# Phase transition in a kinetic model of oriented particles

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## Modelling interactions of self-propelled particles

- Vicsek et. al. (95), Chaté and Ginelli (2007).
  Discrete model (also in time) : simulations present phase transitions.
  New direction of one particle : previous mean direction of the neighbours.
- Degond-Motsch (2008).
  Time-continuous version of this model : hydrodynamic limit without phase transition.
  Direction of one particle continuously relaxing to the mean direction of the neighbours : constant rate $\nu$.
- First try to correct the time-continuous version, with $\nu$ depending on a local density : still no emergence of phase transition.

# Particle dynamics

No focus on the spatial behaviour.
Oriented particles $\omega_1, \dots, \omega_N$ in $\mathbb{S}$, the unit sphere of $\mathbb{R}^n$.

$$\mathrm{d}\omega_k = (\mathrm{Id} - \omega_k \otimes \omega_k)(J_k \,\mathrm{d}t + \sqrt{2D}\,\mathrm{d}B_t^k),$$

$$J_k = \frac{1}{N}\sum_{j=1}^{N} \omega_j.$$

# Kinetic level

After rescaling in time, as $N \to \infty$, the probability density function $f$ (of finding one particle in direction $\omega \in \mathbb{S}$) satisfies (formally) :

$$\partial_t f = Q(f), \tag{1}$$

with

$$Q(f) = -\nabla_\omega \cdot ((\mathrm{Id} - \omega \otimes \omega) J[f] f) + \sigma \Delta_\omega f,$$
$$J[f] = \int_{\omega \in \mathbb{S}} \omega\, f(.,\omega)\, \mathrm{d}\omega\,.$$

(total mass 1 for the unit sphere)

Space-inhomogeneous case : $\sigma = \dfrac{D}{\text{local density}}$.

# Global existence and uniqueness, positivity

**Theorem**

*Given an initial probability density function $f_0$ in $L^2(\mathbb{S})$, then there exists a unique weak solution $f$ of (1) such that $f(0) = f_0$. Moreover, $f(t)$ is a probability density function for all $t \in [0, T]$.*

Sketch of the proof :

- Classical Galerkin method with the spherical harmonics on a small interval $[t_k, t_{k+1}]$.
- Maximum principle for smooth solutions gives positivity.
- Positivity gives a better estimate for the norm at the endpoint $t_{k+1}$.
- A last little trick to show that $t_k \to \infty$.

# Von Mises-Fisher distribution for equilibria

Define, for a constant vector $V \in \mathbb{R}^n$,

$$M_V(\omega) = \frac{\exp(\omega \cdot V)}{\int_{\upsilon \in \mathbb{S}} \exp(\upsilon \cdot V) \mathrm{d}\upsilon}.$$

Then

$$Q(f) = 0 \Longleftrightarrow f = M_{\sigma^{-1} J[f]} = M_{\kappa\Omega}, \Omega \in \mathbb{S}$$

Compatibility condition :

$$|J[f]| = \kappa\sigma = c(\kappa) = \frac{\int_0^\pi \cos\theta \, e^{\kappa \cos\theta} \sin^{n-2}\theta \, \mathrm{d}\theta}{\int_0^\pi e^{\kappa\cos\theta} \sin^{n-2}\theta \, \mathrm{d}\theta}.$$

($0 \leqslant c(\kappa) \leqslant 1$ is the mean speed of the distribution).

# The two phases

**Lemma**

*Define* $\beta = c(\kappa)^2 + n\dfrac{c(\kappa)}{\kappa} - 1$. *Then for any* $\kappa > 0$, *we have* $\beta > 0$.

The function $\kappa \mapsto \dfrac{c(\kappa)}{\kappa}$ ($= \sigma$ for the compatibility condition) is decreasing (its derivative is $-\frac{\beta}{\kappa}$), starting from $\frac{1}{n}$ at 0.

- If $\sigma n \geqslant 1$, only one solution : $\kappa = 0$.
  Equilibrium : constant state 1 (disordered phase).
- If $\sigma n < 1$, either $\kappa = 0$ (same case) or $\kappa$ is the unique positive solution for the compatibility condition (ordered phase).
  Manifold of equilibria (of dimension $n-1$) : $\{M_{\kappa\Omega}, \Omega \in \mathbb{S}\}$.

# Linear stability

Around the constant state : $f = 1 + g$.

$$\partial_t g = \sigma \Delta_\omega g + (n-1)(1+\cancel{g})\,\omega \cdot J[g] - \cancel{(\mathrm{Id} - \omega \otimes \omega) J[g] \cdot \nabla_\omega g},$$
$$\frac{\mathrm{d}}{\mathrm{d}t} J[g] = (n-1)\left(\frac{1}{n} - \sigma\right) J[g] + \cancel{\int_{\mathbb{S}} (\mathrm{Id} - \omega \otimes \omega) J[g] g \,\mathrm{d}\omega}.$$

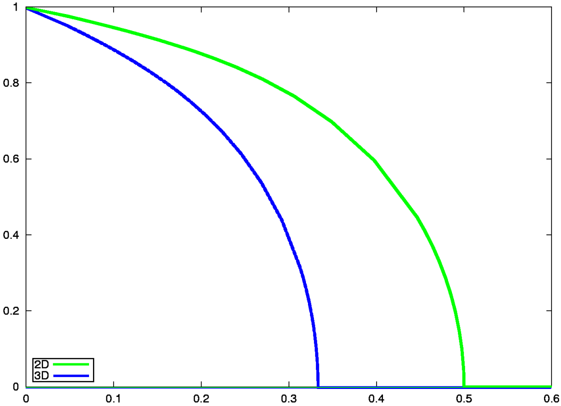
Linearly stable when $n\sigma > 1$, unstable when $n\sigma < 1$.

$$\frac{\mathrm{d}}{\mathrm{d}t} J[f] = M(t)\, J[f].$$

$M$ is a smooth matrix :

- either $\forall t, J[f] = 0$ (simple heat equation)
- either $\forall t, J[f] \neq 0$, then define $\Omega[f] = \dfrac{J[f]}{|J[f]|}$

# Phase transition




Near the critical value of $\sigma$, $c(\kappa(\sigma)) \geqslant \sqrt{1 - n\sigma}$ (thanks to the Lemma) : sharp transition between ordered and disordered phase.

# Free energy

$$\frac{\mathrm{d}}{\mathrm{d}t}L(f) = -D(f) \leqslant 0 \tag{2}$$

where

$$L(f) = \sigma \int_{\mathbb{S}} f \ln f \,\mathrm{d}\omega - \tfrac{1}{2} \left|J[f]\right|^2$$

$$D(f) = \int_{\mathbb{S}} f \left|\nabla_\omega(\sigma \ln f - \omega \cdot J[f])\right|^2 \mathrm{d}\omega.$$

Critical points of $L$, zero points of $D$, are the equilibria, and give the same compatibility condition.

# Around the constant state

Difficulty : logarithmic term.
Problems could occur if $f$ is close to 0.
Expansion : first modes.

$$f = 1 + A(t) \cdot \omega + \widetilde{f}$$

$$\int_{\mathbb{S}} \widetilde{f} = 0, \int_{\mathbb{S}} \omega \widetilde{f} = 0$$

Then $J(f) = \frac{1}{n} A$.
We formally get a variational stability in the case $n\sigma > 1$.
In the case $n\sigma = 1$, we have to expand up to order 4 in $A$.
Idea : control $|A|^4$ by $D(f)$, we get $\int_0^\infty |A(t)|^4 \mathrm{d}t < \infty$.

## Around $M_{\kappa\Omega}$, $\Omega(t)$ being the direction of $J(f)$

If $n\sigma < 1$, fix the corresponding $\kappa > 0$.

$$f = M_{\kappa\Omega}\left(1 + \alpha(t)(\Omega\cdot\omega - \langle\Omega\cdot\omega\rangle_{M_{\kappa\Omega}}) + \widetilde{g}\right)$$
$$\langle\widetilde{g}\rangle_{M_{\kappa\Omega}} = 0,\ \langle\widetilde{g}\omega\rangle_{M_{\kappa\Omega}} = 0$$

ODE for $\alpha$ and $\Omega$ :

$$\dot{\Omega} = \langle\Omega\cdot\omega\widetilde{g}_{\perp_\Omega}\omega\rangle_{M_{\kappa\Omega}}$$
$$\dot{\alpha} = -\alpha\left(\frac{\sigma\beta}{\sigma-\beta} + \alpha\frac{\beta}{\kappa}\right) + \langle(\Omega\cdot\omega)^2\widetilde{g}_{\|_\Omega}\rangle_{M_{\kappa\Omega}}$$

Idea : control $\alpha^2$ and $\langle\widetilde{g}^2\rangle_{M_{\kappa\Omega}}$ by $D(f)$ (again using $\beta > 0$).
And then control the displacement of $\Omega$.

Fix $\Lambda = (\Lambda_l)_{l\in\mathbb{N}}$ a sequence of real numbers.
Pseudo-differential operator $F_\Lambda$ : $F_\Lambda(Y_l) = \Lambda_l Y_l$.
(for any spherical harmonic of degree $l$ : $\Delta_\omega Y_l = -l(l+n-2)Y_l$)
Define the $\|\cdot\|_\Lambda$ norm for $\Lambda > 0$, by $\|g\|_\Lambda^2 = \int_{\mathbb{S}_{n-1}} gF_\Lambda g$

**Lemma**

*For any $g$ with $\int_{\mathbb{S}_{n-1}} g = 0$ :*

$$\left|\int_{\mathbb{S}_{n-1}} \nabla g F_\Lambda g\right| \leqslant \frac{1}{4}\|g\|_{\widetilde{\Lambda}}^2, \tag{3}$$

*where $\widetilde{\Lambda}_l = |(l+n-1)\Lambda_l - l\Lambda_{l+1}| + |(l+n-2)\Lambda_{l-1} - (l-1)\Lambda_l|$.*
*Special case : $\Lambda_l = l(l+1)\dots(l+n-2)$ :*

$$\widetilde{\Lambda} = 0 \textit{ and } \int_{\mathbb{S}_{n-1}} \nabla g F_\Lambda g = 0.$$

# Special case : $\Lambda_l = l(l+1)\dots(l+n-2)$

In this case the $\|\cdot\|_{\Lambda}^{-1}$ norm is equivalent to the $H^{-\frac{n-1}{2}}$ norm (for functions with mean zero).

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f-1\|^2_{\Lambda^{-1}} = -\sigma\|f-1\|^2_{\Lambda^{-1}l(l+n-2)} + (n-1)\Lambda_1^{-1}|J(f)|^2 \quad (4)$$

$$\leqslant (n-1)(\tfrac{1}{n}-\sigma)\|f-1\|^2_{\Lambda^{-1}} \quad (5)$$

In the case $\sigma n > 1$, exponential decay with rate $(n-1)(\frac{1}{n}-\sigma)$ for the $H^{-\frac{n-1}{2}}$ norm !

# Generic case : $\Lambda \sim l^{2s}$

In this case the $\|\cdot\|_\Lambda$ and $\|\cdot\|_{\widetilde{\Lambda}}$ norms are equivalent to the $H^s$ norm. Control the low order terms by the $H^{-\frac{n-1}{2}}$ norm, use the lemma to control the remaining part.

**Theorem**

*Suppose $n\sigma > 1$.*
*In any Sobolev space $H^s(\mathbb{S})$, with $s \geqslant -\frac{n-1}{2}$ we have exponential decay to the constant state $1$ with rate at least $(n-1)(\frac{1}{n} - \sigma)$ : there is a constant $C$ depending only on $s, n, \sigma$ such that for all $f_0 \in H^s(\mathbb{S})$, we have*

$$\|f - 1\|^2_{H^s} \leqslant C\|f_0 - 1\|_{H^s} \exp\left((n-1)(\tfrac{1}{n} - \sigma)t\right). \tag{6}$$

## Summary

- System with phase transition depending on one parameter $\sigma$, if the initial mean speed is non-zero.
- Threshold, which can be viewed as a critical mass ($\sigma$ inversely proportional to a local density).
- Under this threshold ($\sigma n > 1$) : disorder, and strong convergence to the uniform distribution (exponentially in any $H^s$, $2s + n \geqslant 1$).
- Above : formal convergence to a von Mises-Fischer distribution, with parameter $\kappa$ given by $\sigma$, and orientation $\Omega$ close to the initial orientation, provided the initial distribution is sufficiently close to a von-Mises distribution with parameter $\kappa$.

# Work in progress, and future work

- Non homogeneous case, derivation of a hydrodynamic limit.
- More rigorous framework for the case $n\sigma < 1$ and $n\sigma = 1$.
- Understanding the shape in the case $n\sigma < 1$ ("self-similar" solutions)
- Studying the convergence of the particle system to the kinetic equation.

Thanks !