# Statistical mechanics on isoradial graphs & $Z$-invariance

Lorenzo Vigolo
CEREMADE, Université Paris Dauphine - PSL

YRD 2026 - Domaine de la Tour

# Outline

1. **Statistical mechanics**

2. **Isoradial graphs**

3. **$Z$-invariance**

4. **The $Z$-invariant Ising model**

# Statistical mechanics

# 2D Statistical mechanics models

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a finite planar graph. A statistical mechanics model on G is given by the following data:

- a finite set of configurations on G: $\mathcal{C}(\mathsf{G})$;
- some parameters $\nu$ (usually the weights on edges/vertices of G), which allow to define the energy of configurations $\mathcal{E}_\nu(\mathsf{C})$, for $\mathsf{C} \in \mathcal{C}(\mathsf{G})$;
- the Boltzmann probability measure on $\mathcal{C}(\mathsf{G})$:

$$\forall\, \mathsf{C} \in \mathcal{C}(\mathsf{G}),\ \mathbb{P}(\mathsf{C}) = \frac{e^{-\mathcal{E}_\nu(\mathsf{C})}}{Z(\mathsf{G}, \nu)},$$

  where the normalizing constant $Z(\mathsf{G}, \nu) = \sum_{\mathsf{C} \in \mathcal{C}(\mathsf{G})} e^{-\mathcal{E}(\mathsf{C})}$, is called the partition function of our model.

# 2D Statistical mechanics models

We take now G to be infinite and we take an exhaustion with finite graphs $(\mathsf{G}_n)_{n\in\mathbb{N}}$, on which we have our model $(\mathcal{C}(\mathsf{G}_n), \nu, \mathbb{P}_n)$.

# 2D Statistical mechanics models

We take now $\mathsf{G}$ to be infinite and we take an exhaustion with finite graphs $(\mathsf{G}_n)_{n\in\mathbb{N}}$, on which we have our model $(\mathcal{C}(\mathsf{G}_n), \nu, \mathbb{P}_n)$.
Goal 1: study how the behavior of weak limits of $(\mathbb{P}_n)_{n\in\mathbb{N}}$ (infinite volume measures, Gibbs measures) as $n \to \infty$ (uniqueness or not of the limit), changes when we vary the parameters $\nu$.

# 2D Statistical mechanics models

We take now $\mathsf{G}$ to be infinite and we take an exhaustion with finite graphs $(\mathsf{G}_n)_{n\in\mathbb{N}}$, on which we have our model $(\mathcal{C}(\mathsf{G}_n), \nu, \mathbb{P}_n)$.
Goal 1: study how the behavior of weak limits of $(\mathbb{P}_n)_{n\in\mathbb{N}}$ (infinite volume measures, Gibbs measures) as $n \to \infty$ (uniqueness or not of the limit), changes when we vary the parameters $\nu$.
It can happen that there are values $\bar{\nu}$ of the parameters where we have a change in the behavior of the infinite volume measure. Then the model has a phase transition at $\bar{\nu}$. The model at $\bar{\nu}$ is called critical.

# 2D Statistical mechanics models

We take now G to be infinite and we take an exhaustion with finite graphs $(\mathsf{G}_n)_{n\in\mathbb{N}}$, on which we have our model $(\mathcal{C}(\mathsf{G}_n), \nu, \mathbb{P}_n)$.
Goal 1: study how the behavior of weak limits of $(\mathbb{P}_n)_{n\in\mathbb{N}}$ (infinite volume measures, Gibbs measures) as $n \to \infty$ (uniqueness or not of the limit), changes when we vary the parameters $\nu$.
It can happen that there are values $\bar{\nu}$ of the parameters where we have a change in the behavior of the infinite volume measure. Then the model has a phase transition at $\bar{\nu}$.
The model at $\bar{\nu}$ is called critical.
Goal 2: Can we characterize the phase transitions? What is the behavior at $\bar{\nu}$, i.e. at criticality?

# 2D Statistical mechanics models

Generally, there are three ways of characterizing phase transitions (part of the work is to show these are equivalent).

- By uniqueness or non-uniqueness of Gibbs measures;
- By the rate of decay of correlations, i.e. how fast do correlations between two vertices/edges decay as their distance goes to $+\infty$ (no decay, polynomial, exponential).
- If G is $\mathbb{Z}^2$-periodic, we call free energy of the model, the negative of the growth rate of the partition function:

$$F(\nu) := -\lim_{n\to\infty} \frac{1}{n^2} \log Z(\mathsf{G}_n, \nu).$$

  Then, phase transitions are also characterized as points of non-analyticity of the free energy.

# Isoradial graphs

# Isoradial graphs

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a graph. An embedding of G, is a map $\varphi\colon \mathsf{G} \to \mathbb{C}$ that "draws" the graph, i.e. for $v \in \mathsf{V}, xy \in \mathsf{E}$ we have $\varphi(v) \in \mathbb{C}$, $\varphi(xy) \subset \mathbb{C}$ is a simple continuous curve connecting $\varphi(x)$ and $\varphi(y)$.

# Isoradial graphs

Let $G = (V, E)$ be a graph. An embedding of G, is a map $\varphi\colon G \to \mathbb{C}$ that "draws" the graph, i.e. for $v \in V, xy \in E$ we have $\varphi(v) \in \mathbb{C}$, $\varphi(xy) \subset \mathbb{C}$ is a simple continuous curve connecting $\varphi(x)$ and $\varphi(y)$.

An embedding is called planar if, in the drawing, edges are not crossing and $\mathbb{C} \setminus G$ is a disjoint union of topological discs, called faces. The graph is called planar, if it admits a planar embedding.

# Isoradial graphs

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a graph. An embedding of $\mathsf{G}$, is a map $\varphi \colon \mathsf{G} \to \mathbb{C}$ that "draws" the graph, i.e. for $v \in \mathsf{V}, xy \in \mathsf{E}$ we have $\varphi(v) \in \mathbb{C}$, $\varphi(xy) \subset \mathbb{C}$ is a simple continuous curve connecting $\varphi(x)$ and $\varphi(y)$.

An embedding is called planar if, in the drawing, edges are not crossing and $\mathbb{C} \setminus \mathsf{G}$ is a disjoint union of topological discs, called faces. The graph is called planar, if it admits a planar embedding.

**Definition**

We say that an embedding $\varphi$ is isoradial, if it is planar, and all faces are inscribed in circles all of the same radius (usually one takes $1$), and the circumcenter of each face is in the interior of that face.

# Quad-graphs

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a planar graph and $\mathsf{G}^* = (\mathsf{V}^*, \mathsf{E}^*)$ its planar dual, i.e. $\mathsf{V}^*$ are the faces of $\mathsf{G}$ and there's an edge $e^*$ between two faces $u, v \in \mathsf{V}^*$ if and only if $u$ and $v$ are adjacent faces (in this way $\mathsf{E} \cong \mathsf{E}^*$).

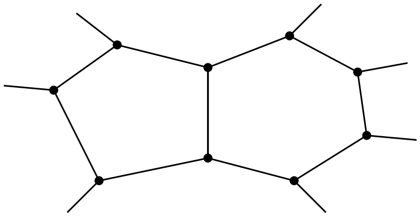
# Quad-graphs

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a planar graph and $\mathsf{G}^* = (\mathsf{V}^*, \mathsf{E}^*)$ its planar dual, i.e. $\mathsf{V}^*$ are the faces of $\mathsf{G}$ and there's an edge $e^*$ between two faces $u, v \in \mathsf{V}^*$ if and only if $u$ and $v$ are adjacent faces (in this way $\mathsf{E} \cong \mathsf{E}^*$).

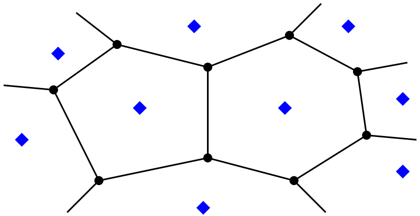
**Definition**

The quad-graph of $\mathsf{G}$, is the graph $\mathsf{G}^\diamond$ constructed as follows. It has vertex set $\mathsf{V} \sqcup \mathsf{V}^*$, and for each face $u \in \mathsf{V}^*$ we have an edge from $u$ to all vertices $x \in \mathsf{V}$ on the boundary of that face.

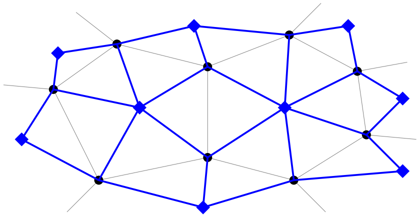
# Quad-graphs



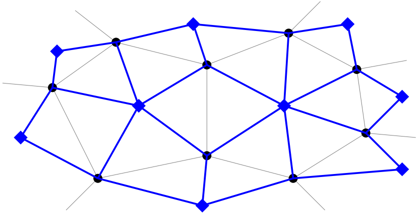
# Quad-graphs



# Quad-graphs

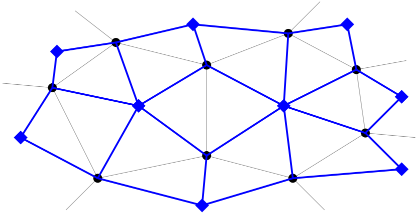


# Quad-graphs



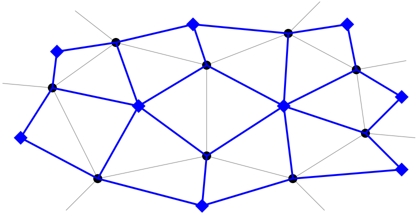
The faces of $G^\diamond$ are quadrilaterals, one for each edge of $G$.

# Quad-graphs



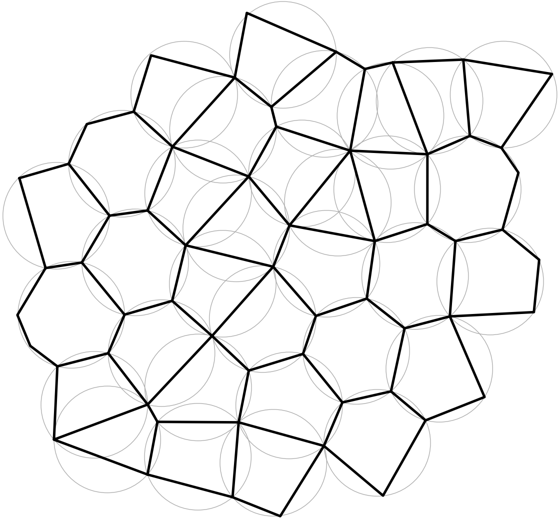
The faces of $G^\diamond$ are quadrilaterals, one for each edge of $G$.
If $G$ is embedded isoradially, we can use as dual vertices the circumcenters of the faces. In this way the edges of $G^\diamond$ are exactly the radii of the circles, which have all the same length. $\implies$ Faces of $G^\diamond$ are rhombii all having the same side-length $1$.

# Quad-graphs

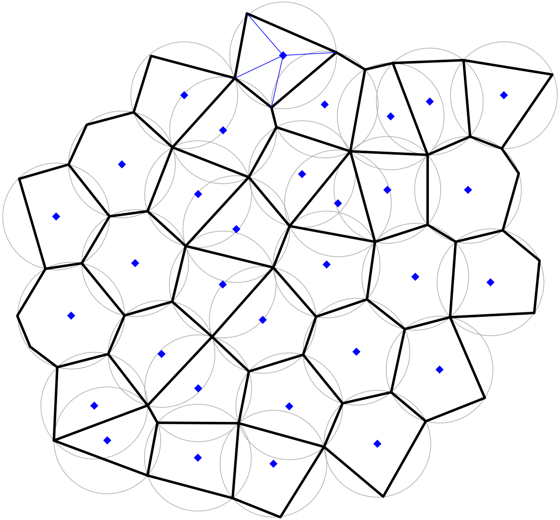


The faces of $G^\diamond$ are quadrilaterals, one for each edge of G.
If G is embedded isoradially, we can use as dual vertices the circumcenters of the faces.
In this way the edges of $G^\diamond$ are exactly the radii of the circles, which have all the same length. $\implies$ Faces of $G^\diamond$ are rhombii all having the same side-length $1$.
Isoradial embedding of G $\cong$ Rhombic embedding of $G^\diamond$.

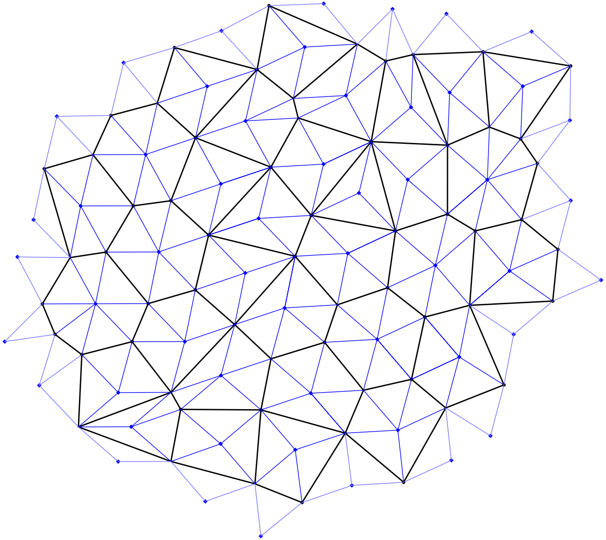
# Isoradial graphs



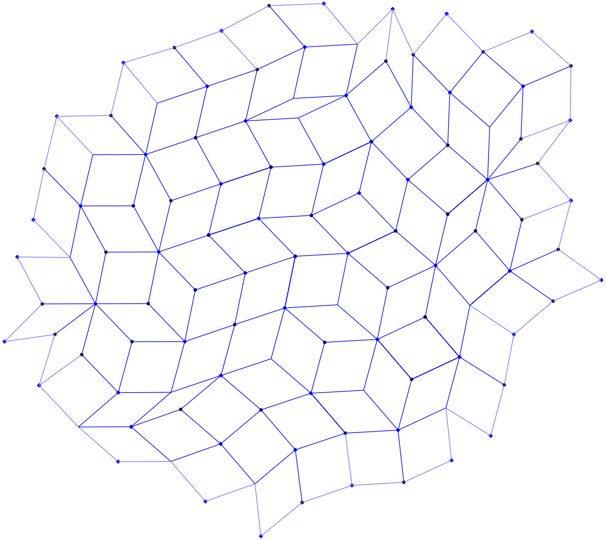
# Isoradial graphs



# Quad-graph

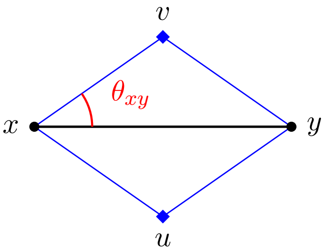


# Quad-graph



# Rhombii half-angles

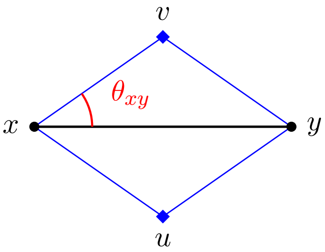
Let G be an isoradial graph embedded in the plane, and $xy \in$ E. Consider its associated rhombus:




$\theta_{xy} =$ the half angle associated to the edge $xy \in (0, \pi/2)$.

# Rhombii half-angles

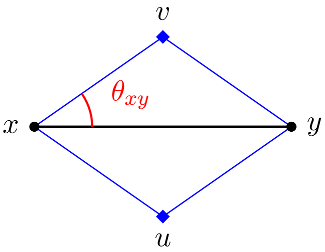
Let G be an isoradial graph embedded in the plane, and $xy \in \mathsf{E}$. Consider its associated rhombus:




$\theta_{xy}$ = the half angle associated to the edge $xy \in (0, \pi/2)$.
The half-angles $(\theta_e)_{e \in \mathsf{E}}$ are completely determined by the embedding of G.

# Rhombii half-angles

Let G be an isoradial graph embedded in the plane, and $xy \in \mathsf{E}$. Consider its associated rhombus:

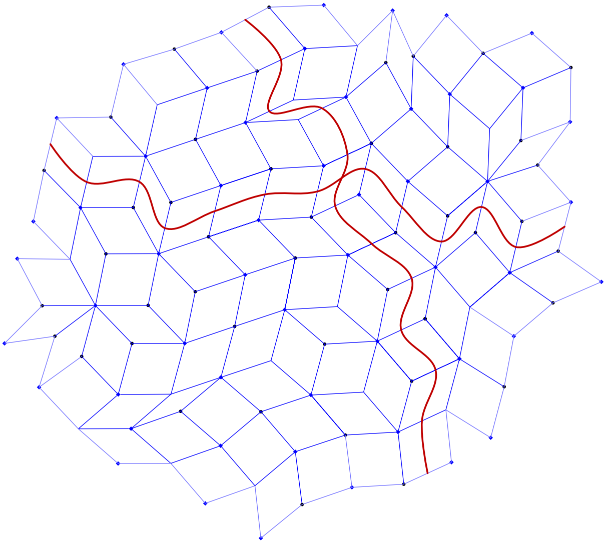



$\theta_{xy}$ = the half angle associated to the edge $xy \in (0, \pi/2)$.
The half-angles $(\theta_e)_{e \in \mathsf{E}}$ are completely determined by the embedding of G.
Idea: If we have a statistical mechanics model on an isoradial graph G, we can ask $\nu = \nu((\theta_e)_{e \in \mathsf{E}})$. $\Longrightarrow$ The parameters and hence the probability measure on isoradial graphs depends on the embedding.

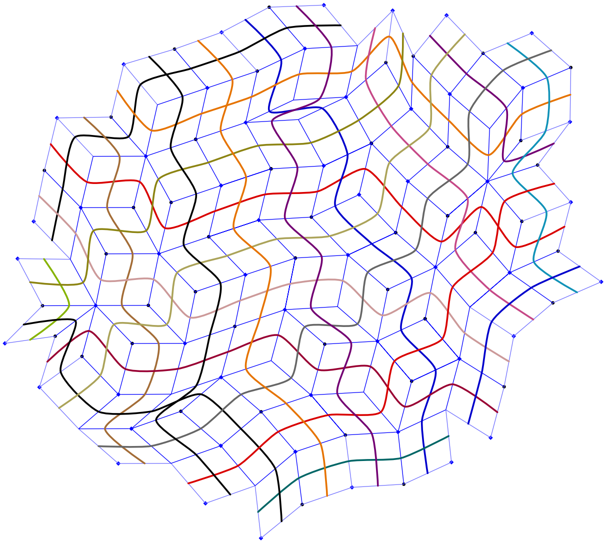
# Train-tracks & isoradial embeddings

A train-track of G, is a maximal path in $(G^\diamond)^*$, i.e. a path moving through the quadrilaterals, that traverses the quadrilaterals from one side to the opposite one.

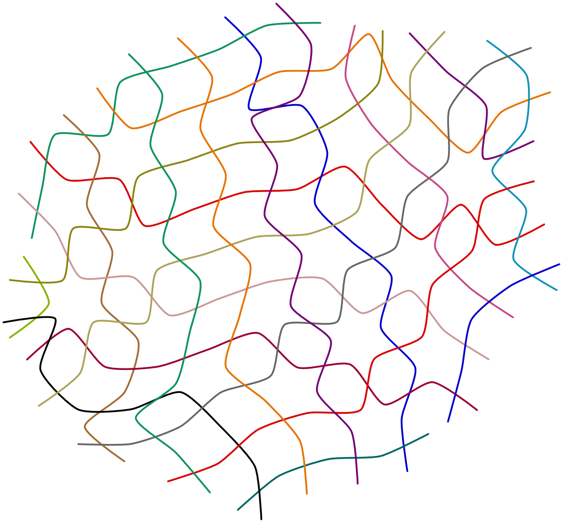
# Train-tracks & isoradial embeddings



# Train-tracks & isoradial embeddings



# Train-tracks & isoradial embeddings

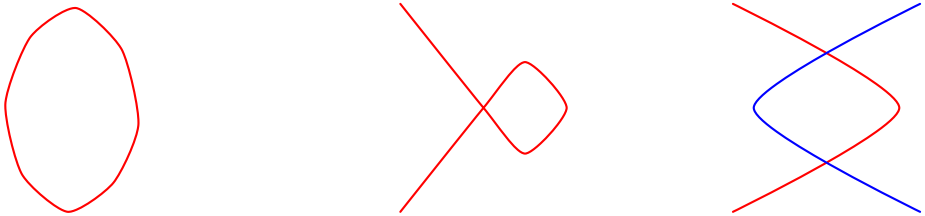


# Train-tracks & isoradial embeddings

**Theorem (R. Kenyon, J. Schlencker '05)**

*A planar graph* $G = (V, E)$ *admits an isoradial embedding if and only if its train-tracks form neither closed loops nor self-intersections, and two train-tracks intersect at most once.*

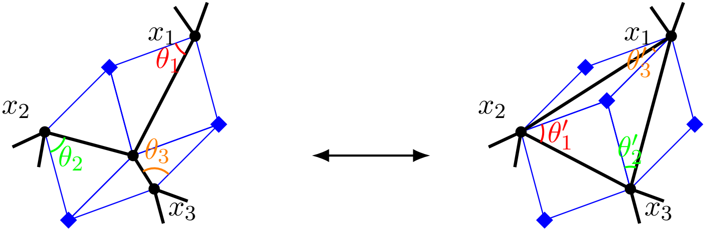
Not permitted:



## $Z$-invariance

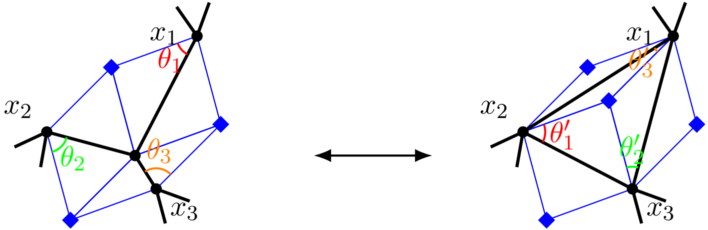
# $Y - \Delta$ transformation

Let G be isoradial, the $Y - \Delta$ transformation changes vertices of degree $3$ to faces of degree $3$, and vice-versa.



# $Y-\Delta$ transformation

Let G be isoradial, the $Y-\Delta$ transformation changes vertices of degree $3$ to faces of degree $3$, and vice-versa.




Then notice, the set of isoradial graphs is closed with respect to this local move. And moreover: for $i = 1, 2, 3$, $\theta'_i = \pi/2 - \theta_i$.

# $Z$-invariance

Assume we have a statistical mechanics model on our isoradial graph G, hence with parameters $\nu = \nu(\theta)$. We say that the model is $Z$-invariant if the following holds.

# $Z$-invariance

Assume we have a statistical mechanics model on our isoradial graph $\mathsf{G}$, hence with parameters $\nu = \nu(\theta)$. We say that the model is $Z$-invariant if the following holds. There exists a universal constant $C > 0$, such that for any fixed configuration $\mathsf{C}(x_1, x_2, x_3)$ on the outside of a star or a triangle in $\mathsf{G}$, with vertices $x_1, x_2, x_3$, we have:

$$Z(\mathsf{G}_Y, \nu(\theta), \mathsf{C}(x_1, x_2, x_3)) = C Z(\mathsf{G}_\Delta, \nu(\theta'), \mathsf{C}(x_1, x_2, x_3)).$$

# $Z$-invariance

Assume we have a statistical mechanics model on our isoradial graph G, hence with parameters $\nu = \nu(\theta)$. We say that the model is $Z$-invariant if the following holds. There exists a universal constant $C > 0$, such that for any fixed configuration $\mathsf{C}(x_1, x_2, x_3)$ on the outside of a star or a triangle in G, with vertices $x_1, x_2, x_3$, we have:

$$Z(\mathsf{G}_Y, \nu(\theta), \mathsf{C}(x_1, x_2, x_3)) = C Z(\mathsf{G}_\Delta, \nu(\theta'), \mathsf{C}(x_1, x_2, x_3)).$$

Consequence: The measure of the model is invariant under the $Y-\Delta$ transformation. This suggests that they should admit some explicit local formulas (depend on the geometry of the graph only locally). These in turn are very useful to do asymptotics and thus to study phase transitions.

# $Z$-invariant models

Studying the different possibilities for the equation above translates into a set of equations for the weights $\nu(\theta)$, called the Yang-Baxter equations. Solving them, means finding the explicit form of $\nu(\theta)$, so that they are satisfied for each possible choice of the vertices $x_1, x_2, x_3$.

# $Z$-invariant models

Studying the different possibilities for the equation above translates into a set of equations for the weights $\nu(\theta)$, called the Yang-Baxter equations. Solving them, means finding the explicit form of $\nu(\theta)$, so that they are satisfied for each possible choice of the vertices $x_1, x_2, x_3$.

If we can find a solution, then we have a well defined $Z$-invariant model on G. However note that this alone is not enough to prove the existence of the local formulas.

# $Z$-invariant models

Although the ideas were already present in works of Kenelly in 1899 and Onsager 1944, the rigorous notion of $Z$-invariance was introduced and developed extensively by Baxter in 1978/1986, were he solved the equations for the eight-vertex model on $\mathbb{Z}^2$ and the Ising model.

# $Z$-invariant models

Although the ideas were already present in works of Kenelly in 1899 and Onsager 1944, the rigorous notion of $Z$-invariance was introduced and developed extensively by Baxter in 1978/1986, were he solved the equations for the eight-vertex model on $\mathbb{Z}^2$ and the Ising model.
The local formulas that are expected for the probabilities essentially derive from local formulas of inverses of discrete differential operators on the infinite graph G. These formula were proven much later and only for models at criticality in '02 by R. Kenyon (spanning trees, dimers) and '10 C. Boutillier and B. de Tilière (for Ising).

# $Z$-invariant models

Although the ideas were already present in works of Kenelly in 1899 and Onsager 1944, the rigorous notion of $Z$-invariance was introduced and developed extensively by Baxter in 1978/1986, were he solved the equations for the eight-vertex model on $\mathbb{Z}^2$ and the Ising model.

The local formulas that are expected for the probabilities essentially derive from local formulas of inverses of discrete differential operators on the infinite graph G. These formula were proven much later and only for models at criticality in '02 by R. Kenyon (spanning trees, dimers) and '10 C. Boutillier and B. de Tilière (for Ising).

This led to the belief that existence of a local formula for an inverse operator is related to the $Z$-invariance and criticality of the underlying model.

# $Z$-invariant models

But this is actually not the case! Criticality isn't needed. In '17 C. Boutillier, B. de Tilière and K. Raschel find the solution to the Yang-Baxter equations for the rooted spanning forest model, which is not critical. The solution is a family of weights depending on a parameter $k \in [0, 1)$. They also show that it has a phase transition at $k = 0$, where the model degenerates to the critical spanning tree of Kenyon's original work. Furthermore this provided access to local formula also for the full $Z$-invariant Ising model not just at criticality ('19), generalizing their work in '10.

## $Z$-invariant models

The key thing to find local formulas is to find a family of local functions $(f_u)_{u\in\mathbb{C}}$ (meromorphic in $u$) in the kernel of the underlying operator $T$ of the model. Then set:

$$\forall\, x,y \in \mathsf{V},\ \tilde{T}^{-1}(x,y) := \int_{\mathsf{C}_{x,y}} f_u(x,y)\, du,$$

where $\mathsf{C}_{x,y}$ is a conveniently chosen contour in $\mathbb{C}$, so that, for $x \neq y$, $T\tilde{T}^{-1}(x,y) = 0$, and $T\tilde{T}^{-1}(x,x) = 1$. The locality property is then inherited by the integrand.

# The $Z$-invariant Ising model

# The Ising model

The Ising model on G is given by the following.

- Spin configurations: $\mathcal{C}(\mathsf{G}) = \{\pm 1\}^{\mathsf{V}}$;
- the parameters are positive edge weights: $\mathsf{J} = (\mathsf{J}_e)_{e \in \mathsf{E}}$, called coupling constants;
- the energy of a spin configuration $\sigma \in \{\pm 1\}^{\mathsf{V}}$ is:

$$\mathcal{E}_{\text{Ising}}(\sigma) := -\sum_{xy \in E} \mathsf{J}_{xy} \sigma_x \sigma_y;$$

- the probability measure is therefore:

$$\forall\, \sigma \in \{\pm 1\}^{\mathsf{V}},\ \mathbb{P}_{\text{Ising}}(\sigma) := \frac{e^{\sum_{xy \in \mathsf{E}} \mathsf{J}_{xy} \sigma_x \sigma_y}}{Z_{\text{Ising}}(\mathsf{G}, \mathsf{J})},$$

  where

$$Z_{\text{Ising}}(\mathsf{G}, \mathsf{J}) = \sum_{\sigma \in \{\pm 1\}^{\mathsf{V}}} e^{\sum_{xy \in \mathsf{E}} \mathsf{J}_{xy} \sigma_x \sigma_y}.$$

Assume G is isoradial, then we need to find right coupling constants $\mathsf{J}(\theta_e)$, so that the model is $Z$-invariant.

# Jacobi elliptic functions

Fix $\tau \in \mathbb{C}$ with $\Re(\tau) = 0, \Im(\tau) > 0$. Denote $q = e^{i\pi\tau}$ the associated nome. The four Jacobi theta functions can be defined by Fourier series as follows.

$$\theta_1(z|\tau) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin((2n+1)z);$$

$$\theta_2(z|\tau) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos((2n+1)z);$$

$$\theta_3(z|\tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz);$$

$$\theta_4(z|\tau) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nz).$$

They are $2\pi$-periodic entire functions of $z$ and analytic in $\tau$ with $\Im(\tau) > 0$.

# Jacobi elliptic functions

Fix an elliptic modulus $k \in \mathbb{C}$, with $k^2 \in (-\infty, 1)$. Its complementary is $k'$ defined by $k^2 + (k')^2 = 1$ The complete elliptic integral associated to $k$ is:

$$K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2(t)}}.$$

Then, denote $\tau = \tau(k) = i\frac{K(k')}{K(k)}$, and $q(k) = e^{i\pi\tau(k)}$.

# Jacobi elliptic functions

Fix an elliptic modulus $k \in \mathbb{C}$, with $k^2 \in (-\infty, 1)$. Its complementary is $k'$ defined by $k^2 + (k')^2 = 1$ The complete elliptic integral associated to $k$ is:

$$K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2(t)}}.$$

Then, denote $\tau = \tau(k) = i\frac{K(k')}{K(k)}$, and $q(k) = e^{i\pi\tau(k)}$.
The Jacobi elliptic functions are in total $12$ functions, denoted in general $\mathrm{pq}(z|k)$, for all possible combinations $\mathrm{p}, \mathrm{q} \in \{\mathrm{s}, \mathrm{c}, \mathrm{d}, \mathrm{n}\}$. They are defined as ratios of theta functions, hence they are periodic meromorphic functions in $z$, for fixed $k$, and meromorphic in $k$ for fixed $z$.

# Jacobi elliptic functions

For the following we'll only need the three below. Set $\zeta := \frac{\pi z}{2K(k)}$. Then

$$\operatorname{sn}(z|k) := \frac{\theta_3(0|\tau)}{\theta_2(0|\tau)} \frac{\theta_1(\zeta|\tau)}{\theta_4(\zeta|\tau)};$$
$$\operatorname{cn}(z|k) := \frac{\theta_4(0|\tau)}{\theta_2(0|\tau)} \frac{\theta_2(\zeta|\tau)}{\theta_4(\zeta|\tau)};$$
$$\operatorname{sc}(z|\tau) := \frac{\operatorname{sn}(z|\tau)}{\operatorname{cn}(z|\tau)} = \frac{\theta_3(0|\tau)}{\theta_4(0|\tau)} \frac{\theta_1(\zeta|\tau)}{\theta_2(\zeta|\tau)}.$$

# $Z$-invariant Ising weights

Baxter solved the $Z$-invariance equations for the Ising model, and found that they admit an infinite family of solutions, parametrized by an elliptic modulus $k$ as before. They are expressed using Jacobi's elliptic functions as follows:

$$\forall\, e \in \mathsf{E},\ \ \mathsf{J}_e = \mathsf{J}(\theta_e|k) = \frac{1}{2}\log\left(\frac{1+\mathrm{sn}(\frac{2K(k)}{\pi}\theta_e|k)}{\mathrm{cn}(\frac{2K(k)}{\pi}\theta_e|k)}\right).$$

# $Z$-invariant Ising weights

Baxter solved the $Z$-invariance equations for the Ising model, and found that they admit an infinite family of solutions, parametrized by an elliptic modulus $k$ as before. They are expressed using Jacobi's elliptic functions as follows:

$$\forall\, e \in \mathsf{E},\ \mathsf{J}_e = \mathsf{J}(\theta_e|k) = \frac{1}{2}\log\left(\frac{1+\mathrm{sn}(\frac{2K(k)}{\pi}\theta_e|k)}{\mathrm{cn}(\frac{2K(k)}{\pi}\theta_e|k)}\right).$$

The functions $\mathrm{sn}(\frac{2K(k)}{\pi}\theta_e|k), \mathrm{cn}(\frac{2K(k)}{\pi}\theta_e|k)$ can be seen as generalizations of $\sin(\theta_e), \cos(\theta_e)$. Indeed, if $k=0$, they reduce to the usual trigonometric functions. Hence, for $k=0$, one recovers:

$$\forall\, e \in \mathsf{E},\ \mathsf{J}_e = \mathsf{J}(\theta_e|k) = \frac{1}{2}\log\left(\frac{1+\sin(\theta_e)}{\cos(\theta_e)}\right),$$

which are exactly the critical isoradial weights for the Ising model. Remark: for $\mathbb{Z}^2$ all half-angles equal $\pi/4$, thus we find: $\frac{1}{2}\log(1+\sqrt{2})$ (Kramers and Wannier '41).

# The local formula for Ising

In the case of the Ising model on G, the associated operator is $\mathsf{K}\colon \mathsf{V}^{\mathrm{F}} \times \mathsf{V}^{\mathrm{F}} \to \mathbb{C}$. Then the coefficients of the inverse operator are:

$$\forall\ x, y \in \mathsf{V}^{\mathrm{F}},\ \mathsf{K}^{-1}_{x,y} = \frac{ik'}{8\pi} \int_{\Gamma_{x,y}} \mathsf{f}_x(u + 2K(k))\mathsf{f}_y(u)\mathsf{e}_{x,y}(u|k)\, du + C_{x,y},$$

where $\mathsf{f}, \mathsf{e}$ are local functions defined in terms of Jacobi elliptic functions, $\Gamma_{x,y}$ is specific a contour on the torus $\mathbb{T}(k) = \mathbb{C}/(\mathbb{Z} + \tau(k)\mathbb{Z})$, and $C_{x,y} = \pm\frac{1}{4}$, if $x$ and $y$ are close, and $0$ otherwise.

Thank you for your attention.

# References


[1] R. J. Baxter. ". Free-fermion, checkerboard and Z-invariant lattice models in statistical mechanics". In: *Proc. Roy. Soc. London* (1986).

[2] C. Boutillier, D. Cimasoni, and B. de Tilière. "Isoradial immersions". In: *Journal of Graph theory* (2022).

[3] C. Boutillier, B. de Tilière, and K. Raschel. "The Z-invariant massive Laplacian on isoradial graphs". In: *Inventiones Mathematicae* (2017).

[4] C. Boutillier, B. de Tilière, and K. Raschel. "The Z-invariant Ising model via dimers". In: *Probability theory and related fields* (2019).

[5] Richard Kenyon. "The Laplacian and Dirac operators on critical planar graphs". In: *Inventiones Mathematicae* (2002).