

# Introduction to MultiFractal Analysis

## Applications in Neuroscience

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- If  $\zeta_f(q)$  nonlinear, multiple Hurst exponent: **MultiFractal**.
- Applications in turbulence flow, financial data, neuroscience (EEG, LFP signals)

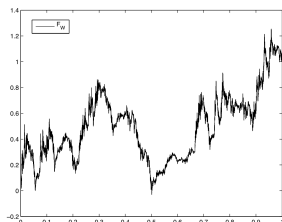


Figure from Barral, Yin, Mandelbrot 2010.

# Multifractal spectrum

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$$|f(x) - f(x_0)| \lesssim |x - x_0|^H.$$

If  $H \geq 1$  one may replace  $f(x_0)$  by a polynomial  $P(x - x_0)$ .

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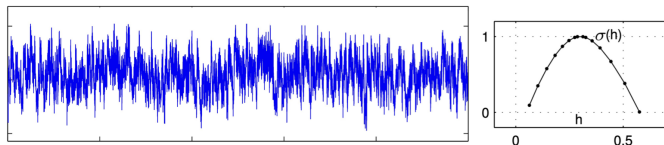


FIGURE 1. Estimated singularity spectrum (right) for the 1D velocity of a turbulent flow (left) - Credit to P. Abry, H. Wendt

Figure from Jaffard, Lashermes, Abry 2006.

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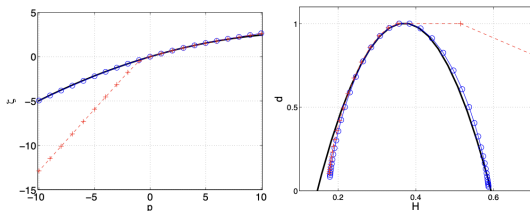


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# Multifractal formalism, heuristic

Let  $\ell \leq 1$ , denote  $N_\ell = \ell\mathbb{N} \cap [0, 1)$ . Recall that

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Thus,

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Thus,

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Since  $g^{**} = g$  under good conditions, if  $D_f = \zeta_f^*$  holds,  $f$  is said to satisfy a **multifractal formalism**.

# Wavelets, a small recap (localized in time and frequency)

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  then we decompose  $f$  on wavelets

$$f = \sum_{j \geq 0} \sum_{k \in \mathbb{Z}_j} W_{jk} \psi_{jk}.$$

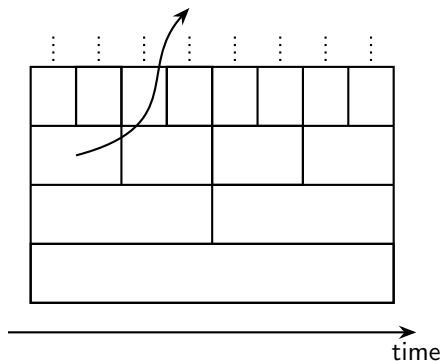
(large scales do not count for local behavior)

$W_{jk}$  tells you the intensity of frequency  $2^{-j}$  localized at time  $k \in 2^{-j}\mathbb{Z} = \mathbb{Z}_j$  in  $f$ .

For multifractal we like the normalisation:

$$f = \sum_{j \geq 0} \sum_{k \in \mathbb{Z}_j} 2^{-j/2} W_{jk} \psi_{jk}.$$

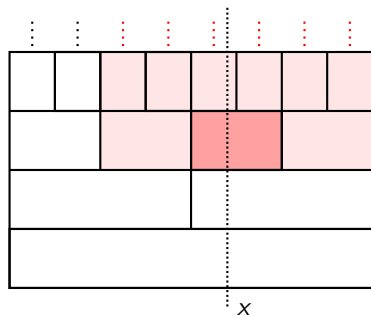
$$\psi_{jk}(t) = 2^{j/2} \psi(2^j(t - k))$$



# Wavelets and MultiFractal

- Simpler notations:  $\lambda = [k, k + 2^{-j})$  stands for  $(j, k)$ .  $\implies \psi_\lambda, W_\lambda \dots$
- For  $\lambda = [k, k + 2^{-j})$  dyadic interval, defined  $3\lambda = [k - 2^{-j}, k + 2 \times 2^{-j})$ .
- **Wavelet Leaders**

$$L_\lambda = \sup_{\lambda' \subset 3\lambda} |W_{\lambda'}|.$$



- For any  $j$ ,  $\exists! \lambda_j(x)$  at scale  $j$  such that  $x \in \lambda_j(x)$ .
- Denote  $W_j(x) = W_{\lambda_j(x)}$  and  $L_j(x) = L_{\lambda_j(x)}$ .

# Wavelets and MultiFractal

Decay of wavelets coefficients is linked to regularity.

## Theorem

$f \in C^H(\mathbb{R}) \iff \forall j, \forall \lambda \in \Lambda_j$  (i.e.  $\lambda$  at scale  $j$ ), we have  $L_\lambda \leq C2^{-jH}$ .

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## Theorem (S. Jaffard)

If  $f \in C^\varepsilon(\mathbb{R})$  for some  $\varepsilon > 0$  then

$$H_f(x_0) = \liminf_{j \rightarrow \infty} \frac{\log_2(L_j(x_0))}{-j}.$$

# Wavelets and MultiFractal

## MultiFractal classical

Scale  $\ell$

$$M_f(\ell, q) = \sum_t |f(t + \ell) - f(t)|^q$$

$$\zeta_f(q) := \liminf_{\ell \rightarrow 0} \frac{\log(M_f(\ell, q))}{\log(\ell)}$$

## MultiFractal Wavelet

Scale  $j$ ,  $\ell = 2^{-j}$

$$M_f(j, q) = \sum_{\lambda \in \Lambda_j} L_\lambda^q$$

$$\zeta_f(q) := \liminf_{j \rightarrow \infty} \frac{\log_2(M_f(j, q))}{-j}$$

Then hope for the best (for example that  $D_f = \zeta_f^*$ ).

## An example: binomial cascade

Consider  $0 < p < 1/2$  and the function  $f_p$  defined by the following wavelet coefficients.

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$p^3$	$p^2(1-p)$	$p^2(1-p)$	$p(1-p)^2$	$p^2(1-p)$	$p(1-p)^2$	$p(1-p)^2$	$(1-p)^3$
$p^2$		$p(1-p)$		$(1-p)p$		$(1-p)^2$	
$p$				$1-p$			
1							

## An example: binomial cascade

- Clearly  $H_f(x_0) \in [-\log_2(1 - p), -\log_2(p)]$ .
- If  $\underline{Y}_0(x)$  is the liminf frequency of 0's in the dyadic expansion of  $x$ , then

$$H_f(x) = -\underline{Y}_0(x) \log_2(p) - (1 - \underline{Y}_0(x)) \log_2(1 - p).$$

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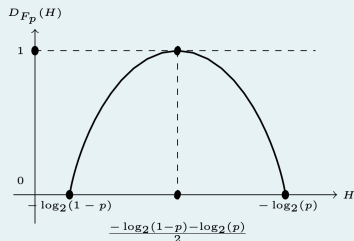
- One can prove that  $\zeta_f(q) = -\log_2(p^q + (1-p)^q)$ , and... the **multifractal formalism holds!** (From slides of S. Seuret, *Multivariate multifractal analysis 2023*)

Theorem (Folklore theorem,  $\sim$  1920 Besicovich-Eggleston, or 1970)

The multifractal spectrum of  $F_p$   
is given by the formula

$$D_{F_p}(H) = \inf_{q \in \mathbb{R}} (qH + \log_2(p^q + (1-p)^q)).$$

In particular,  $D_{F_p}(H) = -\infty$   
as soon as  $H \notin [-\log_2(1-p), -\log_2(p)]$ .



# Frisch-Parisi Conjecture

Besov space  $B_{pq}^s(\mathbb{R})$  defined by the norm (rq.  $B_{2,2}^s = H^s$ ),

$$\|f\|_{spq} = \left[ \sum_{j=0}^{\infty} \left( 2^{j(s+\frac{1}{2}-\frac{1}{p})} \left( \sum_{k \in \mathbb{Z}_j} |2^{-j/2} W_{jk}|^p \right)^{1/p} \right)^q \right]^{1/q}.$$

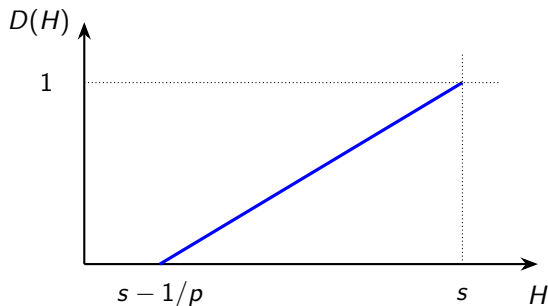
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# Frisch-Parisi Conjecture: Heuristic

- In  $B_{\rho q}^s$ ,  $q$  does not count much so take  $q = p$ . We have

$$\|f\|_{spp}^p = \sum_{j=0}^{\infty} 2^{j(ps-1)} \sum_{k \in \mathbb{Z}_j} |W_{jk}|^p < \infty.$$

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- Let  $N_j(H) = \text{Card}(\{k \mid |W_{jk}| \gtrsim 2^{-jH}\})$ . Since

$$\sum_{j=0}^{\infty} 2^{j(p(s-H)-1)} N_j(H) < \infty,$$

we have  $N_j(H) \lesssim 2^{j[1+p(H-s)]}$ . Thus  $D_f(H)$  is at most  $1 + p(H - s)$ .

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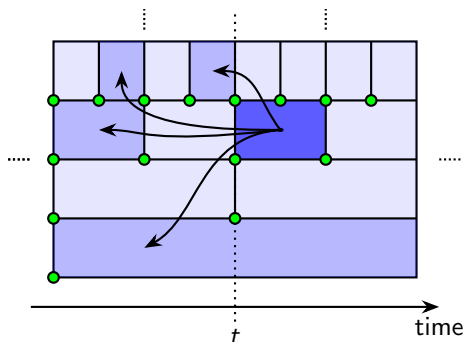
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- There must be  $H$  such that  $D_f(H) = 1$  (ie  $N_j(H) \approx 2^j$ ) and thus  $H \leq s$  so  $H = s$ . If  $s, p$  are optimal (no  $s', p' > s, p$  lead to a finite norm),  $N_j(H) \approx 0$  for  $H > s$ .

# What about my model?



- Each  $W_{jk}$  is associated with the time  $k \in \mathbb{Z}_j$ .

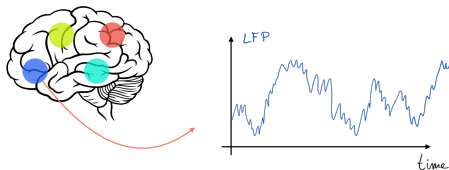
- **Multivariate** and **multiscale** Autoregressive model for  $\mathbf{W}$  on the dyadic tree :

$$W_{jk} = \varepsilon_{jk} + \mu_j + \sum_{j' \in \mathbb{N}} \sum_{k' < k} F_{jj'}(k - k', W_{j'k'}),$$

- $\varepsilon_{jk}$  independent and  $\varepsilon_{jk} \sim \varepsilon_j$ .
- $F_{jj'}(t, x)$  Lipschitz in  $x$  at fixed  $t$ , uniformly (in  $x$ ) bounded in  $L^1$  at fixed  $x$ .

# Motivation from neuroscience

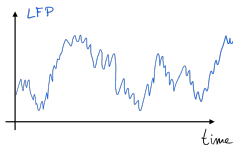
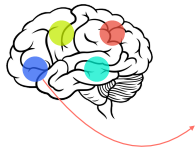
Brain rhythms: Oscillations with specific frequencies of the activity in regions of the brain.  $\rightsquigarrow$  Local Field Potential: mean activity/what remains without spikes.



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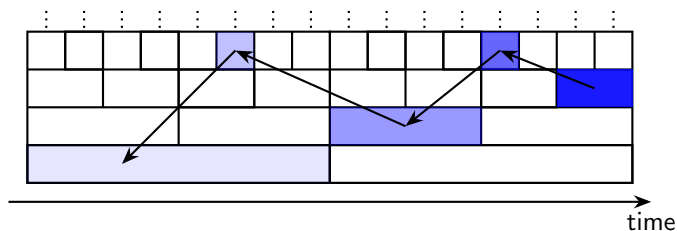
$$LFP(t) = \sum_{j \geq 0} \sum_{k \in \mathbb{Z}_j} W_{jk} \psi_{jk}(t).$$

Localized oscillations in time and frequency  $\Rightarrow$  wavelet decomposition of the LFP.

- $W_{jk}$  is the rhythm of frequency  $2^{-j}$  localized at time  $k \in 2^{-j}\mathbb{Z} = \mathbb{Z}_j$  of region  $\bullet$ . Example:  $\delta$  rhythm, 0.5 – 4Hz, associated with NREM sleep.
- Rhythms interact: slow rhythm trigger a faster one, inhibition from a region to another one...
- **Autoregressive processes** are relevant to model interactions between brain rhythms.

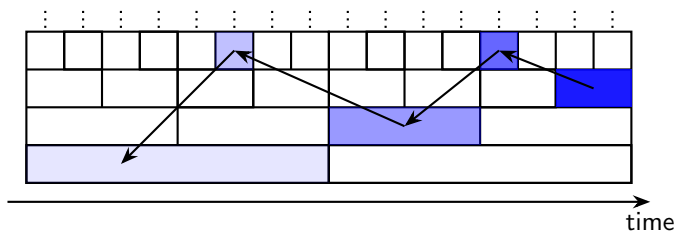
## How to start? Toy model

For any  $(j, k)$  choose randomly  $P(j, k) = (i, \ell)$  with  $i \in \mathbb{N}$  and  $\ell < k$ . The distribution of  $P$  is "1-periodic".



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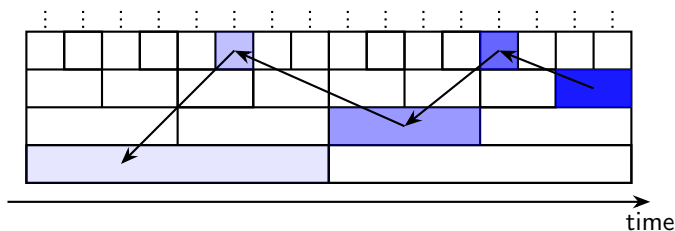
Let  $a, c > 0$  and  $b \in \mathbb{R}$ . Define  $\mathbf{W}$  by

$$W_{jk} = \varepsilon_{jk} + 2^{-c} 2^{b(j-P(j,t)_1)} W_{P(j,t)}$$

where  $\varepsilon_{jk} \sim \mathcal{N}(0, \sigma_j^2)$  independent with  $\sigma_j = 2^{-aj}$ .

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Try to estimate  $N_j(H) = \text{Card}(\{k \in [0, 1) \mid |W_{jk}| \gtrsim 2^{-jH}\})$ .

# First calculations

" $MA(\infty)$ " representation:

$$W_{jk} = \varepsilon_{jk} + \sum_{n \geq 0} 2^{-cn + b(j - P^n(j,k))_1} \varepsilon_{P^n(j,k)}.$$

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Since  $B_{2,2}^s = H^s$ ,

$$f|_{[0,1]} \in H^s([0,1]) \iff \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}_j \cap [0,1]} 2^{-j} 2^{2js} |W_{jk}|^2 < \infty.$$

What is  $s_c$  the largest  $s$  such that  $f \in H^s$ ?

## A surprising result

Let a nonnegative stochastic process  $(X_t)_{t \in \mathcal{T}}$  where the index set  $\mathcal{T}$  is an interval of the real line, or the integers  $\mathbb{N}$ , a Borel measure  $\mu$  on  $\mathcal{T}$  and  $f : \mathcal{T} \rightarrow \mathbb{R}_+$ .

Define

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### Lemma (Jeulin Lemma)

Suppose that there exists  $\varphi : \mathcal{T} \rightarrow (0, \infty)$  measurable and a random variable  $Y$  such that

$$\mathbb{P}(Y > 0) = 1, \quad \mathbb{E}[Y] < \infty, \quad \text{and} \quad \forall t \in \mathcal{T}, \quad \frac{X_t}{\varphi(t)} \sim Y.$$

Then we have  $\mathbb{E}[X(f)] = \mathbb{E}[Y] \int_{\mathcal{T}} f \varphi d\mu$  and the following holds,

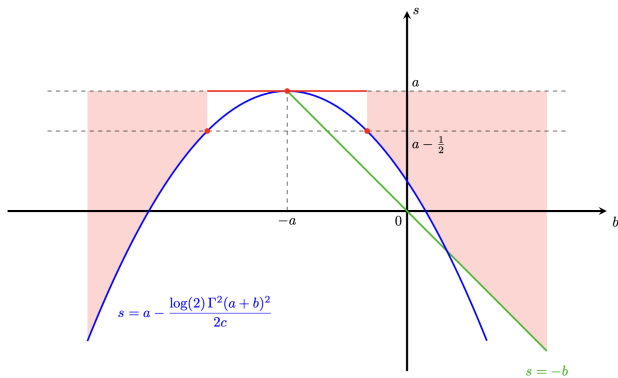
$$\mathbb{P}(X(f) < \infty) > 0 \quad \iff \quad \mathbb{P}(X(f) < \infty) = 1 \quad \iff \quad \mathbb{E}[X(f)] < \infty.$$

# Outcome of Juliana Lemma

We have  $|W_{jk}|^2 \mid (P^n(j, k))_n \sim v_{jk} \mathcal{N}(0, 1)^2$ , thus,

$$f|_{[0,1]} \in H^s([0, 1]) \iff \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}_j \cap [0,1]} 2^{-j} 2^{2js} v_{jk} < \infty.$$

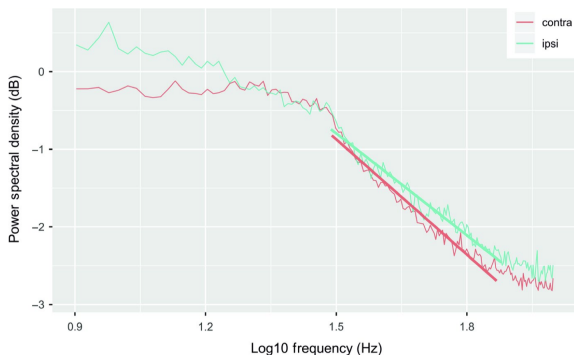
If  $(P^n(j, k))_n$  is a (kind of) martingale with sub-gaussian( $\Gamma^2$ ) increments, then  $s_c$  lies in the red regions,



# Multifractal in Neuroscience

Frequency power-laws in Power Spectral Density (PSD) of brain electric signals are important. One famous algorithm is FOOF.

For example in the article "Pallidal  $1/f$  asymmetry in patients with cervical dystonia"



**FIGURE 1** Example of typical  $1/f$  broadband activity fitting over 30–70 Hz frequency band. It shows that the contralateral (red) PSD is more negatively sloped (thus, more inhibitory), while the ipsilateral (green) PSD is flatter

Figure from Semenova, Popov, Tomskiy, Shaikh, Sedov 2021.

# Conclusion

Take home message:

- Multifractal is about repartition of singularities across scales.
- Wavelets are useful: Theoretical results and analysis, as well as generation of MF processes (wavelets cascades)
- Applications in Neuroscience, need for "horizontal" AR processes on trees (not vertical like cascades).

Thank you for your attention!

# Hausdorff dimension

Let  $A \subset \mathbb{R}^n$ . Let

$$M_r^s(A) = \inf_{\text{diam}(B_i) < r} \left\{ \sum_{i \in \mathbb{N}} \text{diam}(B_i)^s \mid A \subset \bigcup_{i \in \mathbb{N}} B_i \right\}.$$

Then we define  $\dim(A)$  as

$$\dim(A) = \inf \{s \geq 0 \mid \lim_{r \rightarrow 0} M_r^s(A) < \infty\}.$$

**Intuition:** If  $N_r(A)$  is the minimal number of open ball of radius  $r$  to cover  $A$  then  $N_r(A) \approx r^{-\dim(A)}$ .

## Jeulin Lemma, counter example

Let  $X_n \sim \text{Pois}(1/n)$ . Let  $X = \sum_{n \geq 1} X_n$ . We have

$$\mathbb{E}[X] = \sum_n \frac{1}{n} = \infty.$$

However, if the  $X_n$  are independent then

$$\mathbb{P}(X = \infty) = \mathbb{P}((X_n \geq 1) \text{ i.o.}) = 1,$$

and if the  $X_n$  are such that  $X_1 \geq X_2 \geq \dots$  then,

$$\mathbb{P}(X < \infty) = \mathbb{P}(\exists n, X_n = 0) \geq \mathbb{P}(X_n = 0) \geq e^{-1/n} \underset{n \rightarrow \infty}{\geq} 1.$$