Harvesting and habitat in a singular control model: Most rapid approach when varying the Carrying Capacity.

Alain Jean-Marie, INRIA, Sophia Antipolis, France

Mabel Tidball, INRAe, CEE-M, Montpellier, France

MESSH- January 2023

Context of the paper

The Gordon-Schaefer model is extended to take account of the negative impact of fishing on habitats.

The main ingredient: Two interrelated state variables: stock of fish and habitat.
Stock and carrying capacity are positively linked, and the fishing activity has a direct and negative impact on the carrying capacity.

The goal:To extend and characterize Clark's most rapid approach optimal solution to this case

How to proceed:

- We consider a continuous-time, infinite-horizon singular control problem with two state variables and one control: the effort of extraction.

Context of the paper

The problem of habitat degradation is one of the most important causes of the over-exploitation of marine resources (see, e.g., Barbier (2000)).

Considering habitat in the model has been done principally in two ways

- Habitat as a parameter that affects growth function through the intrinsic growth and/or the carrying capacity and/or the standard Schaefer harvest function.
- Carrying capacity is a variable endowed with its own dynamics entirely dependent on habitat dynamics. This is the case in our paper.

Some related papers

- A number of papers have tried to model some aspects of the two-way relationship between fish stock and the habitat stock with the purpose of proposing the implementation of a policy aimed at achieving a given economic and/or ecological objective (see Holland and Schnier (2006), Nichols, Yamayaki and Jennings (2018), Long, Zaccour and Tidball (2020)...).
- The closest work related to our paper is in the thesis of Udumyan (2012); but only some results concerning the steady states of the models are analyzed.

Main results

(i) the extension of the singular control model (Clark's model, most rapid approach) to the case of two state variables,

(ii) the analytical feedback characterization of the optimal extraction and adjoint variables (related to resource and habitat): a) in the singular path, and b) when fishing activity does not affect habitat,

(iii) the numerical approximation that describes the singular path and gives the optimal solution of the problem when fishing activity affect habitat,

(iv) some results of the steady state.

Plan

The outline of the paper

- The optimal singular control problem with two state variables and one control
- The MRAP optimal solution
- How to obtain some analytical optimal solutions
- Numerical simulations to give the optimal solution.

The model

The dynamic model is the following:

$$\dot{X} = F(X, K) - EX \tag{1}$$

$$\dot{K} = G(K) - \beta EK \tag{2}$$

where:

- F, G are the growth functions of X (stock of fish) and K (carrying capacity)
- β is a positive constant that measures the destructive effects of fishing on the habitat
- the function $E(\cdot)$ is a control variable (fishing effort).

The model

The optimization problem is

$$\max_{E(\cdot)} \int_0^\infty e^{-\delta t} P(X)\ E\ \mathrm{d}t \tag{3}$$

where $E(\cdot)$ belongs to the class of measurable functions such that $0 \leq E(t) \leq \overline{E}$ for all $t \geq 0$. The function $P(X)$ is the net marginal profit per unit of effort, when the resource stock is X. A typical form for it is $P(X) = X(p - c(X))$, with p a constant and $c(X)$ is decreasing and convex in X.

$$\left(p - \frac{c}{X}\right) EX, \quad EX = harvesting.$$

First order conditions

The Hamiltonian of the problem is:

$$\begin{aligned}\mathcal{H} &= P(X)E + \lambda_X\left(F(X,K) - EX\right) + \lambda_K\left(G(K) - \beta EK\right)\\ &\quad + \gamma_E E + \mu_E(\overline{E} - E)\ .\end{aligned}$$

Then the first-order conditions are:

$$\frac{\partial \mathcal{H}}{\partial E} = 0 = P(X) - \lambda_X X - \beta\lambda_K K + \gamma_E - \mu_E$$

$$\dot{\lambda}_X = \delta\lambda_X - \frac{\partial \mathcal{H}}{\partial X} = \delta\lambda_X - EP'(X) - \lambda_X\left(\frac{\partial F}{\partial X}(X,K) - E\right)$$

$$\dot{\lambda}_K = \delta\lambda_K - \frac{\partial \mathcal{H}}{\partial K} = \delta\lambda_K - \lambda_X\frac{\partial F}{\partial K}(X,K) - \lambda_K\left(G'(K) - \beta E\right)$$

plus complementarity conditions. We introduce:

$$\gamma_E - \mu_E = \chi := X\lambda_X + \beta K\lambda_K - P(X)\ .$$

Optimal solution

Proposition: Optimal trajectories

On any optimal trajectory, we have necessarily

$$E(t) = \begin{cases} 0 & \text{if } \chi(t) > 0 \\ \overline{E} & \text{if } \chi(t) < 0 \\ \text{some } E \in [0, \overline{E}] & \text{such that } \chi(t) = 0. \end{cases}$$

The logistic growth functions

The analysis is pursued with specific growth functions functions, specified as:

$$F(X,K) = rX\left(1 - \frac{X}{K}\right)$$

$$G(K) = \rho K\left(1 - \frac{K}{\overline{K}}\right).$$

SS for optimal E (if it exists)

The set of stationary states satisfies

$$K^\infty = \overline{K}\left(1 - \frac{\beta E^\infty}{\rho}\right)$$

and: either $X^\infty = 0$, $\lambda_K^\infty = 0$ and $\lambda_X^\infty = \frac{E^\infty P'(0)}{\delta - r + E^\infty}$,
or:

$$X^\infty = K^\infty\left(1 - \frac{E^\infty}{r}\right) \; = \; \overline{K}\left(1 - \frac{\beta E^\infty}{\rho}\right)\left(1 - \frac{E^\infty}{r}\right)$$

$$\lambda_X^\infty = \frac{P'(X^\infty)E^\infty}{\delta + r - E^\infty}$$

$$\lambda_K^\infty = \lambda_X^\infty\left(1 - \frac{E^\infty}{r}\right)^2 \frac{r}{\delta + \rho - \beta E^\infty} \; ,$$

for some effort E^∞. It is necessary that $E^\infty \leq \rho/\beta$ (and $E^\infty \leq r$ in the case $X^\infty \neq 0$) and $0 \leq E^\infty \leq \overline{E}$ for such a stationary solution to exist.

Existence of SS

When $P(X) = pX - c$ with $P(\overline{K}) > 0$

$$\left(p - \frac{c}{X}\right) EX, \quad EX = harvesting.$$

- If $c > 0$ there exists an unique $0 < E^\infty < \min(r, \rho/\beta)$ optimal SS, $(X^\infty > 0, K^\infty > 0)$.
- If $c = 0$ and $\rho/\beta < r$ then

$$E^\infty = \rho/\beta, \quad K^\infty = X^\infty = 0$$

.

All provided that

$$E^\infty \leq \overline{E}.$$

Analytical solutions: feedback characterization of adjoint variables and optimal extraction on the singular curve and trajectories for constant E

Solving

$$\chi = \dot{\chi} = \ddot{\chi} = 0,$$

we can find

$$\lambda_X(X, K), \quad \lambda_K(X, K), \quad E(X, K).$$

But there are not enough equations to find χ.

Yet there are analytical formulas:

- when $\beta = 0$ (see examples to follow)
- for the non-singular pieces of trajectories: optimal control consists of piecewise-constant functions $E(\cdot)$, and we can solve analytically the solution of the differential system where E is constant.

Characterization when $\beta = 0$

Consider the following parameters:

$$\overline{K} = 1,\ r = 8,\ \delta = 0.05,\ \rho = 4,\ \beta = 0,\ p = 1,\ c = \frac{205}{808},\ \overline{E} = 5.$$

The equation of the singular curve is:

$$258560X^2 - 128472XK - 205K - 32800X = 0$$

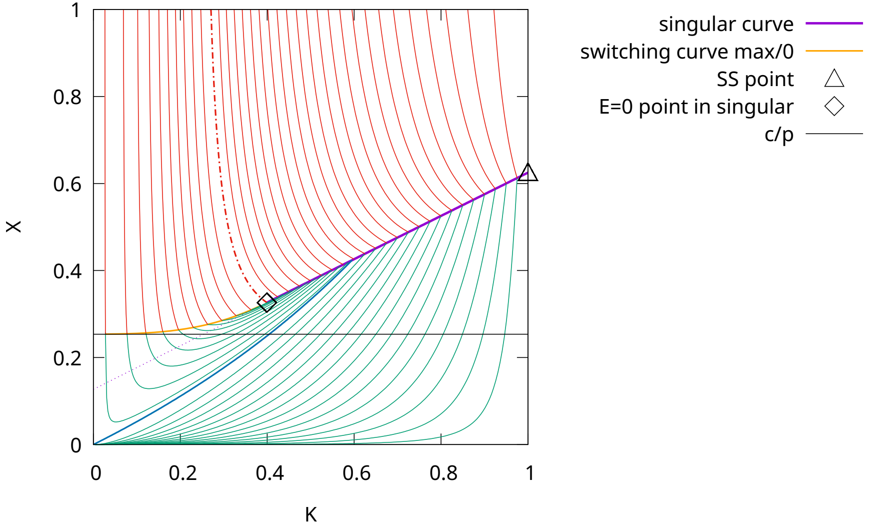
and the stationary point (with non-zero X) is:

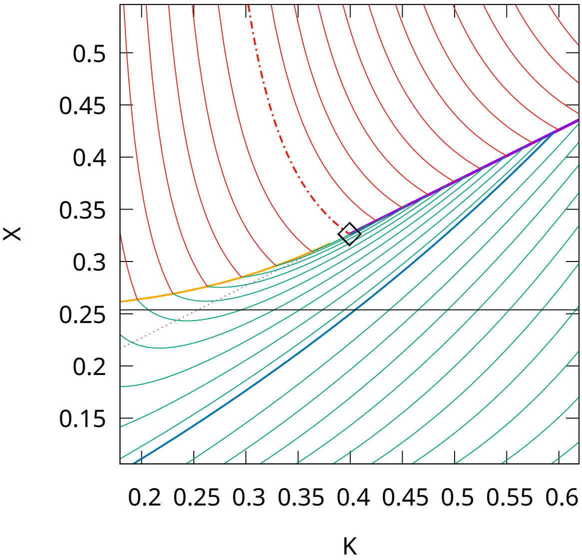
$$E^\infty = 3, \quad K^\infty = 1, \quad X^\infty = \frac{5}{8}.$$

The optimal feedback control on the singular curve has the form:

$$E^*(K, X) = 8\left(1 - \frac{X}{K}\right) + \frac{4K\,(1-K)\,(128472X + 205)}{X\,(128472K - 517120X + 32800)}.$$

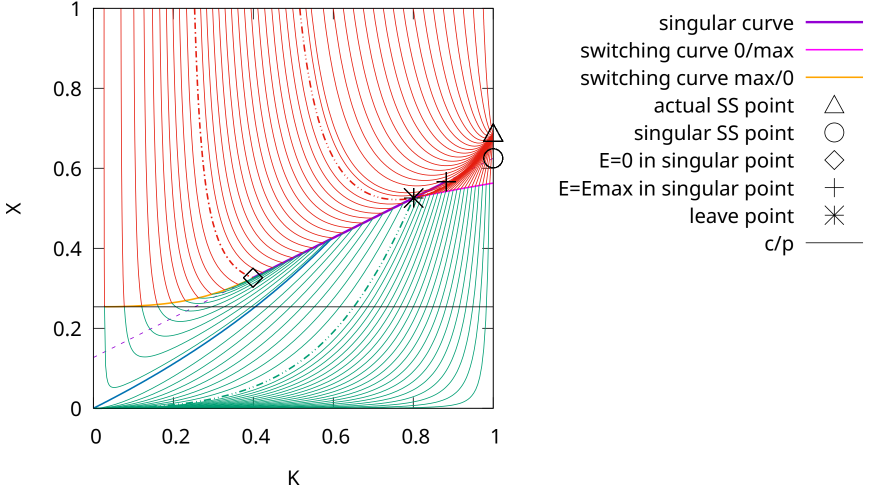
$\beta = 0$, large $\overline{E}$. Optimal solutions



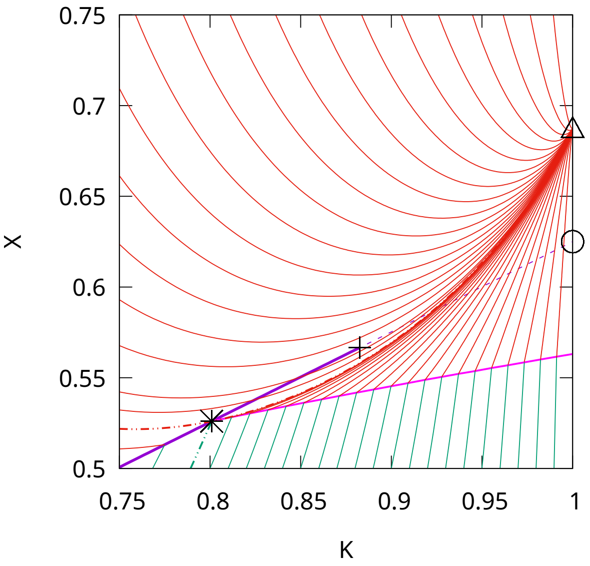


0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
X
0.2 0.25 0.3 0.35 0.4 0.45 0.5 0.55 0.6
K

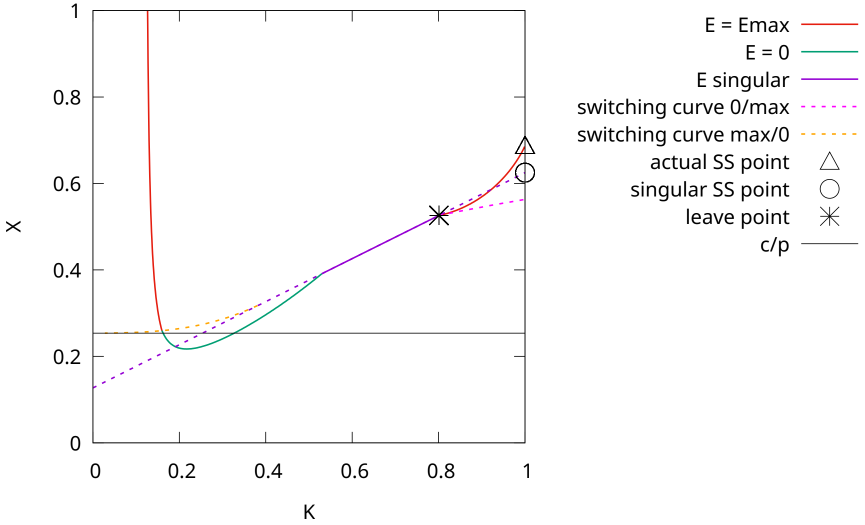
$\beta = 0$, small $\overline{E}$. Optimal solutions



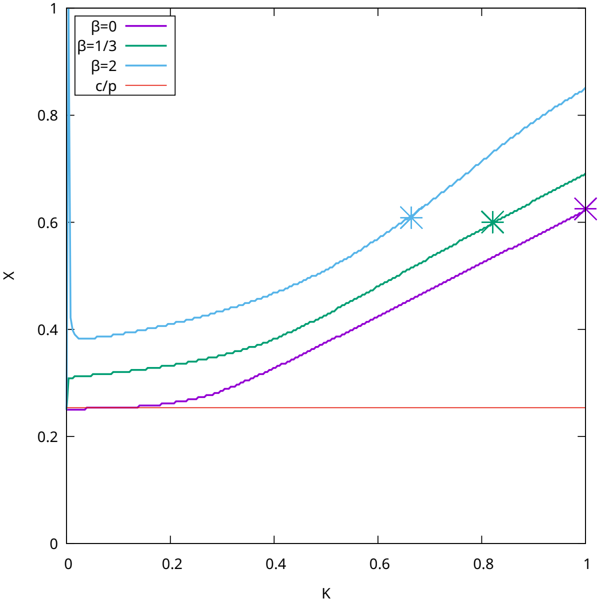
$\beta = 0$, small $\overline{E}$. Optimal solutions (ctd)



$\beta = 0$, small $\overline{E}$. One particular trajectory



Numerical examples with $\beta > 0$. Comparison



Conclusions

- We have made an extension of the singular control model (Clark's model, most rapid approach) when habitat is taken into account.
- We have obtained some analytical optimal solutions, completed with simulations, that give the optimal solution of the problem.
- Clark solution follows when some parameters are zero ($\rho = \beta = 0$)
- Case $\beta = 0$, analytical characterization when $E < \bar{E}$
- Case $\beta > 0$, Value Iteration converges quickly.

Extensions

- Find a way to compute the singular curve analytically
- Consider β as a control.
- ???