Benign landscapes for synchronization

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1. Orthogonal synchronization
2. Benign landscapes for low-dimensional relaxations
3. A sharp condition-number criterion
4. Applications: $\mathbb{Z}_2$, signed Kuramoto and Kuramoto

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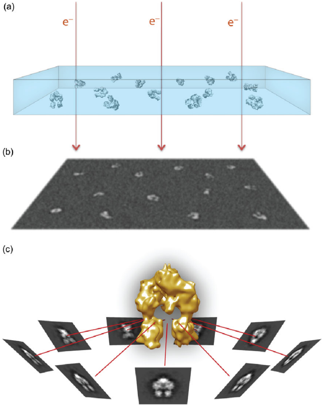

Figure: Single-particle cryo-EM workflow. Adapted from Skiniotis and Southworth [1].

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Synchronization viewpoint

Estimate globally consistent orientations $Z_1, \dots, Z_n \in \mathrm{SO}(3)$ from noisy pairwise relative information $\widehat{R}_{ij}$.

From cryo-EM images to synchronization over $\mathrm{SO}(3)$

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This viewpoint is used in the cryo-EM viewing-direction estimation method of Shkolnisky and Singer [2].

Unknowns

We give ourselves n orthogonal matrices

$$Z_1, \ldots, Z_n \in \mathrm{O}(d), \qquad Z_i Z_i^\top = I_d.$$

And a graph

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$O(d)$-synchronization

The goal is to recover $Z_1, \ldots, Z_n$ from corrupted pairwise measurements

$$C_{ij} = Z_i Z_j^\top + \Delta_{ij} \in \mathbb{R}^{d \times d}, \ \forall \{i, j\} \in E.$$

Here Δ_{ij} is noise.

A natural estimator is

$$\min_{Y_1,\ldots,Y_n \in \mathbb{R}^{d\times d}} \sum_{(i,j)\in E} \left\| C_{ij} - Y_i Y_j^\top \right\|_F^2 \quad \text{s.t.} \quad Y_i Y_i^\top = I_d.$$

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What the objective measures

For each edge, the fitted relative transformation is

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Computational issue

This is a nonconvex optimization problem with many coupled orthogonality constraints. For $d = 1$, it contains binary quadratic optimization problems such as MaxCut-type problems, so it is **NP-hard** in general.

For the square problem $Y_i \in \mathrm{O}(d)$, the least-squares objective is equivalent to a trace maximization. Indeed,

$$\left\| C_{ij} - Y_i Y_j^\top \right\|_F^2 = \|C_{ij}\|_F^2 + d - 2 \left\langle C_{ij}, Y_i Y_j^\top \right\rangle .$$

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The question of this talk

Does increasing p improve the nonconvex landscape?

More precisely: can a small amount of overparameterization remove spurious stable critical points?

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Why second-order points matter

Gradient-based methods naturally settle at stable critical points. A stable local maximum must satisfy the second-order optimality conditions, so we study SOCPs.

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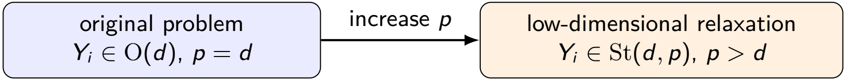

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original problem	increase p →	low-dimensional relaxation
$Y_i \in \mathrm{O}(d),\ p = d$		$Y_i \in \mathrm{St}(d,p),\ p > d$

Landscape question

For which p is the implication

$$\mathrm{SOCP} \implies \text{global optimum}$$

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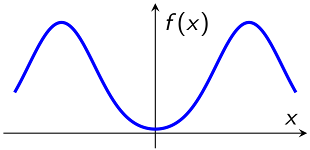

Example: neither convex nor concave on $\mathbb{R}$, but it has no spurious local maximum. This is benign but non-concave.

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Second-order critical point

A feasible point Y is a SOCP if

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Benign landscape

The landscape is benign if

$$\mathrm{SOCP} \quad \Longrightarrow \quad Y \text{ is globally optimal.}$$

Equivalently, there are no spurious local maxima.

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Sharpness

For general connected graphs, $p = d$ and $p = d + 1$ can have spurious local optima because $\mathrm{St}(d, p)$ is connected but **not simply connected**.

Let Δ be the block noise matrix and L_G the graph Laplacian of G :

$$L_G = \mathrm{diag}(A\mathbf{1}) - A, \qquad 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n.$$

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- $\widehat{Z}$ is the unique solution of the original $\mathrm{O}(d)$ problem, up to a global $\mathrm{O}(d)$ transformation.

$\Longrightarrow$ If the SNR is small, the landscape is benign.

In the noiseless gauge $Z_i = I_d$, the synchronization energy can be written as the quantity we want to **minimize**

$$E(Y) = \frac{1}{2} \sum_{i,j=1}^{n} A_{ij} \left\| Y_i - Y_j \right\|_F^2 , \qquad Y_i \in \mathrm{St}(d, p).$$

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Synchronized configurations

$$Y_1 = Y_2 = \cdots = Y_n$$

are exactly the zero-energy configurations when G is connected.

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Gradient flow selects stable critical points

Along the flow,

$$\frac{d}{dt} E(Y(t)) = -\left\|\mathrm{grad}\, E(Y(t))\right\|^2 \leq 0.$$

Since $\mathrm{St}(d, p)^n$ is compact and E is analytic, trajectories converge to critical points; stable equilibria satisfy the second-order necessary conditions, hence are SOCPs for $\min E$, equivalently for $\max F$.

Corollary 1 (McRae–Boumal, 2023)

For any connected graph G, the $\mathrm{St}(d, p)$-valued Kuramoto network synchronizes when

$$p \geq d + 2.$$

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Optimality

For connected graphs in general, this threshold is optimal: it is equivalent to simple connectedness of $\mathrm{St}(d,p)$.

McRae–Boumal (2023) gives a clean answer in the noiseless case:

$$p \geq d + 2 \quad \Longrightarrow \quad \text{no spurious stable points.}$$

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Next specialization

We now focus on $d = 1$, where

$$\mathrm{O}(1) = \{\pm 1\}.$$

The geometry becomes simpler, and the landscape can be controlled by a deterministic matrix attached to the ground truth: a certificate Laplacian.

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Same question, simpler geometry

Increasing p replaces signs by vectors on the sphere. We ask whether this extra dimension removes non-global stable critical points.

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Ground-truth certificate

The conditions

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Nonconvex question

Assuming this certificate holds, how large must p be to ensure

$$Y \text{ SOCP} \quad \Longrightarrow \quad YY^\top = xx^\top?$$

Theorem 2.2 (Rakoto Endor–Waldspurger, 2024)

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$$L \succeq 0, \qquad \lambda_2(L) > 0.$$

If

$$\boxed{p > \frac{\lambda_n(L)}{\lambda_2(L)},}$$

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Sharpness

The threshold cannot be improved in general: there are instances with

$$\frac{\lambda_n(L)}{\lambda_2(L)} = p$$

and a non-global second-order critical point.

Let $D \succ 0$ be any positive diagonal matrix and define

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Why diagonal scaling helps

A good choice of D can reduce the effective condition number. In graph applications, this often corresponds to using a normalized Laplacian.

Setting	Criterion	Object controlling the landscape
McRae–Boumal	$p \geq d + 2$	connected noiseless graph
Rakoto Endor–Waldspurger	$p > \kappa(L)$	sharp certificate condition
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$$\kappa(L) = \frac{\lambda_n(L)}{\lambda_2(L)}, \qquad \kappa(L_D) = \frac{\lambda_n(D^{-1/2} L D^{-1/2})}{\lambda_2(D^{-1/2} L D^{-1/2})}.$$

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Applications

In probabilistic and graph models, concentration estimates show that these condition numbers are small in the regimes of interest. This gives benign landscapes for:

$\mathbb{Z}_2$-synchronization, signed Kuramoto networks.

Consequence: $\mathbb{Z}_2$-synchronization

We want to retrieve n numbers $x_1, \ldots, x_n \in O(1) = \{\pm 1\}^n$ from corrupted measurments

$$x_i x_j + \sigma W_{ij}, \; W_{ij} \sim N(0,1).$$

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Gaussian observation model

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Consequence (McRae, 2025)

If for some $\varepsilon > 0$,

$$\sigma \leq \sqrt{\frac{n}{(2+\varepsilon)\log n}},$$

then, with probability tending to 1 as $n \to \infty$, for every fixed $p \geq 2$, every SOCP satisfies

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Sharpness of the noise level

The threshold $\sigma \asymp \sqrt{n/(2\log n)}$ is sharp for exact recovery in this model, as stated in Bandeira (2018).

Take the complete measurement graph (that is, the edge-observation probability is fixed equal to one), and let $\delta \in [0, 1]$ be the signal strength. For each pair $i < j$, set

$$C_{ij} = \begin{cases} z_i z_j, & \text{with probability } \frac{1+\delta}{2}, \\ -z_i z_j, & \text{with probability } \frac{1-\delta}{2}. \end{cases}$$

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In the gauge $z = \mathbf{1}$, taking $A = C$ gives a complete signed Kuramoto network:

$$\dot{\theta}_i = K \sum_j A_{ij} \sin(\theta_j - \theta_i),$$

with positive attractive edges and negative repulsive edges.

Theorem 3.2 (McRae, 2025), with complete measurements

Suppose that, for some $\varepsilon > 0$,

$$\frac{n}{\log n}\left(1 - \sqrt{1 - (1 - \varepsilon)\delta^2}\right) \geq 1.$$

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Simpler sufficient condition

For small δ, it is enough to have

$$\delta \gtrsim \sqrt{\frac{2\log n}{n}}.$$

Thus the complete signed network tolerates a vanishing average sign bias of order $\sqrt{\log n / n}$.

Thank you for your attention !

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