Partially Observed Mean Field Games with a Major Player

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Mean Field Games and Related Topics
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Work with Nevroz Şen
Completely Observed Major Minor MFG
Nonlinear MFG Theory Involving Major Agents

Extension of the LQG MFG model for Major and Minor agents (Huang, SIAM 2010) to the case of nonlinear dynamical systems (Nourian and PEC, SIAM 2013)

- Dynamic game models with (i) a major agent, and (ii) a large population of minor agents
- Minor agents react to both mass effect and a major agent.

Motivation and Applications:
- Economic and social models with both minor and massive agents
- Power markets with large consumers and large utilities together with many domestic consumers and generators using smart meters
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- Dominating player: Bensoussan-Yam, 2014
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MFG Nonlinear Major-Minor Agent Formulation

Problem Formulation:

- Notation: Subscript 0 for the major agent $A_0$ and an integer valued subscript for minor agents $\{A_i : 1 \leq i \leq N\}$.
- The states of $A_0$ and $A_i$ are $\mathbb{R}^n$ valued and denoted $z_0^N(t)$ and $z_i^N(t)$.

Dynamics of the Major and Minor Agents:

$$dz_0^N(t) = \frac{1}{N} \sum_{j=1}^{N} f_0(t, z_0^N(t), u_0^N(t), z_j^N(t))dt$$

$$+ \frac{1}{N} \sum_{j=1}^{N} \sigma_0(t, z_0^N(t), z_j^N(t))dw_0(t), \quad z_0^N(0) = z_0(0), \quad 0 \leq t \leq T,$$

$$dz_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} f(t, z_i^N(t), u_i^N(t), z_j^N(t))dt$$

$$+ \frac{1}{N} \sum_{j=1}^{N} \sigma(t, z_i^N(t), z_j^N(t))dw_i(t), \quad z_i^N(0) = z_i(0), \quad 1 \leq i \leq N.$$
MFG Nonlinear Major-Minor Agent Formulation

Cost Functions for Major and Minor Agents:

\[ J^N_0(u^N_0; u^N_{-0}) := E \int_0^T \left( \frac{1}{N} \sum_{j=1}^{N} L_0[t, z^N_0(t), u^N_0(t), z^N_j(t)] \right) dt, \]

\[ J^N_i(u^N_i; u^N_{-i}) := E \int_0^T \left( \frac{1}{N} \sum_{j=1}^{N} L[t, z^N_i(t), z^N_0(t), u^N_i(t), z^N_j(t)] \right) dt. \]

- The major agent has non-negligible influence on the mean field (mass) behaviour of the minor agents: A consequence is that the mean field is no longer a deterministic function of time.

- \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\): a complete filtered probability space

- \(\mathcal{F}^N_t := \sigma\{z_j(s), w_j(s) : 0 \leq j \leq N, 0 \leq s \leq t\}.\)

- \(\mathcal{F}^{w_0}_t := \sigma\{z_0(0), w_0(s) : 0 \leq s \leq t\}.\)

- \(\omega_0\) represents sample point of \(\mathcal{F}^{w_0}_t\).
McKean-Vlasov Approximation

Analyze in the infinite population:

Let \( \varphi_0(t, \omega_0, x), \varphi(t, \omega_0, x) \) be two arbitrary \( F_{t_0}^{\omega_0} \)-measurable processes.

**Stochastic Coefficient McKean-Vlasov (MV) SDE:**

\[
\begin{align*}
\text{dz}_0(t) &= f_0[t, z_0(t), \varphi_0(t), \mu_t(\omega_0)] \, dt + \sigma_0[t, z_0(t), \mu_t(\omega_0)] \, dw_0(t) \\
\text{dz}(t) &= f[t, z(t), \varphi(t), \mu_t(\omega_0)] \, dt + \sigma[t, z(t), \mu_t(\omega_0)] \, dw(t), \quad 0 \leq t \leq T
\end{align*}
\]

\((z_0(\cdot), z(\cdot), \mu(\cdot))\) is a consistent solution if \( \forall \alpha \in \mathbb{R}^n \)

\[
P(z(t) \leq \alpha | F_{t_0}^{\omega_0}) = \int_{-\infty}^{\alpha} \mu_t(\omega_0, dx), \text{ a.s.}
\]

Generate \( N + 1 \) independent samples of the MV SDE system:

\[
\text{dz}_i(t) = f[t, z_i(t), \varphi(t), \mu_t] \, dt + \sigma[t, z_i(t), \mu_t] \, dw_i(t), \quad z_i(0) = z_i^N(0)
\]

**Theorem (McKean-Vlasov Convergence, Nourian and PEC, SIAM’13)**

Denote the closed loop dynamics in the finite population by \( \hat{z}_j^N \). S.t.t.c,

\[
\sup_{0 \leq j \leq N} \sup_{0 \leq t \leq T} \mathbb{E}|\hat{z}_j^N(t) - z_j(t)| = O\left(\frac{1}{\sqrt{N}}\right).
\]
Major-Minor Agents’ Non-Standard SOCPs

Major Agent’s SOCP for an Infinite Population:

\[
\begin{align*}
dz_0(t) &= f_0[t, z_0(t), u_0(t), \mu_t(\omega_0)]dt + \sigma_0[t, z_0(t), \mu_t(\omega_0)]dw_0(t) \\
\inf_{u_0 \in U_0} J_0(u_0) &= \inf_{u_0 \in U_0} \mathbb{E} \int_0^T L[t, z_0(t), u_0(t), \mu_t(\omega_0)] dt. \\
U_0 &= \{u(\cdot) \in U_0 : u \text{ is adapted to } F_t^{w_0} \text{ and } \mathbb{E} \int_0^T |u(t)|^2 dt < \infty\}
\end{align*}
\]

Generic Minor Agent’s SOCP for an Infinite Population:

\[
\begin{align*}
dz_i(t) &= f_i[t, z_i(t), u(t), \mu_t(\omega_0)]dt + \sigma_i[t, z_i(t), \mu_t(\omega_0)]dw_i(t) \\
\inf_{u \in U} J_i(u) &= \inf_{u \in U} \mathbb{E} \int_0^T L[t, z_i(t), z_0(t, \omega_0), u(t), \mu_t(\omega_0)] dt. \\
U &= \{u(\cdot) \in U : u \text{ is adapted to } F_t^{w_0, w_i} \text{ and } \mathbb{E} \int_0^T |u(t)|^2 dt < \infty\}
\end{align*}
\]

\[
\mu_t(\omega_0) = \mathcal{L}(z_i(t)|F_t^{w_0}) \quad \omega_0\text{-dependent Mean Field (Distribution)}
\]

- SOCP with random parameters: SHJB theory by Peng, SIAM’92.
Summary of the Major Agent’s Stochastic MFG (SMFG) System:

M-SHJB
\[-d\phi_0(t, \omega_0, x) = \left[ \inf_{u \in U_0} H_0[t, \omega_0, x, u, D_x \phi_0(t, \omega_0, x)] \right. \]
\[\left. + \langle \sigma_0[t, x, \mu_t(\omega_0)], D_x \psi_0(t, \omega_0, x) \rangle + \frac{1}{2} \text{tr}(a_0[t, \omega_0, x]D_{xx}^2 \phi_0(t, \omega_0, x)) \right\} dt \]
\[-\psi_0^T(t, \omega_0, x) dw_0(t, \omega_0), \quad \phi_0(T, x) = 0 \]

M-SBR
\[u_0^0(t, \omega_0, x) = \arg \inf_{u \in U_0} H_0[t, \omega_0, x, u, D_x \phi_0(t, \omega_0, x)] \]

M-SMV
\[d z_0^0(t, \omega_0) = f_0[t, z_0^0(t, \omega_0), u_0^0(t, \omega_0, z_0^0), \mu_t(\omega_0)] dt \]
\[+ \sigma_0[t, z_0^0(t, \omega_0), \mu_t(\omega_0)] dw_0(t, \omega_0) \]

where \[a_0[t, \omega_0, x] := \sigma_0[t, x, \mu_t(\omega_0)] \sigma_0^T[t, x, \mu_t(\omega_0)] \], and Hamiltonian \[H_0\] is
\[H_0[t, \omega_0, x, u, p] := \langle f_0[t, x, u, \mu_t(\omega_0)], p \rangle + L_0[t, x, u, \mu_t(\omega_0)].\]
Major-Minor Agent Stochastic MFG System

Summary of the Minor Agents’ SMFG System:

\[ -d\phi(t, \omega_0, x) = \left[ \inf_{u \in U} H[t, \omega_0, x, u, D_x \phi(t, \omega_0, x)] \right. \]
\[ + \frac{1}{2} \text{tr} \left( a[t, \omega_0, x] D_{xx}^2 \phi(t, \omega_0, x) \right) \right] dt - \psi^T(t, \omega_0, x) dw_0(t, \omega_0), \quad \phi(T, x) = 0 \]

m-SBR: \[ u^o(t, \omega_0, x) = \arg \inf_{u \in U} H[t, \omega_0, x, u, D_x \phi(t, \omega_0, x)] \]

m-SMV: \[ d\bar{z}^o(t, \omega_0) = f[t, \bar{z}^o(t, \omega_0), u(t, \omega_0, \bar{z}^o), \mu_t(\omega_0)] dt \]
\[ + \sigma[t, \bar{z}^o(t, \omega_0), \mu_t(\omega_0)] dw(t) \]

where \( a[t, \omega_0, x] := \sigma[t, x, \mu_t(\omega_0)] \sigma^T[t, x, \mu_t(\omega_0)] \), Hamiltonian \( H \) is

\[ H[t, \omega_0, x, p] := \langle f[t, x, u, \mu_t(\omega_0)], p \rangle + L[t, x, u, z_0(t, \omega_0), \mu_t(\omega_0)]. \]

- Backward in time SDEs and solutions consist of a pair.
Existence and uniqueness of Solutions to the Major and Minor (MM) Agents' SMFG System: A fixed point argument with random parameters in the space of stochastic probability measures.

\[ \mu(\cdot)(\omega_0) \xrightarrow{\text{M-SHJB}} (\phi_0(\cdot, \omega_0, x), \psi_0(\cdot, \omega_0, x)) \xrightarrow{\text{M-SBR}} u_0(\cdot, \omega_0, x) \]

\[ u^0(\cdot, \omega_0, x) \xleftarrow{\text{m-SBR}} (\phi(\cdot, \omega_0, x), \psi(\cdot, \omega_0, x)) \xleftarrow{\text{m-SHJB}} z_0(\cdot, \omega_0) \]

**Theorem (Existence and Uniqueness of Solutions (Nourian and PEC, SIAM 2013))**

*Under technical conditions including a contraction gain condition there exists a unique fixed point for the map \( \Gamma \), and hence a unique solution to the major and minor agents’ MM-SMFG system.*
Define
\[
U_j := \left\{ u_j(\cdot, \omega_0) \in U_0 := u_i(\cdot, \omega_0, z_0(\cdot, \omega_0), \cdots, z_N(\cdot, \omega_0)) \in C_{\text{Lip}}(z_0, \cdots, z_N) : 
\begin{array}{l}
u_j(t, \omega_0) \text{ is an } \mathcal{F}^w_t \text{ measurable process and adapted to} \\
\sigma\{z_i(\tau, \omega_0), 0 \leq i \leq N, 0 \leq \tau \leq t\} \text{ such that } \mathbb{E} \int_0^T |u_j(t)|^2 dt < \infty
\end{array}\right\}
\]

**Definition (\(\epsilon\)-Nash Equilibrium)**

Given \(\epsilon > 0\), the set of controls \(\{u_0^0; 0 \leq j \leq N\}\) generates an \(\epsilon\)-Nash equilibrium w.r.t. the costs \(J^N_j, 1 \leq j \leq N\) if, for each \(j\),
\[
J^N_j (u_j^0, u_{-j}^0) - \epsilon \leq \inf_{u_j \in U_j} J^N_j (u_j, u_{-j}^0) \leq J^N_j (u_j^0, u_{-j}^0).
\]

**Theorem (Nourian and PEC, SIAM 2013)**

*Subject to technical conditions, the set of infinite population MF best response control processes \((u_0^0, \cdots, u_N^0)\) in a finite \(N\) population system of minor agents generates an \(\epsilon_N\)-Nash equilibrium where \(\epsilon_N = O(1/\sqrt{N})\).*
Recall: Information Structures of Major Minor MFG Best Response Policies:

\[ u^0_0(t, \omega_0, x) \equiv u^0_0 \left( t, x \mid (\mu_s(\omega_0))_{0 \leq s \leq t} \right) \]
\[ = \arg \inf_{u_0 \in U_0} \left\langle f_0[t, x, u_0, \mu_t(\omega_0)], D_x \phi_0(t, \omega_0, x) \right\rangle + L_0[t, x, u_0, \mu_t(\omega_0)] \]
\[ u^0(t, \omega_0, x) \equiv u^0 \left( t, x \mid (\mu_s(\omega_0), z^0_0(s))_{0 \leq s \leq t} \right) \]
\[ = \arg \inf_{u \in U} \left\langle f[t, x, u, \mu_t(\omega_0)], D_x \phi(t, \omega_0, x) \right\rangle + L[t, x, u, z^0_0(t), \mu_t(\omega_0)]. \]

Recall: Information Structures of Minor Only MFG Best Response Policies:

\[ u(t, x) \equiv u \left( t, x \mid (\mu_s)_{0 \leq s \leq t} \right) \]
\[ = \arg \inf_{u \in U} \left\langle f[t, x, u, \mu_t], D_x \phi(t, x) \right\rangle + L[t, x, u, \mu_t]. \]

\[ \triangleright \text{ Estimation or observation by } u_j \text{ of } z^0_{-j} \text{ has at most } \epsilon \text{ value.} \]
Partially Observed Major Minor MFG
Major Agent’s State Estimation Problem for $A_i$:

Dynamics of the signal and the observation

\[
\begin{align*}
    dz_0(t, \omega_0) &= f_0 [t, z_0, u_0, \mu_t(\omega_0)] \, dt + \sigma_0 [t, z_0, \mu_t(\omega_0)] \, dw_0(t, \omega_0) \\
    dy_i(t, \omega_i) &= h (t, z_0(t)) \, dt + d\nu_i(t, \omega_i).
\end{align*}
\]

Seek recursive expressions for $E [\ell(z_0(t))|\mathcal{F}_t^{y_i}]$ with $\ell \in C^2_b (\mathbb{R}^n)$. 

Major Agent's State Estimation Problem for $A_i$:

Dynamics of the signal and the observation

\[ dz_0(t, \omega_0) = f_0 [t, z_0, u_0, \mu_t(\omega_0)] dt + \sigma_0 [t, z_0, \mu_t(\omega_0)] dw_0(t, \omega_0) \]
\[ dy_i(t, \omega_i) = h(t, z_0(t)) dt + d\nu_i(t, \omega_i). \]

Seek recursive expressions for $E[\ell(z_0(t)) | F_{t_i}^{y_i}]$ with $\ell \in C_0^2 (\mathbb{R}^n)$.

Nonlinear filtering theory for SMV Systems

- In order to convert the partially observed MM-MFG System into a fully observed one, we shall find a (tractable) expression for the conditional density.
Define the adjoint operator on $C^2(\mathbb{R}^n)$:

$$L^*(t, \omega_0)\theta = \frac{1}{2} \sum_{j,l=1}^{n} \frac{\partial^2}{\partial x_j \partial x_l} (\sigma_0 \sigma_0^T)_{jl} \theta - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} [f_0 \theta].$$

Assume that for $A \in \mathcal{B}(\mathbb{R}^n)$, $P(z_0(t) \in A|\mathcal{F}_t^{y_i}) = \int_A \varphi_i(t, x, \omega_i) dx$, where $\varphi_i(\cdot)$ is $(t, x)$-measurable and $\mathcal{F}_t^{y_i}$ adapted for each $t \in [0, T]$. Then, s.t.t.c., $\varphi_i(t, x, \omega_i)$ satisfies the following stochastic PDE: For every $t$,

$$\varphi_i(t, x, \omega_i) = \varphi_i(0, x) + \int_0^t L^*(s, \omega_0)\varphi_i(s, x, \omega_i) ds + \int_0^t \varphi_i(s, x, \omega_i) \left\{ h^T(t, x) - \int_{\mathbb{R}^n} h^T(s, v) \varphi_i(s, v, \omega_i) dv \right\} dI_i(s)$$

for a.e. $x$ with probability 1 where $I_i(t) = y_i(t) - \int_0^t \mathbb{E} [h(s, z_0(s))|\mathcal{F}_s^{y_i}] ds$. 
Define the adjoint operator on $C^2(\mathbb{R}^n)$:

$$
\mathcal{L}^*(t, \omega_0)\theta = \frac{1}{2} \sum_{j,l=1}^{n} \frac{\partial^2}{\partial x_j \partial x_l} (\sigma_0 \sigma_0^T)_{jl} \theta - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} [f_{0j} \theta].
$$

Assume that for $A \in \mathcal{B}(\mathbb{R}^n)$, $P(z_0(t) \in A|\mathcal{F}_{t}^{y_i}) = \int_A \varphi_i(t, x, \omega_i) dx$, where $\varphi_i(\cdot)$ is $(t, x)$-measurable and $\mathcal{F}_{t}^{y_i}$ adapted for each $t \in [0, T]$. Then, s.t.t.c., $\varphi_i(t, x, \omega_i)$ satisfies the following stochastic PDE: For every $t$,

$$
\varphi_i(t, x, \omega_i) = \varphi_i(0, x) + \int_0^t \mathcal{L}^*(s, \omega_0)\varphi_i(s, x, \omega_i) ds + \int_0^t \varphi_i(s, x, \omega_i) \left\{ h^T(t, x) - \int_{\mathbb{R}^n} h^T(s, v)\varphi_i(s, v, \omega_i) dv \right\} dI_i(s)
$$

for a.e. $x$ with probability 1 where $I_i(t) = y_i(t) - \int_0^t \mathbb{E} [h(s, z_0(s)) | \mathcal{F}_{s}^{y_i}] ds$.

- For some $k \geq 1$, $\varphi_i(t)$ is $\mathcal{F}_{t}^{y_i}$-adapted, $G_k$-valued (Beneš-Karatsaz’83):

$$
G_k := \{ p \in L^1(\mathbb{R}^n) : \|p\|_k = \int_{\mathbb{R}^n} (1 + |x|^k) |p(x)| dx < \infty \}.
$$
Partially Observed MM-MFG Systems

Analysis of the Minor Agents’ PO SOCP:

- State dynamics:
  \[ dz_i(t) = f \[ t, z_i(t), u_i(t), \mu_t(\omega_i) \] dt + \sigma \[ t, z_i(t), \mu_t(\omega_i) \] dw_i(t). \]
  
  where \( \mu_t(\omega_i) = \mathcal{L}(z_i(t)|\mathcal{F}^i_t) \).

- Cost function:
  \[
  \inf_{u_i \in \mathcal{U}} J_i(u_i) := \inf_{u_i \in \mathcal{U}} \mathbb{E} \int_0^T L \[ t, z_0(t), z_i(t), u_i(t), \mu_t(\omega_i) \] dt
  \]
  
  where \( \mathcal{U} := \{ u_i(\cdot) \in U : u_i(t) \text{ is } \mathcal{F}^{z_i}_t \text{ - adapted} \} \), \( z_i(t) := (z_i(t), y_i(t)) \).

Apply Separation Theorem:

- Standard approach to PO SOCP: Replace state with its density, obtain FO SOCP with infinite dimensional state.

\[ \mathbb{E} \int_0^T L[s, z_0^i(s), z_i(s), u_i(s), \mu_s(\omega_i)]ds = \mathbb{E} \int_0^T L[s, \varphi_i(s), z_i(s), u_i(s), \mu_s(\omega_i)]ds. \]

A SOCP with random parameters and infinite dimensional state.
Partially Observed MM-MFG Systems

Analysis of Minor Agents’ PO SOCP:

The value function for agent $i$, $V_i(t, y(t))$:

$$V_i(t, y(t)) = \inf_{u_i \in \mathcal{U}} \mathbb{E}_{\mathcal{F}_t} \int_t^T L[s, \varphi_i(s), z_i(s), u_i(s), \mu_s(\omega_i)] \, ds$$

where $y(t) := (x(t), p(t))$ is the initial condition of $\bar{z}_i(\cdot)$

By the Principle of Optimality and the Martingale Representation Theorem:

$$V_i(t, y(t)) = \int_t^T \Gamma_i(s, y(s)) \, ds - \int_t^T \Psi_i^T(s, y(s)) \, dI_i(s), \quad t \in [0, T]$$

where $\Gamma_i(s, y(s))$ and $\Psi_i^T(s, y(s))$ are $\mathcal{F}_{s_i}^i$-adapted processes.
Analysis of Minor Agents’ PO SOCP:

The value function for agent $i$, $V_i(t, y(t))$:

$$V_i(t, y(t)) = \inf_{u_i \in U} \mathbb{E}_{\mathcal{F}_t} \int_t^T L\left[s, \varphi_i(s), z_i(s), u_i(s), \mu_s(\omega)\right] ds$$

where $y(t) := (x(t), p(t))$ is the initial condition of $z_i(\cdot)$

By the Principle of Optimality and the Martingale Representation Theorem:

$$V_i(t, y(t)) = \int_t^T \Gamma_i(s, y(s)) ds - \int_t^T \Psi_i^T(s, y(s)) dI_i(s), \; t \in [0, T]$$

where $\Gamma_i(s, y(s))$ and $\Psi_i^T(s, y(s))$ are $\mathcal{F}_{s}^{I_i}$-adapted processes.

To obtain SHJB: Extend Peng’92 to density valued state setting.
Partially Observed MM-MFG Systems

Analysis of Minor Agents' PO SOCP:

The value function for agent $i$, $V_i(t, y(t))$:

$$V_i(t, y(t)) = \inf_{u_i \in \mathcal{U}} \mathbb{E} \int_t^T L[s, \varphi_i(s), z_i(s), u_i(s), \mu_s(\omega_i)] ds$$

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By the Principle of Optimality and the Martingale Representation Theorem:

$$V_i(t, y(t)) = \int_t^T \Gamma_i(s, y(s)) ds - \int_t^T \Psi_i^T(s, y(s)) dI_i(s), \ t \in [0, T]$$

where $\Gamma_i(s, y(s))$ and $\Psi_i^T(s, y(s))$ are $\mathcal{F}_s^{I_i}$-adapted processes.

To obtain SHJB: Extend Peng'92 to density valued state setting.

Essential tool: Extend Itô-Kunita formula to the density-valued SDEs.
SHJB for Minor Agents’ PO SOCP:

**Theorem (N$\mathcal{S}$ and PEC, CDC 2014)**

Assume that $V_i(t, x, p)$, $\Psi_i(t, x, p)$, $\nabla_x V_i(t, x, p)$, $\nabla^2_x V_i(t, x, p)$, $D_3 V_i(t, x, p)$ and $D_2^3 V_i(t, x, p)$, $(t, x, p) \in [0, T] \times \mathbb{R}^n \times G_k$, are all a.s. continuous in $(t, x, p)$ where $D$ denotes the partial Fréchet derivative. Set $H^* : [0, T] \times \Omega \times \mathbb{R}^n \times G_k \times \mathbb{R}^n \to \mathbb{R}$

$$H^*(t, \omega_i, y, q) := \inf_{u \in U} \left\{ \langle f[t, x, u, \mu_t(\omega_i)], q \rangle + L[t, y, u, \mu_s(\omega_i)] + D_3 V_i(t, \omega_i, y) \cdot L^*(t, \omega_0)p(t) \right\}$$

and set $\hat{p}(t) = p(t)(h^T(t) - h^T \circ p(t))$. Then the pair $(V_i(t, y(t)), \Psi_i(t, y(t)))$ satisfies the following SHJB:

$$-dV_i(t, \omega_i, y) = \left[ H^*(t, \omega_i, y, \nabla_x V_i(t, \omega_i, y)) + \frac{1}{2} \operatorname{tr} \left( a[t, \omega_i, x] \nabla^2_{xx} V_i(t, \omega_i, y) \right) \right] dt - \Psi_i^T(t, \omega_i, y) dI_i(t), \quad V_i(T, y) = 0$$

where $(t, y) \in [0, T] \times \mathbb{R}^n \times G_k$, $(\cdot, \cdot)$ denotes the second-level map operation and $a[t, \omega_i, x] := \sigma[t, x, \mu_t(\omega_i)] \sigma^T[t, x, \mu_t(\omega_i)]$ (Note the retrieval of fully observation case).
MM-MFG System for the Generic Minor Agent:

[Observation and m-MF]
\[ dy(t, \bar{\omega}) = h(t, z_0^o(t)) \, dt + d\nu(t, \bar{\omega}), \quad \mu_t(\bar{\omega}) := \mathcal{L} \left( z^o(t) | \mathcal{F}^\bar{\omega}_t \right). \]

[m-SHJB]
\[ -dV(t, \bar{\omega}, y) = \left[ H^*(t, \bar{\omega}, y, \nabla_x V(t, \bar{\omega}, y)) + \frac{1}{2} \operatorname{tr} (a[t, \bar{\omega}, x] \nabla^2_{xx} V(t, \bar{\omega}, y)) \right. \]
\[ + \frac{1}{2} D_3^2 V(t, \bar{\omega}, y) \cdot (\hat{\rho}(t), \hat{\rho}(t)) \bigg] dt - \Psi^T(t, \bar{\omega}, y) d\bar{I}(t), \quad V(T, y) = 0 \]

[m-BR] \[ u^o(t, \bar{\omega}, y) \equiv u^o(t, \bar{\omega}, y | \mu_s(\bar{\omega})_{0 \leq s \leq t}) \]
\[ := \arg \inf_{u \in U} H^u [t, \bar{\omega}, y, u, \nabla_x V(t, \bar{\omega}, y)] \]
\[ \equiv \arg \inf_{u \in U} \{ \langle f[t, x, u, \mu_t(\bar{\omega})], \nabla_x V(t, \bar{\omega}, y) \rangle + \mathbf{L}[t, y, u, \mu_t(\bar{\omega})] \} \]

[m-SMV] \[ dz^o(t, \bar{\omega}, \tilde{\omega}) = f[t, z^o, u^o(t, z^o, \varphi), \mu_t(\bar{\omega})] \, dt + \sigma[t, z^o, \mu_t(\bar{\omega})] \, dw(t, \tilde{\omega}). \]

Remark: The solution to the SHJB equations are forward in time $\mathcal{F}^{\bar{\omega}}_t$-adapted.
Partially Observed MM-MFG Systems

The major agent has complete observation on its own state.

**SOCP for the Major Agent:**

\[
\begin{align*}
&M-\text{SHJB} \quad -dV_0(t, \omega_0, x) = \\
&\quad \left[ H_0(t, \omega_0, x, \nabla_x V_0(t, \omega_0, x)) \\
&\quad + \langle \sigma_0[t, x, \mu_t(\omega_0)], \nabla_x \psi_0(t, \omega_0, x) \rangle + \frac{1}{2} \text{tr}(a_0(t, \omega_0, x) \nabla_{xx}^2 V_0(t, \omega_0, x)) \right] dt \\
&\quad - \Psi_T^T(t, \omega_0, x) dw_0(t), \quad V_0(T, x) = 0
\end{align*}
\]

\[
\begin{align*}
&M-\text{BR} \quad u_0^\circ(t, \omega_0, x) \equiv u_0^\circ(t, x|\mu_t(\omega_0))_{0 \leq s \leq t} \\
&\quad := \arg \inf_{u_0 \in U_0} \left\{ f_0[[t, x, u_0, \mu_t(\omega_0)], \nabla_x V_0(t, \omega_0, x)] + L_0[t, x, u_0, \mu_t(\omega_0)] \right\}
\end{align*}
\]

\[
\begin{align*}
&M-\text{SMV} \quad dz_0^\circ(t, \omega_0) = f_0[t, z_0^\circ, u_0^\circ(t), \mu_t(\omega_0)] dt + \sigma_0[t, z_0^\circ(t), \mu_t(\omega_0)] dw_0(t)
\end{align*}
\]

where \( \mu_t(\omega_0) := \mathcal{L}(z^\circ(t)|F_t^{\omega_0}) \).
\[ \text{M-FPK}] \text{ Assume } \mu_t(\omega_0, dx) = p_0(t, \omega_0, x)dx \text{ (a.s.), } 0 \leq t \leq T \\
\begin{align*}
\frac{dp_0(t, \omega_0, x)}{dt} &= -\left( \langle \nabla_x, f[t, x, u^o(t, x, p), \mu_t(\omega_0)]p_0(t, \omega_0, x) \rangle \\
&\quad + \frac{1}{2} \text{tr} \langle \nabla^2_{xx}, a[t, \omega_0, x]p_0(t, \omega_0, x) \rangle \right) \right) dt, \quad p_0(0, \omega_0, x) = p_0(0, x)
\end{align*}

\[ \text{m-FPK}] \text{ Assume } \mu_t(\bar{\omega}, dx) = p(t, \bar{\omega}, x)dx \text{ (a.s.), } 0 \leq t \leq T \\
\begin{align*}
\frac{dp(t, \bar{\omega}, x)}{dt} &= -\left( \langle \nabla_x, f[t, x, u^o(t, x, p), \mu_t(\bar{\omega})]p(t, \bar{\omega}, x) \rangle \\
&\quad + \frac{1}{2} \text{tr} \langle \nabla^2_{xx}, a[t, \bar{\omega}, x]p(t, \bar{\omega}, x) \rangle \right) \right) dt, \quad p(0, \bar{\omega}, x) = p(0, x)
\end{align*}
Solution to the PO MM-MFG System:

- Similar to the completely observed situation, the solution of the PO MM-MM-MFG system consists of the following 9-tuple adapted processes:
  
  \[
  \left( V_0(t, \omega_0, x), \Psi_0(t, \omega_0, x), u_0^0(t, \omega_0, x), z_0^0(t, \omega_0),
  \right.
  
  \[\varphi(t), V(t, \bar{\omega}, y), \Psi(t, \bar{\omega}, y), u^o(t, \bar{\omega}, y), z^o(t, \bar{\omega}) \] 
  
  with \( \mu_t(\bar{\omega}) := \mathcal{L} \left( z^o(t) | \mathcal{F}_t^\bar{\omega} \right) \) and \( \mu_t(\omega_0) := \mathcal{L} \left( z^o(t) | \mathcal{F}_t^{\omega_0} \right) \).

- One needs to prove the coupled system has a unique solution.
Analysis of the PO MM-SMFG System

Existence and Uniqueness of Solutions to the PO MM SMFG System: A fixed point argument with random parameters in the space of stochastic probability measures.

\[
\mu(\cdot)(\bar{\omega}) \xrightarrow{m-\text{SHJB}} (V_0(\cdot, \omega_0, x), \Psi_0(\cdot, \omega_0, x)) \xrightarrow{m-\text{SBR}} u_0^o(\cdot, \omega_0, x) \downarrow \xrightarrow{m-\text{SMV}} \]

\[
u_0^o(\cdot, \bar{\omega}, y) \xleftarrow{m-\text{SBR}} (V(\cdot, \bar{\omega}, y), \Psi(\cdot, \bar{\omega}, y)) \xleftarrow{m-\text{SHJB}} \phi(\cdot, x) \xleftarrow{\text{NLF}} z_0^o(\cdot, \omega_0)
\]

where

\[
\phi(t, x) = \phi(0, x) + \int_0^t \mathcal{L}^*(s, \omega_0) \phi(s, x) ds + \int_0^t \phi(s, x) \left\{ h^T(t, x) - \int_{\mathbb{R}^n} h^T(s, v) \phi(s, v) dv \right\} d\bar{I}(s)
\]

**Theorem (Existence and Uniqueness of Solutions and \(\epsilon\)-Nash Equilibrium, NŠ, and PEC, CDC 2014, 2015)**

Under the conditions on the Lipschitz coefficients of \(h, f_0, f, \sigma_0, \sigma, L_0, L, u_0^o\) and \(u^o\), the PO MM SMFG system has a unique solution. Furthermore, s.t.t.c, for all \(N \geq 1\), \((u_0^o, \cdots, u_N^o)\) generates an \(O(1/\sqrt{N})\)-Nash equilibrium w.r.t. the cost functions \(J_j^N(u_j; u_{-j}), 0 \leq j \leq N\), defined on the finite \(N + 1\) population system.
Concluding Remarks

Current and Future Work

Extend the theory to the cases:

- Where the **major agent's state** is also partially observed, which involves **minor agents' estimates of the major estimates** (Progress in the LQG case by Dena Firoozi and PEC).

- Where the initial conditions of the mean field (i.e. the initial state distribution of the mass of minor agents) is not known to the minor agents.
  
  Each requires NLF for the conditional measure (or density) of partially observed conditional measure valued processes.

- **Applications to markets (e.g. power)** where minor agents (customers and suppliers) receive **intermittent observations on active major agents** (such as utilities and international energy prices) and on **passive major agents** (such as wind and ocean behaviour).

- More than one major player: Decentralized stochastic control, information structures.