

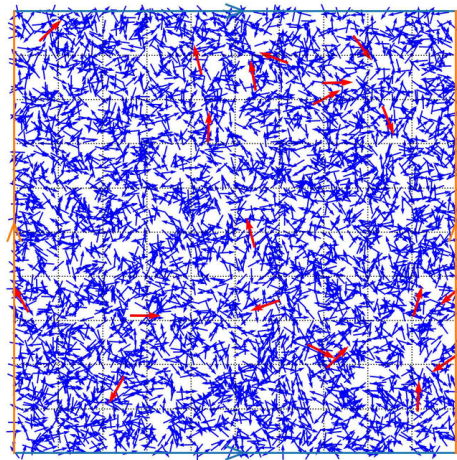
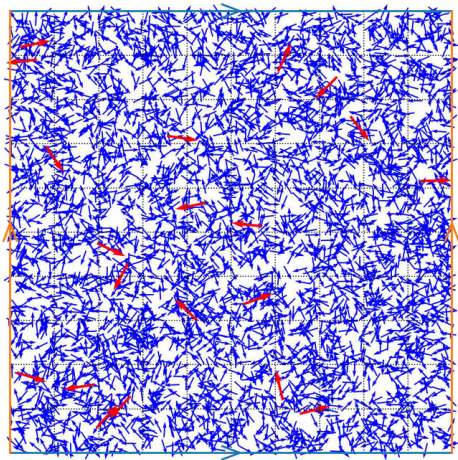
Hypocoercivity in a model of aligning self-propelled particles

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Work in collaboration with Emeric Bouin (CEREMADE)

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Motivation : aligning self-propelled particles ^{1,2,3}



¹Vicsek *et al.*, *Phys. Rev. Lett.*, 1995 [VCBJ⁺95]

²Degond, Motsch, *M3AS*, 2008 [DM08]

³Degond, F, Liu, *J. Nonlin. Sci.*, 2013 [DFL13]

The self-propelled particles model

System of coupled SDEs

Particles at positions $X_k = \mathbb{R}^d$ (or a flat torus \mathbb{T}), speeds $V_k \in \mathbb{S}$ (unit sphere), $1 \leq k \leq N$.

$$\begin{cases} dX_k = c V_k dt \\ dV_k = - \sum_{j=1}^N \nu_{j,k} \frac{1}{2} \nabla_{V_j} \|V_j - V_k\|^2 dt + \sqrt{2\sigma} P_{V_k^\perp} \circ dB_{t,k}. \end{cases}$$

Careful : gradients on the sphere (for instance $\nabla_v(\mathbf{u} \cdot v) = P_{v^\perp} \mathbf{u}$), Laplace Beltrami, Stratonovich formulation for brownian motion on the sphere.

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Empirical distribution $f^N = \frac{\rho}{N} \sum_j \delta_{X_j} \otimes \delta_{V_j}$.

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Parameters : N, K, \mathbb{T}, ρ (hidden in f^N).

The mean-field or moderate interaction limit

Mean-field limit ⁴ : convergence of f^N to a density f , solution of a kinetic equation

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (P_{v^\perp} (K *_x J_f) f) = \Delta_v f.$$

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Moderate interaction limit expected if the limit kinetic equation is well posed ⁵

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→ Only remaining parameters : the shape of \mathbb{T} , and ρ (hidden in f).

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Theorem : (local in time) existence and uniqueness, initial condition f_0 in $L^\infty(\mathbb{R}^d \times \mathbb{S})$.

There exists a unique weak solution in $\mathcal{C}([0, T], L^\infty(\mathbb{R}^d \times \mathbb{S}))$ for all $T < \frac{1}{(d-1)\|f_0\|_\infty}$. It is nonnegative and satisfies the following estimate (maximum principle):

$$\forall t \in [0, T], \quad \|f(t)\|_\infty \leq \|f_0\|_\infty + (d-1) \int_0^t \|J_f(s)\|_\infty \|f(s)\|_\infty ds.$$

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Fokker–Planck formulation via von Mises distributions, free energy

Define the von Mises distribution $M_J(v) = \frac{e^{v \cdot J}}{\int_{\mathbb{S}} e^{v' \cdot J} dv'}$, then

$$\partial_t f = -\nabla_v \cdot (P_{v^\perp} J_f f) + \Delta_v f = \nabla_v \cdot \left(M_{J_f} \nabla_v \left(\frac{f}{M_{J_f}} \right) \right) = \nabla_v \cdot (f \nabla_v (\ln f - v \cdot J_f)).$$

Dissipation of the free energy $\mathcal{F}[f] = \int_{\mathbb{S}} f \ln f - \frac{1}{2} |J_f|^2$: Fisher information (w.r.t ρM_{J_f}).

$$\frac{d}{dt} \mathcal{F} = -\mathcal{D} = - \int_{\mathbb{S}} |\nabla_v (\ln f - v \cdot J_f)|^2 f dv = -\mathcal{I}(f | \rho M_{J_f}).$$

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Criteria for steady states, compatibility equation.

$\mathcal{D}[f] = 0 \Leftrightarrow$ critical point of \mathcal{F} under mass $\rho \Leftrightarrow f = \rho M_J$, with $J_{\rho M_J} = \rho \langle v \rangle_{M_J} = J$.

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Compatibility equation : $J = \kappa \Omega$ with $\Omega \in \mathbb{S}$ and $\kappa = \rho c(\kappa)$ for $c(\kappa) = \frac{\int_0^\pi \cos \theta e^{\kappa \cos \theta} \sin^{d-2} \theta d\theta}{\int_0^\pi e^{\kappa \cos \theta} \sin^{d-2} \theta d\theta}$.

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Behaviour : $\frac{\kappa}{c(\kappa)} \nearrow +\infty$ as $\kappa \rightarrow +\infty$, and $\searrow \rho_c = d$ as $\kappa \rightarrow 0$.

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Our main goal : around isotropic state, $\rho < \rho_c = d$

Stability/instability for the space-homogeneous model

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Main result (spoiler) : it is stable ! In $H^{s,1}$ norm (s derivatives in x , one in v).

Assume that $s \geq d$ if d is odd or $s \geq d+1$ if d is even. If $g_0 \in L^\infty(\mathbb{R}^d \times \mathbb{S}) \cap H^{s,1}(\mathbb{R}^d \times \mathbb{S})$ is small, the solution is global. There exists an energy, equivalent to the $H^{s,1}(\mathbb{R}^d \times \mathbb{S})$ norm of g , that decays in time. On \mathbb{T} , the energy is exponentially decreasing.

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- Related work⁷ for BGK instead of Fokker-Planck : $\partial_t f + v \cdot \nabla_x f = \rho_f M_{J_f} - f$.

⁷Merino-Aceituno, Schmeiser, Winter *ArXiv* 2024 [MASW24]

Back to basics : one particle (Velocity Spherical Brownian Motion¹⁰)

A single self-propelled particle exploring around, no interaction.

$$X \in \mathbb{R}^d \text{ (or } \mathbb{T}), V \in \mathbb{S}, \begin{cases} dX = V dt \\ dV = P_{V^\perp} \circ dB_t. \end{cases}$$

Its law satisfies $\partial_t f + v \cdot \nabla_x f = \Delta_v f$.

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Looks simple, can't we do à la Villani⁸ ? Define an energy equivalent to the square of the H^1 norm (if $\beta < \alpha\gamma$) : $\mathcal{F} = \|f\|_2^2 + \alpha\|\nabla_v f\|_2^2 + 2\beta\langle\nabla_v f, \nabla_x f\rangle + \gamma\|\nabla_x f\|_2^2$, and then get

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Furthermore, if we want a quantitative regularising estimate for short times à la Hérau⁹ (that is (α, β, γ) replaced by $(\alpha t, \beta t^2, \gamma t^3)$) it does not work. Why ?

⁸Villani, 2009 [Vil09]

⁹Hérau, *JFA* 2007 [Hé07]

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The trouble is the sphere — but there is a nice algebraic framework

We want to write our equation as $\partial_t f + \mathbb{T}f = A^2 f$.

Fancy decomposition of the Laplace-Beltrami on the sphere

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- If $f, g \in C^1(\mathbb{S})$, then $\nabla_v f \cdot \nabla_v g = \sum_{i < j} A_{i,j} f A_{i,j} g$. Consequently $\Delta_v f = A^2 f$.

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Evolution of quadratic quantities and commutators

Write $T = v \cdot \nabla_x$. If X is a smooth differential operator and $Q_X = \int_{\mathbb{R}^d \times \mathbb{S}} f X f \, dx dv$, then $\frac{d}{dt} Q_X = Q_{\Phi(X)}$, where the operator $\Phi(X)$ goes as follows:

$$\Phi(X) = A^2 X + X A^2 + [T, X] = 2AXA + [A[A, X]] + [T, X].$$

Villani's chain of commutators : start from $C_0 = A$ and then $C_{i+1} = [T, C_i]$, hoping to get all the missing "directions". Here it stops at $C_1 = [T, A] := S = -v \wedge \nabla_x$, since then $[T, S] = 0$.

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→ Hörmander theory : commute as you can ! We get $[A, S] = (d-1)T$. And we are happy since $T^2 + S^2 = \Delta_x$.

Weights of operators

We will always take operators X composed thanks to A (weight $\frac{1}{2}$), S (weight $\frac{3}{2}$) and Δ_x (weight 4). Weights of compositions are the sum of weights.

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Set $\mathcal{F}_0 = Q_{\text{Id}}$, $\mathcal{F}_1(\tau, \cdot) = \alpha\tau Q_{-A^2} + \beta\tau^2 Q_{SA+AS} + \gamma\tau^3 Q_{-S^2} + \delta\tau^4 Q_{-\Delta_x}$.

Then there exists coefficients α, β, γ such that $\mathcal{F}(t) = \mathcal{F}_0 + \mathcal{F}_1(\min(t, 1), f)$ is decreasing in time. Furthermore, on \mathbb{T} , this quantity is equivalent to the H^1 norm of f if f has mean zero, and is controlled by its dissipation at positive time, leading to an exponential decay.

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Higher order (in x only) : $\mathcal{F}_k(\tau, \cdot) = \nu_k \tau^{4(k-1)} \sum_{|m|=k-1} \binom{k-1}{m} \mathcal{F}_1(\tau, \partial_x^m)$.

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In the case of the torus in space (and $d = 3$), see also the recent work on the model of Saintillan–Shelley model¹¹.

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Back to our model, another nice algebraic view

Our equation on the perturbation g ($f = \rho + g$) :

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We then get

$$\partial_t g + Tg = A^2g - \frac{\rho + g}{d}U^2g - \frac{1}{d}UgAg = (A^2 - \frac{\rho}{d}U^2)g - \frac{1}{d}(A(gUg)).$$

To simplify notations, we note $L = A - (1 - \sqrt{1 - \frac{\rho}{d}})U$, so that $L^2 = A^2 - \frac{\rho}{d}U^2$.

$$\partial_t g + \mathbb{T}g = \mathbb{L}^2 g - \frac{1}{d}(\mathbb{A}(gUg)).$$

Same functional $\mathcal{F}(\tau, g(t, \cdot))$, new terms in the dissipation.

$$\frac{d}{dt} Q_X = Q_{\Phi^\rho(X)} + R_X, \text{ where this time}$$

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and where the non-linear term produces

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Special case : if f only depends on one space variable. Then, by small time regularity, we have stability in $L^2 \cap L^\infty$.



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Thanks