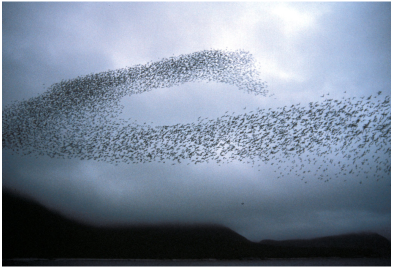
Alignment of self-propelled particles : from microscopic to ~~hydrodynamic~~ kinetic models.

Amic Frouvelle – CEREMADE – Université Paris Dauphine

Mathematical Methods for the Study of Self-organization in the Biological Sciences, ESI, Vienna, November 14th-18th, 2022



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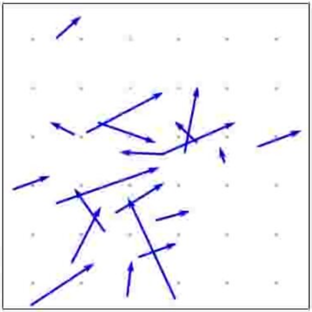
First example: Cucker–Smale Model

N particles, at positions $x_i \in \mathbb{R}^d$, aligning their velocities $v_i \in \mathbb{R}^d$.

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = v_i,$$

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{1}{N}\sum_{j=1}^{N} K(|x_j - x_i|)(v_j - v_i).$$



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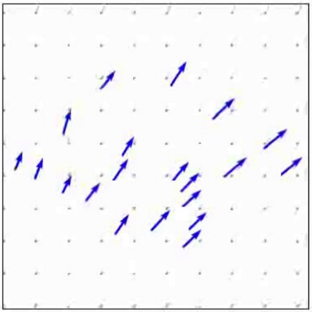

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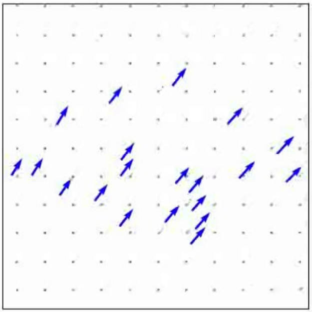

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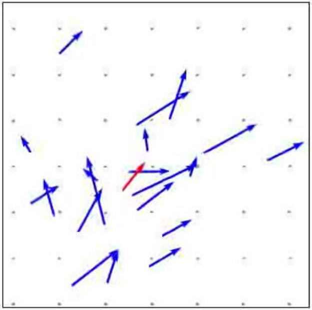
Theorem : Cucker-Smale 2007 [CS07, HL09]

Flocking if $\int_0^{+\infty} K(r)\mathrm{d}r = \infty$.
For any initial condition, there exists $C, \lambda, M > 0$ and $v_\infty \in \mathbb{R}^d$ such that $|v_i(t) - v_\infty| \leqslant Ce^{-\lambda t}$ and $|x_i(t) - x_j(t)| \leqslant M$ (for all i, j and $t \in \mathbb{R}$).



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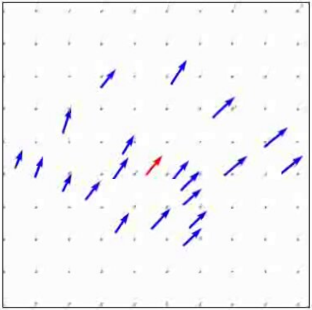

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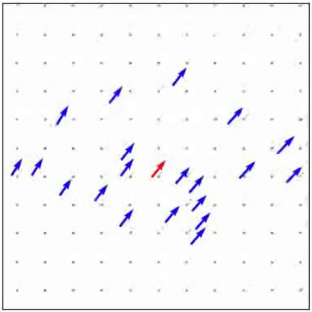
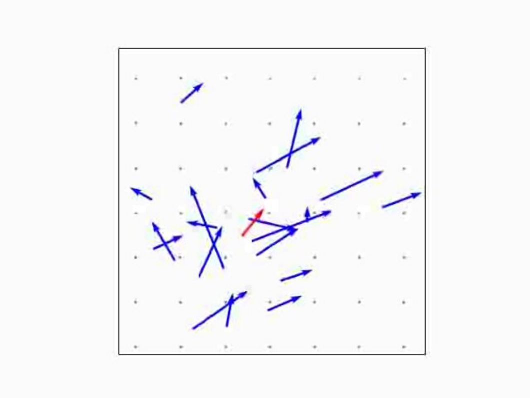


First example: Cucker–Smale Model

Exercise : a strong observation kernel is needed

For 2 particles, with $x_1 = x_2 = 0$ initially, prove that there exists some "observation kernel" $K > 0$ such that $|x_1(t) - x_2(t)| \to +\infty$.
Idea : reverse engineering (start by given trajectories $x_1(t)$ and $x_2(t)$ and find K).



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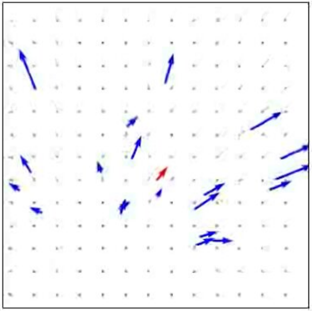

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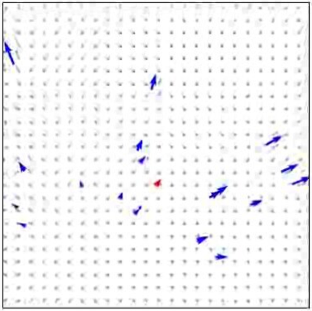
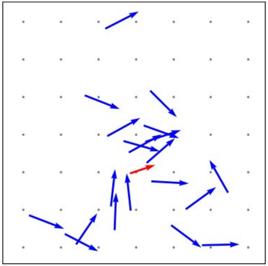


Second example: aligning self-propelled particles (2D)

N particles, at positions $x_i \in \mathbb{R}^2$, with speed 1 and angle $\theta_i \in \mathbb{R}$.

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix} = \tau(\theta_i), \qquad \frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \frac{1}{N}\sum_{j=1}^{N} K(x_j - x_i)\sin(\theta_j - \theta_i).$$



Exercise : well-posed ODE in $\mathbb{R}^{3N}$

This is an ODE of the form $\frac{\mathrm{d}Z}{\mathrm{d}t} = B(Z)$. If K is bounded and Lipschitz, then B is globally Lipschitz and Cauchy-Lipschitz theorem applies.

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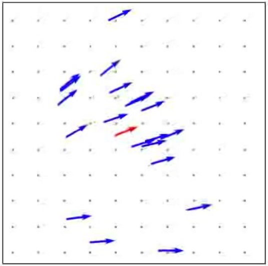

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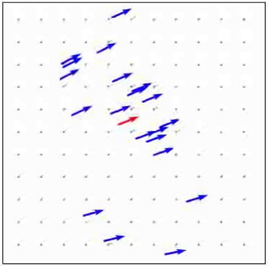

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Empirical measure, indistinguishable particles

Definition : Empirical measure $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \otimes \delta_{\theta_i}$

$$\forall \varphi \in C^b(\mathbb{R}^2 \times \mathbb{R}), \langle \varphi \rangle_{\mu_t^N} = \int_{\mathbb{R}^2\times\mathbb{R}} \varphi \, \mathrm{d}\mu_t^N = \frac{1}{N}\sum_{i=1}^N \varphi(x_i(t), \theta_i(t))$$

Exchangeability : μ_t^N invariant under label permutations, all the particles $z_i = (x_i, \theta_i)$ follow the (non-autonomous) flow

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \tau(\theta) \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} = b[\mu_t^N](x, \theta), \end{cases}$$

where we have set $b[f](x, \theta) = \int_{\mathbb{R}^2\times\mathbb{R}} K(x' - x) \sin(\theta' - \theta) \, \mathrm{d}f(x', \theta')$.

Exercise : close equation for μ_t^N !

Prove that μ_t^N is a weak solution of the following kinetic equation :

$$\partial_t \mu_t^N + \tau(\theta) \cdot \nabla_x \mu_t^N + \partial_\theta (b[\mu_t^N]\mu_t^N) = 0.$$

Vlasov equation

Useful notation : $\mathcal{J}_f = \int_{\mathbb{R}} \tau(\theta) \mathrm{d}f(\theta)$, and $\mathcal{J}_f^K = K * \mathcal{J}_f$. We then get $b[f](x,\theta) = \partial_\theta(\tau(\theta) \cdot \mathcal{J}_f^K)$.

Theorem : global well-posedness

If K is bounded and Lipschitz, then the Vlasov equation

$$\partial_t f + \tau(\theta) \cdot \nabla_x f + \partial_\theta(\partial_\theta(\tau(\theta) \cdot \mathcal{J}_f^K) f) = 0$$

is well-posed in $C([0,T], \mathcal{P}_1(\mathbb{R}^2 \times \mathbb{R}))$.

Ref.: H. Spohn, "Large scale dynamics of interacting particles" [Spo91].

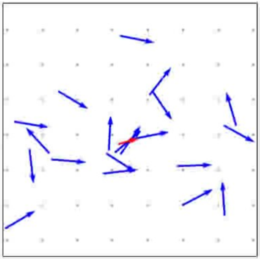
- Solve the linear (nonautonomous) equation $\partial_t f + \tau(\theta) \cdot \nabla_x f + \partial_\theta(\partial_\theta(\tau(\theta) \cdot J(x,t)) f) = 0$ by characteristics : $f = \Phi(J)$.
- Fixed point in $C([0,T], \mathcal{P}_1(\mathbb{R}^2 \times \mathbb{R}))$ of $f \mapsto \Phi(\mathcal{J}_f^K)$. Links between the Lipschitz character of K and contraction in Wasserstein distance.

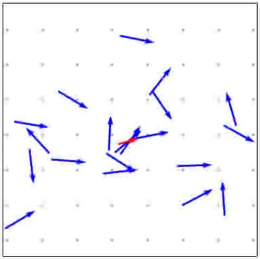
Third example: same thing, with noise !

System of stochastic differential equations (SDE).

$$\mathrm{d}x_i = \begin{pmatrix} \cos\theta_i(t) \\ \sin\theta_i(t) \end{pmatrix} \mathrm{d}t,$$

$$\mathrm{d}\theta_i = \frac{1}{N}\sum_{j=1}^{N} K(|x_j - x_i|)\sin(\theta_j - \theta_i)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_t^i.$$





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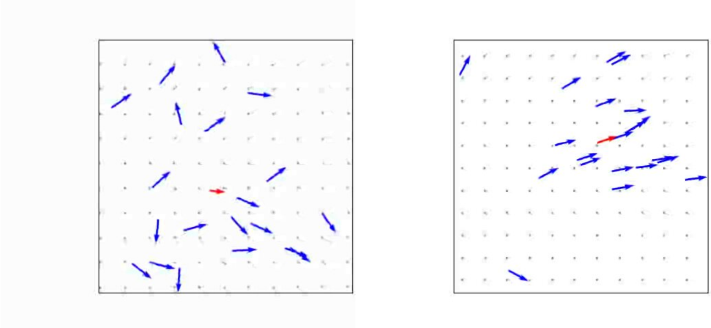

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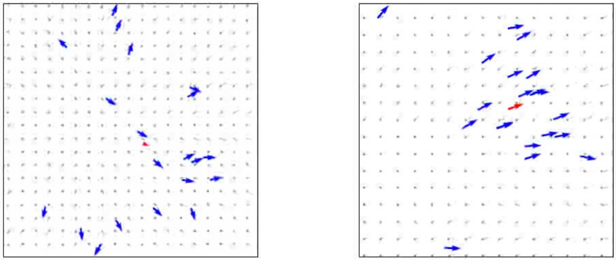
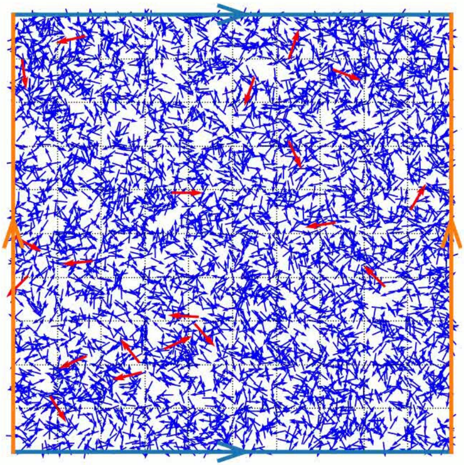


Artificial confinement

If $x \mapsto K(|x|)$ is periodic (think with small compact support + periodic extension), then everything works in $C([0,T], \mathcal{P}(\mathbb{R}^2/\Gamma \times \mathbb{R}/2\pi\mathbb{Z}))$.



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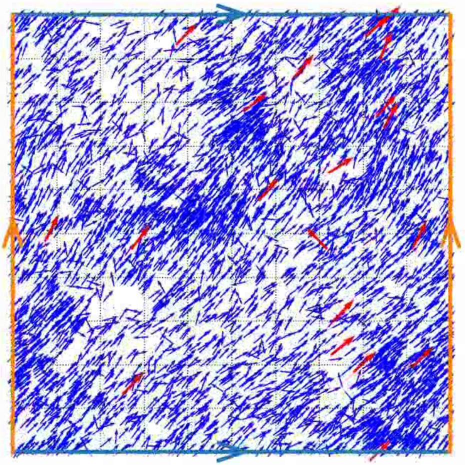

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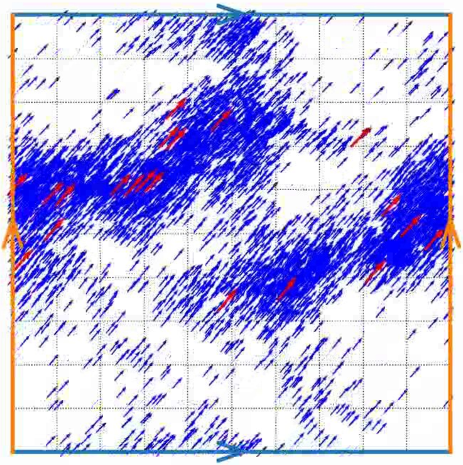

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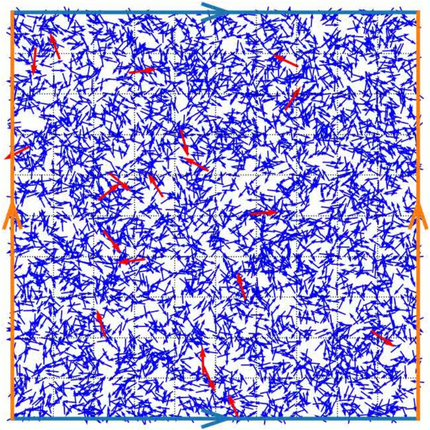

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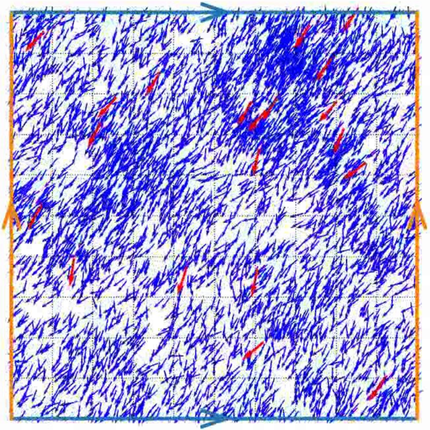

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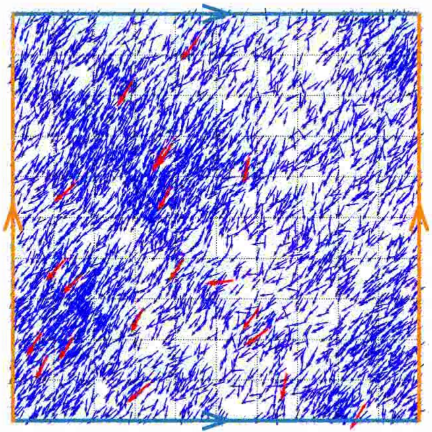

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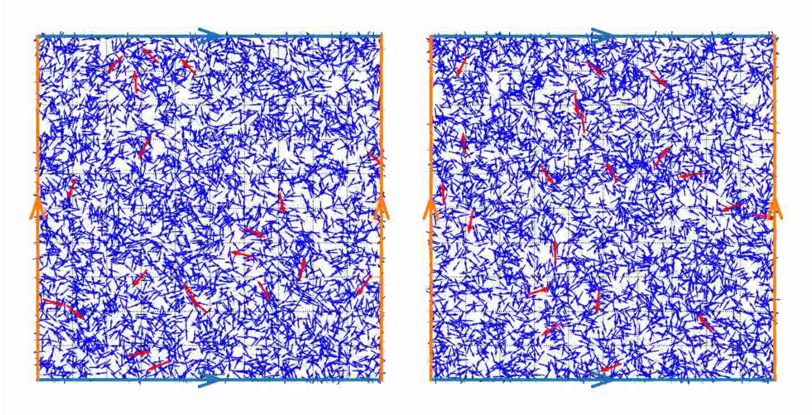

Now μ_t^N is a probabilistic object (realisations of the brownian motions).
How to get a kinetic equation for that ?

$\rightsquigarrow$ Law of large numbers. . . $\langle \varphi \rangle_{\mu_t^N} = \frac{1}{N} \sum_{i=1}^{N} \varphi(x_i(t), \theta_i(t)) \xrightarrow{?} \langle \varphi \rangle_{\mu_t}$.

Main ref.: A.-S. Sznitman, "Topics in propagation of chaos" [Szn91].

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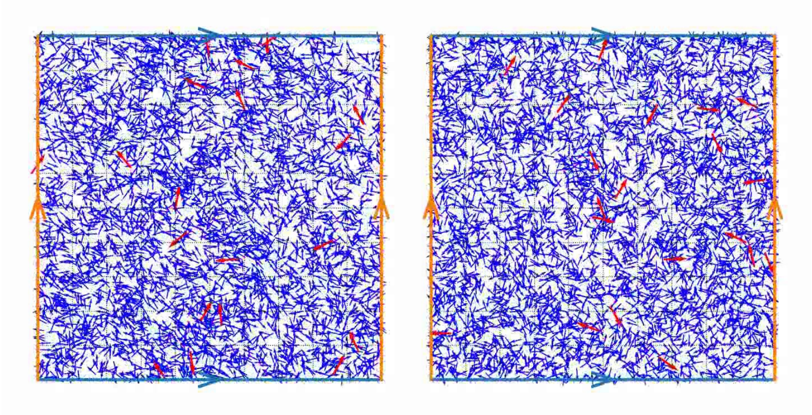

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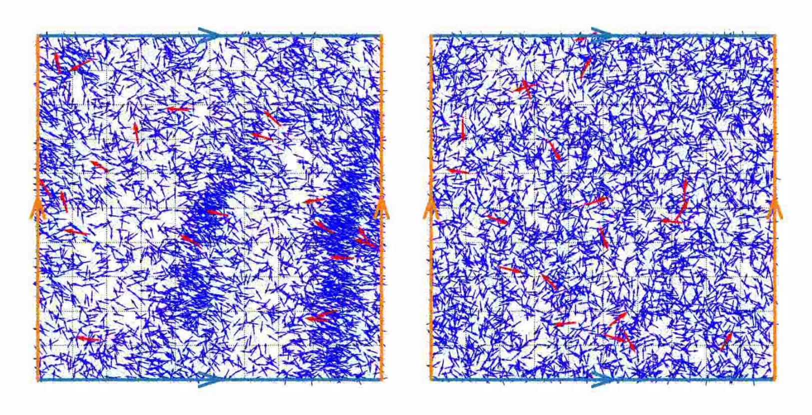

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Existence theory for SDEs

Rewrite the system in integral formulation, and with $b[\mu_t^N]$ (exchangeability).

$$x_i(t) = x_i^0 + \int_0^t \tau(\theta_i(s))\,\mathrm{d}s,$$

$$\theta_i(t) = \theta_i^0 + \int_0^t b[\mu_s^N](x_i(s), \theta_i(s))\,\mathrm{d}s + \sqrt{2\sigma}\,B_t^i.$$

If K is bounded and Lipschitz : fixed point argument as in Cauchy-Lipschitz theory. Pathwise global existence and uniqueness of solutions.

Law of the $3N$-dimensional process (linear), Ito formula

Probability density $f^N(x_1, \theta_1, \dots)$ of finding particles around (x_i, θ_i) :

$$\partial_t f^N + \sum_{i=1}^N [\tau(\theta_i) \cdot \nabla_{x_i} f^N + \partial_{\theta_i}(b[\mu^N](x_i, \theta_i) f^N)] = \sigma \sum_{i=1}^N \partial^2_{\theta_i \theta_i} f^N,$$

where $b[\mu^N](x, \theta) = \dfrac{1}{N} \sum_{j=1}^N K(x_j - x) \sin(\theta_j - \theta)$.

Marginals, as in BBGKY hierarchy ?

Exercise : equation of the first marginal

If $f^{N,1}(x,\theta) = \int f^N(x,\theta,x_2,\theta_2\dots)\mathrm{d}x_2\mathrm{d}\theta_2\dots\mathrm{d}x_n\mathrm{d}\theta_n$
and $f^{N,2}(x,\theta,x',\theta') = \dots$, then

$$\partial_t f^{N,1} + \tau(\theta)\cdot\nabla_x f^{N,1} + \tfrac{N-1}{N}\partial_\theta(b[f^{N,2}(x,\theta,\cdot,\cdot)](x,\theta)) = \sigma\partial^2_{\theta\theta}f^{N,1}.$$

If $f^{N,1}\to f$ and $f^{N,2}\to f\otimes f$ (asymptotic indep. of two particles), then

$$\partial_t f + \tau(\theta)\cdot\nabla_x f + \partial_\theta(b[f]f) = \sigma\partial^2_{\theta\theta}f.$$

This is very formal, but we recover the (nonlinear) Vlasov equation with a diffusion. Can we rigorously derive this equation ?
Notice that it can be written (since $b[f] = \partial_\theta(\mathcal{J}^K_f\cdot\tau(\theta))$) as

$$\partial_t f + \tau(\theta)\cdot\nabla_x f = \sigma\partial_\theta\left(e^{\frac{1}{\sigma}\mathcal{J}^K_f\cdot\tau(\theta)}\partial_\theta\left(\frac{f}{e^{\frac{1}{\sigma}\mathcal{J}^K_f\cdot\tau(\theta)}}\right)\right).$$

The nonlinear coupling processes

We look at processes which are the natural limits of (x_i, θ_i) as $N \to \infty$.

Theorem [Szn91]

There exist a unique solution, pathwise and in law, to

$$\bar{x}(t) = x^0 + \int_0^t \tau(\bar{\theta}(s))\,\mathrm{d}s,$$

$$\bar{\theta}(t) = \theta^0 + \int_0^t b[\bar{\mu}_s](\bar{x}(s), \bar{\theta}(s))\,\mathrm{d}s + \sqrt{2\sigma}\,B_t,$$

whee $\bar{\mu}_t$ is the law of the process $\bar{x}(t), \bar{\theta}(t)$: $\mathbb{E}[\varphi(\bar{x}(t), \bar{\theta}(t))] = \langle \varphi \rangle_{\bar{\mu}_t}$.

- No more index i, no more N, only one particle interacting with its own law.
- Fixed point problem for the law $\bar{\mu}_t$.
- PDE satisfied by $\bar{\mu}_t$? by Itō formula, we get

$$\partial_t \bar{\mu}_t + \tau(\theta) \cdot \nabla_x \bar{\mu}_t + \partial_\theta (b[\bar{\mu}_t]\bar{\mu}_t) = \sigma \partial^2_{\theta\theta} \bar{\mu}_t.$$

Distance between processes

We therefore introduce $(\bar{x}_i, \bar{\theta}_i)$, the nonlinear coupling processes associated to the same initial conditions x_i^0, θ_i^0 and Brownian motion B_t^i. Their law is always $\bar{\mu}_t$ (if the initial condition are iid), and they are independent. They satisfy

$$x_i - \bar{x}_i(t) = \int_0^t [\tau(\theta_i) - \tau(\bar{\theta}_i)](s)\, \mathrm{d}s,$$

$$\theta_i - \bar{\theta}_i(t) = \int_0^t \left(b[\mu_s^N](x_i(s), \theta_i(s)) - b[\bar{\mu}_s](\bar{x}_i(s), \bar{\theta}_i(s))\right) \mathrm{d}s,$$

For convenience : $z_t^i = (x_i(t), \theta_i(t))$, $\bar{z}_t^i = (\bar{x}_i(t), \bar{\theta}_i(t))$, and the new empirical distribution $\bar{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\bar{z}_t^i}$. We obtain

$$\begin{aligned} |\theta_i - \bar{\theta}_i(t)| \leqslant \int_0^t |b[\mu_s^N](z_s^i) - b[\mu_s^N](\bar{z}_s^i)| \mathrm{d}s \quad & (\text{use } b[f](\cdot) \text{ Lipschitz}) \\ + \int_0^t |b[\mu_s^N](\bar{z}_s^i) - b[\bar{\mu}_s^N](\bar{z}_s^i)| \mathrm{d}s \quad & (\text{use } b[\cdot](z) \text{ Lipschitz}) \\ + \int_0^t |b[\bar{\mu}_s^N](\bar{z}_s^i) - b[\mu_s](\bar{z}_s^i)| \mathrm{d}s \quad & (\text{LLN for indep. processes}). \end{aligned}$$

Distance between processes (continued)

We obtain, after using exchangeability and taking expectation

$$\mathbb{E}[|z_t^1 - \bar{z}_t^1|] \leqslant C \int_0^t \mathbb{E}[|z_s^1 - \bar{z}_s^1|]\mathrm{d}s + \int_0^t \mathbb{E}[|b[\bar{\mu}_s^N](\bar{z}_s^1) - b[\mu_s](\bar{z}_s^1)|]\mathrm{d}s.$$

Recall: $b[\bar{\mu}_s^N](\bar{z}_s^1) = \frac{1}{N}\sum_{i=1}^N \beta(\bar{z}_s^i, \bar{z}_s^1)$ and $b[\mu_s](\bar{z}_s^1) = \int \beta(z, \bar{z}_s^1)\mathrm{d}\mu_s(z)$, with $\beta(z, z') = K(x - x')\sin(\theta - \theta')$, so β is bounded and Lipschitz. Therefore $\mathbb{E}[|b[\bar{\mu}_s^N](\bar{z}_s^1) - b[\mu_s](\bar{z}_s^1)|] = \mathbb{E}\Big[\Big|\frac{1}{N}\sum_{i=1}^N Y_i\Big|\Big]$, where the random variables $Y_i = \beta(\bar{z}_s^i, \bar{z}_s^1) - \int \beta(z, \bar{z}_s^1)\mathrm{d}\mu_s(z)$ are centered, bounded, pairwise independent. We get $\mathbb{E}\Big[\Big|\frac{1}{N}\sum_{i=1}^N Y_i\Big|\Big]^2 \leqslant \frac{1}{N^2}\mathbb{E}\Big[\Big(\sum_{i=1}^N Y_i\Big)^2\Big] \leqslant \dfrac{C}{N}$.

Final estimation (with Gronwall)

$$\sup_{t\in[0,T]} W_1(f^{N,1}(t), \bar{\mu}_t) \leqslant \sup_{t\in[0,T]} \mathbb{E}[|z_t^1 - \bar{z}_t^1|] \leqslant \frac{e^{CT}}{\sqrt{N}}.$$

Remarks, to go beyond

- More details in [Szn91] : convergence of the empirical measure, case of non independent initial conditions, but "chaotic".
- No propagation of chaos uniformly in time : $\frac{e^{CT}}{\sqrt{N}}$ is small only if $T \ll \ln N$.
- Can be adapted to non binary collisions, when $b[\mu^N]$ is not linear in μ^N, but only Lipschitz, and when σ also depends on μ^N [Oel84].
- Same model in higher dimensions (on the sphere) : [BCC12]

Unit speed particles $(X_i, V_i) \in \mathbb{R}^d \times \mathbb{S}^{d-1}$

$$\mathrm{d}X_i = V_i \mathrm{d}t,\ \mathrm{d}V_i = \frac{1}{N}\sum_{j=1}^N K(X_i - X_j) P_{V_i^\perp} V_j + \sqrt{2\sigma} P_{V_i^\perp} \circ \mathrm{d}B_t^i.$$

Careful : orthogonal projection to keep unit speed + Stratonovich formulation for diffusion on the sphere. Limit law, density $f(t, x, v)$:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (\nabla_v(\mathcal{J}_f^K \cdot v) f) = \sigma \Delta_v f \\ \mathcal{J}_f^K = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} K(x - y) v\, f(t, y, v) \mathrm{d}y \mathrm{d}v. \end{cases}$$

Careful : divergence and gradient on the sphere, Laplace-Beltrami.

Space-homogeneous model

No space, or $K = 1$: only equations on angles (f is 2π-periodic now).

$$\begin{cases} \partial_t f + \partial_\theta(\partial_\theta(\mathcal{J}_f \cdot \tau(\theta))f) = \sigma \partial_{\theta\theta} f \\ \mathcal{J}_f = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tau(\theta)\, f(t,\theta)\mathrm{d}\theta. \end{cases}$$

Fokker-Planck factorized formulations :

$$\partial_t f = \sigma \partial_\theta (e^{\frac{1}{\sigma}\mathcal{J}_f \cdot \tau(\theta)} \partial_\theta (e^{-\frac{1}{\sigma}\mathcal{J}_f \cdot \tau(\theta)} f)) = \sigma \partial_\theta \Big(M_{\mathcal{J}_f} \partial_\theta \big(\frac{f}{M_{\mathcal{J}_f}} \big) \Big),$$

with von-Mises distribution

$$M_J(\theta) = \frac{e^{\frac{1}{\sigma} J \cdot \tau(\theta)}}{\mathcal{Z}(J)}, \quad \mathcal{Z}(J) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\frac{|J|}{\sigma}\cos\theta}\mathrm{d}\theta.$$

Useful : $\partial_t f = \sigma\, \partial_\theta(\partial_\theta(\ln f - \frac{1}{\sigma}\mathcal{J}_f \cdot \tau(\theta))f)$.

Dissipation of free energy $\mathcal{F}(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f \ln f \mathrm{d}\theta - \frac{1}{2\sigma}\|\mathcal{J}_f\|^2$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(f) = -\frac{\sigma}{2\pi}\int_{-\pi}^{\pi} |\partial_\theta(\ln f - \tfrac{1}{\sigma}\mathcal{J}_f \cdot \tau(\theta))|^2 f \mathrm{d}\theta := -\sigma \mathcal{D}[f].$$

Homogeneous steady-states

Equivalent conditions for steady states of FP equation

- $\mathcal{D}[f] = 0$.
- $f = M_J$ where $\mathcal{J}_{M_J} = \langle \tau \rangle_{M_J} = \frac{1}{2\pi} \int_{\pi}^{\pi} \tau(\theta) M_J(\theta) \mathrm{d}\theta = J$.
- $f = M_J$ where J is a critical point of $V(J) = \frac{1}{2\sigma} \|J\|^2 - \ln \mathcal{Z}(J)$.
- f is a critical point of $\mathcal{F}$ under unit mass constraint.
- $f = M_J$ where J is a critical point of $W(J) = \mathcal{F}[M_J]$.
- $f = M_{\kappa\Omega}$ for $\Omega \in \mathbb{S}^1$ and $\kappa = c(\kappa)$ for $c(\kappa) = \dfrac{\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos\theta e^{\frac{\kappa}{\sigma}\cos\theta} \mathrm{d}\theta}{\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\frac{\kappa}{\sigma}\cos\theta} \mathrm{d}\theta}$.

$\frac{\kappa}{c(\kappa)} \nearrow +\infty$ as $\kappa \to +\infty$, and $\searrow 2\sigma$ as $\kappa \to 0$.

Families of equilibria : threshold of phase transition

- Case $\sigma \geqslant \frac{1}{2}$. Only solution : $\kappa = 0$. Isotropic equilibrium (global minimizer of $\mathcal{F}$, 0 is global min for W and V).
- Case $\sigma < \frac{1}{2}$. Either isotropic (saddle for $\mathcal{F}$, local max for W, V), or unique solution $\kappa(\sigma) > 0$: family of von-Mises distributions (global min of $\mathcal{F}$, same for W, V). Close to Diracs if σ small.

Long-time behaviour : modified entropy methods

Relative entropy and Fisher information.

$$\mathcal{H}(f|g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f \ln\left(\frac{f}{g}\right)\mathrm{d}\theta, \quad \mathcal{I}(f|g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f \left|\partial_\theta \ln\left(\frac{f}{g}\right)\right|^2 \mathrm{d}\theta.$$

Free energy / relative entropy identities :

- $\mathcal{F}[f] - \mathcal{F}[f_{\mathrm{eq}}] = \mathcal{H}(f|M_{\mathcal{J}_f}) + V(\mathcal{J}_f) - V(J_{\mathrm{eq}})$,
- $\mathcal{H}(f|f_{\mathrm{eq}}) = \mathcal{F}[f] - \mathcal{F}[f_{\mathrm{eq}}] + \frac{1}{2\sigma}\|J_{\mathrm{eq}} - \mathcal{J}_f\|^2$,
- $\mathcal{D}[f] = \mathcal{I}(f|M_{\mathcal{J}_f})$.

Csiszár-Kullback-Pinsker and log-Sobolev inequalities

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta) - g(\theta)|\,\mathrm{d}\theta \leqslant \sqrt{2\,\mathcal{H}(f|g)}.$$

There exists $\lambda > 0$ such that for all $\|J\| \leqslant 1$ and proba f on $\mathbb{S}^1$ we have

$$\mathcal{H}(f|M_J) \leqslant \frac{1}{2\lambda}\mathcal{I}(f|M_J).$$

Exponential convergence to a stable steady-state

Coercivity properties of V : there exists $C > 0$ such for J close the set $\mathcal{V}_\infty$ of minimizers, there exists $J_\infty \in \mathcal{V}_\infty$ such that (using CKP):

$$\frac{1}{C}\|J - J_\infty\|^2 \leqslant |V(J) - V(J_\infty)| \leqslant C\|\nabla V(J)\|^2 = \frac{C}{\sigma}\|\mathcal{J}_{M_J} - J\|^2$$

For $J = \mathcal{J}_f$, using CKP: $|V(J) - V_\infty| \leqslant \frac{2C}{\sigma}\mathcal{H}(f|M_{\mathcal{J}_f})$.
Therefore by the free energy / relative entropy identities and LSI :

$$0 \leqslant \mathcal{F}[f] - \mathcal{F}_\infty \leqslant (1 + \tfrac{2C}{\sigma})\mathcal{H}(f|M_{\mathcal{J}_f}) \leqslant \widetilde{C}\mathcal{D}[f].$$

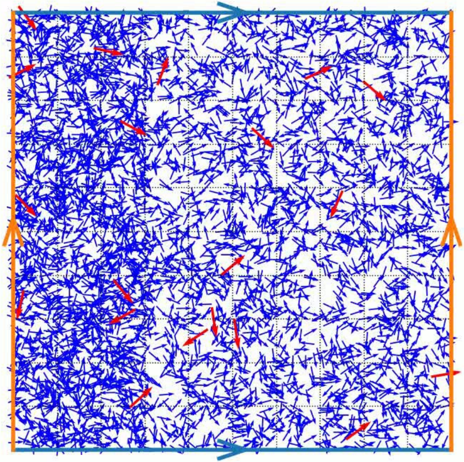
Gronwall : exponential decay of free energy, controlled by $\mathcal{H}(f|f_{eq,0})$ thanks to the identities. Stability: $\mathcal{J}_f$ stays close to $\mathcal{V}_\infty$ (coercivity + identities). Then $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_f$ exponentially controlled (CKB + identities).

Theorem (see also [FL12]): exponential stability

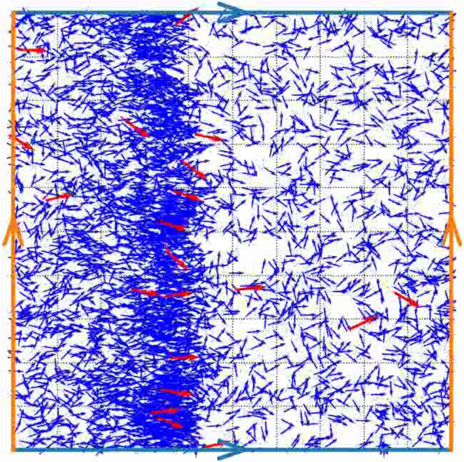
Let $\mathcal{E}_\infty$ be the family of stable equilibria (in the sense of free energy), then there exists $\delta, \lambda, C > 0$ such that if $\mathcal{H}(f_0|f_{\mathrm{eq},0}) < \delta$ for $f_{\mathrm{eq},0} \in \mathcal{E}_\infty$, then there exists $f_\infty \in \mathcal{E}_\infty$ such that

$$\forall t \geqslant 0, \mathcal{H}(f(t,\cdot)|f_\infty) \leqslant C\, e^{-2\lambda t}\,\mathcal{H}(f_0|f_{\mathrm{eq},0}).$$

Movie time : the travelling bands (the phase transition picture for the inhomogeneous model is hard !)



Movie time : the travelling bands (the phase transition picture for the inhomogeneous model is hard !)



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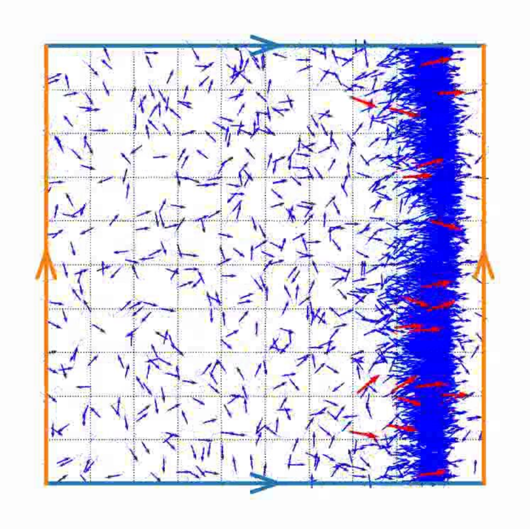
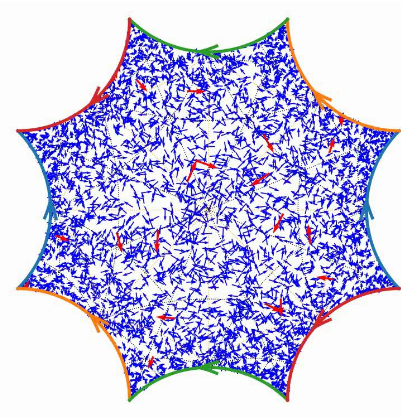
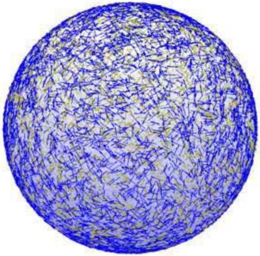


More geometries, more movies (in progress, with S. Motsch and A. Diez) !





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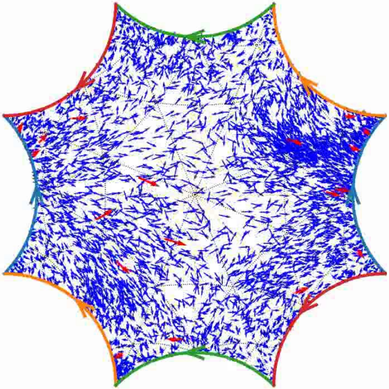

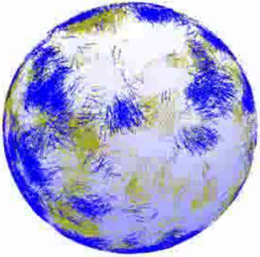

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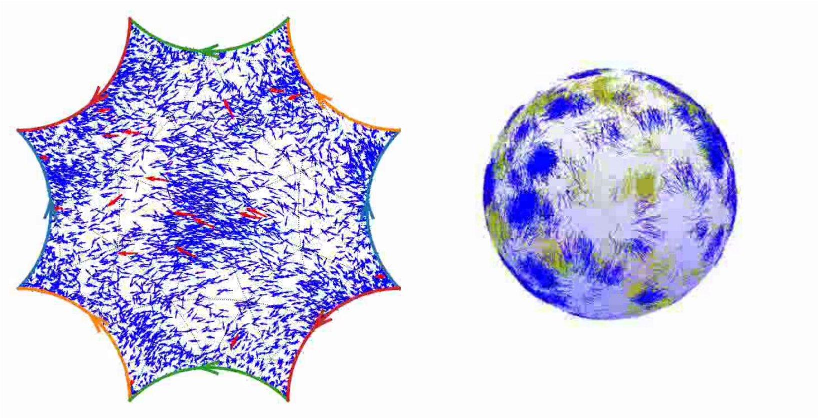

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