrm{d}x\mathrm{d}s = \frac{P}{\gamma}\int_0^t \varphi(s,L(s))\mathrm{d}s + \int_0^{+\infty} R_0(x)\varphi(0,x)\mathrm{d}x.$$

where $\varphi = -\partial_x\psi$.

Identification of the limit

It is the weak formulation of the equation

$$
\begin{array}{rcll}
\partial_t R & = & \frac{T}{\gamma}\partial_x^2 R & t \in \mathbb{R}_+, x \in [0, L(t)] \\
R(t,0) & = & 0 & t \in \mathbb{R}_+ \\
\partial_x R(t, L(t)) & = & \frac{P}{T} & t \in \mathbb{R}_+ \\
R(0,x) & = & R_0(x) & x \in [0, L_0] \\
L'(t) & = & -\frac{T^2}{\gamma P}\partial_x^2 R(t, L(t)) & t \geq 0
\end{array}
$$

This equation admits a unique weak solution (R, L).

The border

What is new in this system ? The border.

Problem : we don't know how to show that the sequence $\left(\frac{q_n(n^2\cdot)}{n}\right)_{n\in\mathbb{N}}$ is relatively compact.

Solution ? Setting $\beta = \delta_{L(t)}(\mathrm{d}x)\mathrm{d}t$, the weak formulation becomes

$$-\int_0^t\int_0^{+\infty} R(s,x)\partial_t\varphi(s,x)\mathrm{d}x\mathrm{d}s - \frac{T}{\gamma}\int_0^t\int_0^{+\infty} R(s,x)\partial_x^2\varphi(s,x)\mathrm{d}x\mathrm{d}s \\ = \frac{P}{\gamma}\int_{[0,t]\times\mathbb{R}_+}\varphi\mathrm{d}\beta + \int_0^{+\infty} R_0(x)\varphi(0,x)\mathrm{d}x.$$

and it is easier to show that the sequence $(\beta^n)_n$, defined by

$$\beta^n(\mathrm{d}x,\mathrm{d}t) = \delta_{\frac{q_n(n^2 t)}{n}}(\mathrm{d}x)\mathrm{d}t$$

is relatively compact.

Relative Entropy

We start at an initial configuration ν_0^n and we note ν_t^n the distribution of the system at time t. We assume that ν_t^n is absolutely continuous with respect to $\mu_{P,T}^n$, with density $f_n(t,\cdot)$. We define

$$H_n(t) = H(\nu_t^n|\mu_{\beta,P}^n) = \int_{\Gamma_n} f_n(t,q,p)\log(f_n(t,q,p))\mathrm{d}\mu_{P,T}^n.$$

Proposition

H_n is a non-increasing function.

Application of the relative entropy

Theorem

For all functions φ such that the quantities are well-defined, if X is a random variable with law ν, we have

$$\forall \alpha > 0, \quad \mathbb{E}[\varphi(X)] \leq \frac{1}{\alpha}\Big(H(\nu|\mu) + \log\Big(\mathbb{E}_\mu\left[\exp\left(\alpha\varphi\right)\right]\Big)\Big)$$

Application :

$$\forall \alpha > 0, \quad \mathbb{E}[\varphi(q(t), p(t))] \leq \frac{1}{\alpha}\Big(H_n(t) + \log \mathbb{E}_{\mu^n_{P,T}}\left[\exp\left(\alpha\varphi(q,p)\right)\right]\Big)$$

Since the entropy is non-increasing, we can write :

$$\forall \alpha > 0, \quad \mathbb{E}[\varphi(q(t), p(t))] \leq \frac{1}{\alpha}\Big(H_n(0) + \log \mathbb{E}_{\mu^n_{P,T}}\left[\exp\left(\alpha\varphi(q,p)\right)\right]\Big)$$

Application : tightness of $(\beta^n)_n$

We apply the inequality with $\alpha = cn$.

$$\mathbb{E}\left[\frac{q_n(n^2 t)}{n}\right] \leq \frac{1}{cn} H_n(0) + \frac{1}{cn} \log\left(\mathbb{E}_{\mu_{P,T}}\left[e^{cq_n}\right]\right).$$

and for $c \leq P/T$,

$$\frac{1}{cn} \log \mathbb{E}_{\mu^n_{P,T}}[e^{cq_n}] = \frac{1}{c} \log\left(\frac{P}{P - cT}\right).$$

Then, if $H_n(0) = O(n)$, we obtain that this quantity is bounded by a constant $C > 0$.

Application : tightness of $(\beta^n)_n$

It implies that

$$\sup_{n\in\mathbb{N}} \sup_{s\in\mathbb{R}_+} \mathbb{P}\left(\frac{q_n(n^2 s)}{n} \geq r\right) \leq \frac{C}{r}$$

and then by Markov inequality, we can show that for every $\varepsilon > 0$, it exists $r > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{P}(\beta^n([0,t]\times[r,+\infty]) > \varepsilon) \leq \varepsilon.$$

It is sufficient to use Prokhorov theorem and show that the sequence of the laws of β^n is relatively compact.

The End

Thank you for your attention

♥ ♥ ♥ ♥ ♥ ♥

It remains 17 talks !