Stability of the damped wave equation

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1 Introduction

Consider the wave equation

$$\begin{cases} \partial_{tt} w(t,x) - \Delta w(t,x) + a(x)u(t,x) = 0, & 0 < t < T, \quad x \in \Omega, \\ w(t,x) = 0, & 0 < t < T, \quad x \in \partial\Omega, \\ (w(0,x), \partial_t w(0,x)) = (w^0(x), w_t^0(x)), & x \in \Omega, \end{cases} \tag{1}$$

where

- $\Omega \subset \mathbb{R}^d$ is open bounded connected and $\partial\Omega \in C^\infty$,
- $a \in L^\infty(\Omega)$
- $w(t,x)$ represents the elevation with respect to Ω of a wave propagating along Ω,
- $u(t,x)$ is the control,
- $(w^0(x), w_t^0(x))$ are initial data.

The control problem is:

> Given a final time $0 < T < \infty$ and initial/terminal conditions $(w^0, w_t^0), (w^T, w_t^T)$, find a control law $u : [0,T] \times \Omega \to \mathbb{R}$ such that the solution w of (1) satisfies
>
> $$(w(T,x), \partial_t w(T,x)) = (w^T(x), w_t^T(x)), \quad \forall x \in \Omega.$$

There are two trivial situations:

- For $d = 1$ we can use the characteristics method: put

  $$\alpha(t,x) := w_t(t,x) + w_x(t,x), \quad \beta(t,x) := w_t(t,x) - w_x(t,x),$$

  since

  $$\partial_{tt} - \partial_{xx} = (\partial_t - \partial_x)(\partial_t + \partial_x),$$

  one gets

  $$\alpha_t - \alpha_x = 0, \quad \beta_t + \beta_x = 0.$$

  This allows to reduce the control of the wave equation to the control of a system of two transport equations, and one easily sees that the controllability holds whenever $\operatorname{supp} a$ contains a non trivial interval.

- If $a(x) \equiv 1$ one can use the control $u(t,x)$ to completely change the dynamics (and choose whichever!).

The interesting situation is therefore when $d \geq 2$ and $\operatorname{supp} a \subset\subset \Omega$.

2 Control theory of linear and time-invariant-systems

Consider the energy

$$E(t) = \frac{1}{2}\int_\Omega \left\{|\nabla w(t,x)|^2 + |\partial_t w(t,x)|^2\right\} dx.$$

For w a solution of (1) one computes

$$\frac{d}{dt}E(t) = -\int_\Omega \partial_t w(t,x)a(x)u(t,x)dx.$$

This suggests to take $u(t,x) = a(x)\partial_t w(t,x)$, as we will see later. For the moment we wish to take advantage of the above dissipation law and agree to see $(w(t,\cdot), \partial_t w(t,\cdot))$ as an element of the Hilbert space $X := H_0^1(\Omega) \times L^2(\Omega)$. We endow $H_0^1(\Omega)$ with the norm

$$\|f\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla f(x)|^2 dx,$$

which is equivalent to the standard norm on $H^1(\Omega)$, when restricted to the closed subspace $H_0^1(\Omega)$, owing to the Poincaré inequality. We then have

$$E(t) = \frac{1}{2}\|(w(t), \partial_t w(t)\|^2_{H_0^1(\Omega)\times L^2(\Omega)}.$$

In fact, with

$$z(t) := \begin{pmatrix} w(t,\cdot) \\ \partial_t w(t,\cdot) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -\Delta & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -a(\cdot) \end{pmatrix}, \quad z^0 = (w^0, w_t^0),$$

we obtain that A is an unbounded operator on the state space X and B is bounded $U \to X$, with $U := L^2(\Omega)$. The PDE (1) becomes the non-homogeneous "ODE"

$$\left\{\begin{array}{rcl} \dot{z}(t) & = & Az(t) + Bu(t), \\ z(0) & = & z^0. \end{array}\right. \tag{2}$$

One shows that the operator A is skew-adjoint, that is $A^* = -A$, or equivalently

$$\forall z \in D(A), \quad \operatorname{Re}\langle Az, z\rangle_X = 0.$$

This has the following important consequence: the solution $z(t) = e^{tA}z^0$ of (2) with $u(t) \equiv 0$ satisfies

$$\frac{d}{dt}\|z(t)\|_X^2 = 2\operatorname{Re}\langle \dot{z}(t), z(t)\rangle_X = 2\operatorname{Re}\langle Az(t), z(t)\rangle_X = 0.$$

In other words, the semigroup $(e^{tA})_{t\geq 0}$ preserves the norm, and for each $t \geq 0$ the operator e^{tA} is a surjective isometry $X \to X$. As a consequence, the semigroup has no regularization property, which makes the control of (2) hard.

Now consider the adjoint system

$$\begin{cases} -\dot{\varphi}(t) &= A^*\varphi(t), \\ \varphi(T) &= \varphi^T. \end{cases} \tag{3}$$

Integration by parts shows that

$$0 = \int_0^T \langle \dot{z}(t) - Az(t) - Bu(t), \varphi(t)\rangle_X dt = \langle z(T), \varphi^T\rangle_X - \langle z^0, \varphi(0)\rangle_X - \int_0^T \langle u(t), B^*\varphi(t)\rangle_U dt,$$

and using functional analysis one deduces the following.

Theorem 1. *Fix $0 < T < \infty$. Then, the system (1) is controllable in time T, i.e.*

$$\forall z^0, z^T \in X, \quad \exists u \in L^2(0,T;U), \quad z(T) = z^T,$$

if and only if we have the observability inequality

$$\int_0^T \|B^*\varphi(t)\|_U^2 dt \geq c\|\varphi^T\|_X^2,$$

with a constant $c > 0$ independent of $\varphi^T \in X$.

Roughly speaking, the above observability inequality means that one can reconstruct the initial data φ^T of (3) by observing the signal $B^*\varphi(t)$. In our case $B^* = B$ is the multiplication by $(0, -a)$ and if $\operatorname{supp} a$ is small we have limited information.

3 Stability of damped semigroups

Consider $e^{t\mathcal{A}}$ a semigroup on X. We shall consider two stability concepts.

Definition 2. $e^{t\mathcal{A}}$ *is strongly stable if*

$$\forall z^0 \in X, \quad e^{t\mathcal{A}}z^0 \xrightarrow[t\to\infty]{X} 0.$$

Definition 3. $e^{t\mathcal{A}}$ *is exponentially stable if*

$$\exists C, \lambda > 0, \quad \forall z^0 \in X, \quad \|e^{t\mathcal{A}}z^0\|_X \leq Ce^{-\lambda t}\|z^0\|_X.$$

It is clear that exponential stability is stronger than strong stability. Recall that for the wave equation e^{tA} is made of isometries, whence it is not strongly stable. In the abstract framework introduced above, for a solution $z(\cdot)$ of (2) the dissipation law becomes

$$\frac{d}{dt}\frac{1}{2}\|z(t)\|_X^2 = \operatorname{Re}\langle Bu(t), z(t)\rangle_X = \operatorname{Re}\langle u(t), B^*z(t)\rangle_U,$$

which suggests to take

$$u(t) = -B^*z(t). \tag{4}$$

For (1) this yields $u(t) = a\partial_t w(t)$, which is what we have guessed earlier. We call (4) the collocated feedback of the pair (A, B). The word "feedback" means that the command (at time t) is computed as a function of the state (at time t). The word "collocated" means that the measurement we make to synthesis the command is made at the same place the system is actuated (for (1), on $\operatorname{supp} a$). It is therefore of interest to study the stability properties of the semigroup generated by $\mathcal{A} := A - BB^*$, in fact such properties are closely tied to the controllability of (A, B).

Proposition 4. *Let A be skew-adjoint with compact resolvent on X and $B : U \to X$. The following assertions are equivalent to each others:*

1. *The semigroup $e^{t\mathcal{A}}$ is strongly stable.*
2. *For all eigenvector φ of A, we have $B^*\varphi \not\equiv 0$.*
3. *The pair (A, B^*) is approximately observable in infinite time (AOIT):*

$$\forall z^0 \in X, \quad B^* e^{tA} z^0 \equiv 0 \text{ on } (0, \infty) \Longrightarrow z^0 = 0.$$

Remark 5.

- *The hypothesis of A having compact resolvent imposes that it is diagonalizable in a Hilbert basis. This makes the second condition meaningful: it may be that A has no eigenvector (e.g. (1) with unbounded Ω).*
- *For (1), the second condition is obviously equivalent to: for any eigenvector φ of the Dirichlet Laplacian on Ω, we have $\varphi(x)a(x) \not\equiv 0$. Since such φ are real analytic, a sufficient condition is therefore that $\operatorname{supp} a$ has non empty interior.*
- *One shows that (A, B^*) being AOIT is weaker than the approximate controllability of (A, B) in some time:*

$$\exists 0 < T < \infty, \quad \forall z^0, z^T \in X, \quad \forall \epsilon > 0, \quad \exists u \in L^2(0, T; U), \quad \|z(T) - z^T\|_X < \epsilon.$$

  This makes AOIT a weak property.

Proof. We only show that $4 \Longrightarrow 1$: We assume 4 and show that

$$\forall z^0 \in X, \quad e^{t\mathcal{A}} z^0 \xrightarrow[t\to\infty]{X} 0.$$

The semigroup $e^{t\mathcal{A}}$ is bounded, hence we may take $z^0 \in D(\mathcal{A}) = D(A)$ without loss of generality. We rely on the LaSalle invariance principle: set

$$\omega(z^0) := \{z_\infty \in X : \exists 0 \le t_n \uparrow \infty, \quad e^{t_n\mathcal{A}} z^0 \xrightarrow[t\to\infty]{X} z_\infty\},$$

it is enough to show that

$$\emptyset \neq \omega(z^0) \subset \{0\}.$$

That $\omega(z^0) \neq \emptyset$ is a consequence of the compactness of $D(\mathcal{A}) = D(A) \subset X$, in fact by weak compactness we have $\omega(z^0) \subset D(A)$. The dissipation law brings

$$\forall z \in X, \quad \int_0^\infty \|B^* e^{t\mathcal{A}} z\|_U^2 dt \le \frac{1}{2}\|z\|_X^2,$$

and in particular for $z \in D(A)$ the observation $t \mapsto B^* e^{t\mathcal{A}} z$ is of class $H^1(0,\infty;U)$, hence

$$B^* e^{t\mathcal{A}} z \xrightarrow[t\to\infty]{U} 0.$$

We deduce that $\omega(z^0) \subset \ker B^*$. Now for $z_\infty \in \omega(z^0)$ the curve $\gamma(t) := e^{t\mathcal{A}} z_\infty$ satisfies

$$\gamma(t) \in e^{t\mathcal{A}} \omega(z^0) \subset \omega(z^0) \subset \ker B^*,$$

hence

$$\dot{\gamma}(t) = A\gamma(t) - BB^*\gamma(t) = A\gamma(t).$$

Thus $\gamma(t) = e^{tA} z_\infty$ and

$$0 \equiv B^*\gamma(t) = B^* e^{tA} z_\infty,$$

whence by hypothesis $z_\infty = 0$. □

The corresponding result for exponential stabilization is

Proposition 6. *Let A be skew-adjoint on X and $B : U \to X$. The following assertions are equivalent to each others:*

1. *The semigroup $e^{t\mathcal{A}}$ is exponentially stable.*
2. *There is $0 < T < \infty$ such that (A, B) is controllable.*

Proof. Assume that $e^{t\mathcal{A}}$ is exponentially stable, that is

$$\|e^{t\mathcal{A}} z^0\|_X \le C e^{-\lambda t} \|z^0\|_X,$$

for some $C, \lambda > 0$ independent of z^0, t. Let $T > 0$ be such that $\mu := Ce^{-\lambda T} < 1$, the dissipation law brings

$$-\int_0^T \|B^* e^{t\mathcal{A}} z^0\|_U^2 dt = \frac{\|e^{T\mathcal{A}} z^0\|_X^2 - \|z^0\|_X^2}{2} \le \frac{\mu^2 - 1}{2} \|z^0\|_X^2,$$

hence

$$\int_0^T \|B^* e^{t\mathcal{A}} z^0\|_U^2 dt \ge \frac{1 - \mu^2}{2} \|z^0\|_X^2.$$

Using that A is skew-adjoint, one finds

$$\int_0^T \|B^* e^{t\mathcal{A}} z^0\|_U^2 dt \asymp \int_0^T \|B^* e^{tA^*} z^0\|_U^2 dt, \tag{5}$$

hence the observability and the controllability.

Conversely, assume that (A, B) is controllable in time T, we thus have

$$\int_0^T \|B^* e^{t\mathcal{A}} z^0\|_U^2 dt \gtrsim \int_0^T \|B^* e^{tA^*} z^0\|_U^2 dt \gtrsim \|z^0\|_X^2,$$

where we have used (5). Fix

$$E(t) = \frac{1}{2} \|z(t)\|_X^2, \quad z(t) = e^{t\mathcal{A}} z^0.$$

Using the dissipation law and the previously established inequality, one finds

$$E(T) - E(0) = -\int_0^T \|B^* z(t)\|_U^2 dt \leq -cE(0),$$

hence

$$E(T) \leq (1-c)E(0), \quad 1-c < 1.$$

By induction, $E(nT) \leq (1-c)^n E(0)$, and using that the energy $t \mapsto E(t)$ is non-increasing one finds that $e^{t\mathcal{A}}$ is exponentially stable. □

4 The geometric control condition

From the previous discussion we are interested in a criterion for the controllability of (1). This is the purpose of the celebrated geometric control condition, to be introduced in a very sketchy way. We accept that given $p \in \Omega$ and $v \in \mathbb{S}^{d-1}$ it is possible to construct a ray $t \mapsto r(t)$ such that

$$r(0) = p, \quad \dot{r}(0) = v, \quad r(t) \in \overline{\Omega},$$

which propagates linearly while in Ω, and reflects according to the laws of the geometric optics when it touches $\partial\Omega$.

Definition 7. *Let $\omega \subset \overline{\Omega}$ be a subset. We say that that the pair (Ω, ω) satisfies the geometric control condition (GCC) if the following holds: for all ray $t \mapsto r(t)$ constructed as above, there exists $0 < t_0 < \infty$ such that $r(t_0) \in \omega$. If in the latter one can take $t_0 \leq T$ with T independent of the ray, we say that (Ω, ω) satisfies the GCC at time T.*

Theorem 8. *Assume that $0 \leq a \in C^0(\overline{\Omega})$. Then,*

- *(1) is controllable (in some time) if and only if $(\Omega, \{a > 0\})$ satsfies the GCC.*
- *The minimal controllability time is the minimal time to have the GCC:*

$$\inf\{T > 0 : (1) \ \ \textit{controllable}\} = \inf\{T > 0 : (\Omega, \{a > 0\}) \ \textit{satisfies GCC at time } T\}.$$

Interestingly, when $a = 1_\omega$ with $\omega \subset \Omega$ open, the GCC of (Ω, ω) still implies the controllability of (1), but in general the converse if wrong. A typical example is

$$\Omega = \mathbb{S}^{d-1}, \quad \omega = \{(x_1, ..., x_d) \in \mathbb{S}^{d-1} : x_1 > 0\},$$

for which one can show that (1) is controllable, but obviously (Ω, ω) does not satisfy the GCC.