Hypocoercivity in a model of aligning self-propelled particles

Amic Frouvelle – CEREMADE – Université Paris Dauphine PSL

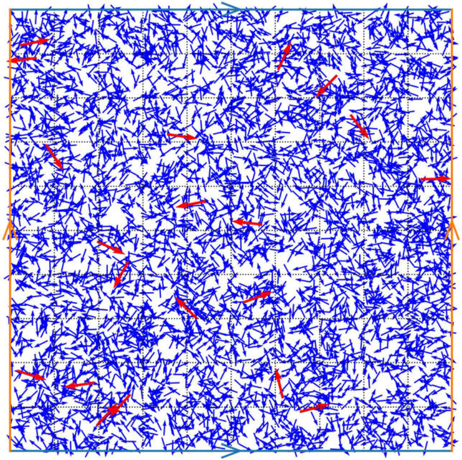
Work in collaboration with Emeric Bouin

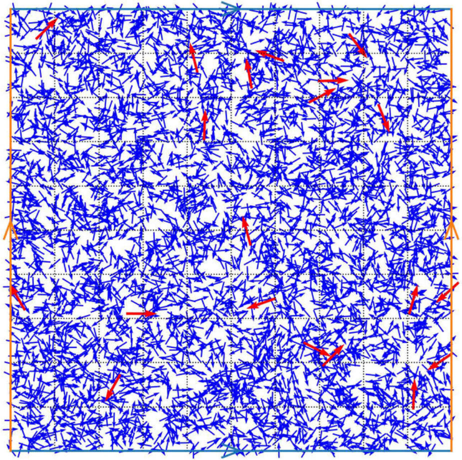
ReaDiNet workshop on reaction-diffusion systems and population dynamics,

“a workshop on the mathematical analysis of reaction-diffusion systems (propagation phenomena, self propelled motion, pattern formation...), as well as connections between PDEs, probability theory and the life sciences (biology, epidemiology, population dynamics...)”

Parent, June 11$^{\text{th}}$ 2024

Motivation : aligning self-propelled particles [1,2,3]





[1]Vicsek *et al.*, *Phys. Rev. Lett.*, 1995 [VCBJ+95]
[2]Degond, Motsch, *M3AS*, 2008 [DM08]
[3]Degond, F, Liu, *J. Nonlin. Sci.*, 2013 [DFL13]

The self-propelled particles model

System of coupled SDEs

Particles at positions $X_k = \mathbb{R}^d$ (or a flat torus $\mathbb{T}$), speeds $V_k \in \mathbb{S}$ (unit sphere), $1 \leqslant k \leqslant N$.

$$\begin{cases} \mathrm{d}X_k = c\, V_k \mathrm{d}t \\ \mathrm{d}V_k = -\sum_{j=1}^{N} \nu_{j,k} \frac{1}{2} \nabla_{V_j} \|V_j - V_k\|^2 \mathrm{d}t + \sqrt{2\sigma} P_{V_k^{\perp}} \circ \mathrm{d}B_{t,k}. \end{cases}$$

Careful : gradients on the sphere (for instance $\nabla_v (\mathbf{u} \cdot v) = P_{v^\perp} \mathbf{u}$), Laplace Beltrami, Stratonovich formulation for brownian motion on the sphere.

The self-propelled particles model

System of coupled SDEs

Particles at positions $X_k = \mathbb{R}^d$ (or a flat torus $\mathbb{T}$), speeds $V_k \in \mathbb{S}$ (unit sphere), $1 \leqslant k \leqslant N$.

$$\begin{cases} \mathrm{d}X_k = c\, V_k \mathrm{d}t \\ \mathrm{d}V_k = -\sum_{j=1}^{N} \nu_{j,k} \frac{1}{2} \nabla_{V_j} \|V_j - V_k\|^2 \mathrm{d}t + \sqrt{2\sigma} P_{V_k^\perp} \circ \mathrm{d}B_{t,k}. \end{cases}$$

Careful : gradients on the sphere (for instance $\nabla_v(\mathbf{u} \cdot v) = P_{v^\perp}\mathbf{u}$), Laplace Beltrami, Stratonovich formulation for brownian motion on the sphere.
Change scales : $c = \sigma = 1$. Assumption : $\nu_{j,k} = \frac{\rho}{N} K(X_j - X_k)$, $\int_{\mathbb{R}^d} K(y)\mathrm{d}y = 1$.

The self-propelled particles model

System of coupled SDEs

Particles at positions $X_k = \mathbb{R}^d$ (or a flat torus $\mathbb{T}$), speeds $V_k \in \mathbb{S}$ (unit sphere), $1 \leqslant k \leqslant N$.

$$\begin{cases} \mathrm{d}X_k = c\, V_k \mathrm{d}t \\ \mathrm{d}V_k = -\sum_{j=1}^{N} \nu_{j,k} \frac{1}{2} \nabla_{V_j} \|V_j - V_k\|^2 \mathrm{d}t + \sqrt{2\sigma} P_{V_k^\perp} \circ \mathrm{d}B_{t,k}. \end{cases}$$

Careful : gradients on the sphere (for instance $\nabla_v(\mathbf{u} \cdot v) = P_{v^\perp}\mathbf{u}$), Laplace Beltrami, Stratonovich formulation for brownian motion on the sphere.
Change scales : $c = \sigma = 1$. Assumption : $\nu_{j,k} = \frac{\rho}{N} K(X_j - X_k)$, $\int_{\mathbb{R}^d} K(y)\mathrm{d}y = 1$.

Empirical distribution $f^N = \frac{\rho}{N} \sum_j \delta_{X_j} \otimes \delta_{V_j}$.

$$\mathrm{d}V_k = P_{V_k^\perp}\Big(\int_{\mathbb{T}\times\mathbb{S}} K(x - X_k) v \mathrm{d}f^N(x, v)\Big)\mathrm{d}t + \sqrt{2} P_{V_k^\perp} \circ \mathrm{d}B_{t,k}$$

The self-propelled particles model

System of coupled SDEs

Particles at positions $X_k = \mathbb{R}^d$ (or a flat torus $\mathbb{T}$), speeds $V_k \in \mathbb{S}$ (unit sphere), $1 \leqslant k \leqslant N$.

$$\begin{cases} \mathrm{d}X_k = c\, V_k \mathrm{d}t \\ \mathrm{d}V_k = -\sum_{j=1}^{N} \nu_{j,k} \frac{1}{2} \nabla_{V_j} \|V_j - V_k\|^2 \mathrm{d}t + \sqrt{2\sigma} P_{V_k^\perp} \circ \mathrm{d}B_{t,k}. \end{cases}$$

Careful : gradients on the sphere (for instance $\nabla_v(\mathbf{u} \cdot v) = P_{v^\perp}\mathbf{u}$), Laplace Beltrami, Stratonovich formulation for brownian motion on the sphere.
Change scales : $c = \sigma = 1$. Assumption : $\nu_{j,k} = \frac{\rho}{N} K(X_j - X_k)$, $\int_{\mathbb{R}^d} K(y)\mathrm{d}y = 1$.

Empirical distribution $f^N = \frac{\rho}{N} \sum_j \delta_{X_j} \otimes \delta_{V_j}$.

$$\mathrm{d}V_k = P_{V_k^\perp} K *_x J_{f^N} \,\mathrm{d}t + \sqrt{2} P_{V_k^\perp} \circ \mathrm{d}B_{t,k},$$

where for a measure f, its first moment (in v) is denoted $J_f = \int_{\mathbb{S}} v f(v)\mathrm{d}v$.

The self-propelled particles model

System of coupled SDEs

Particles at positions $X_k = \mathbb{R}^d$ (or a flat torus $\mathbb{T}$), speeds $V_k \in \mathbb{S}$ (unit sphere), $1 \leqslant k \leqslant N$.

$$\begin{cases} \mathrm{d}X_k = c\, V_k \mathrm{d}t \\ \mathrm{d}V_k = -\sum_{j=1}^{N} \nu_{j,k} \frac{1}{2} \nabla_{V_j} \|V_j - V_k\|^2 \mathrm{d}t + \sqrt{2\sigma} P_{V_k^\perp} \circ \mathrm{d}B_{t,k}. \end{cases}$$

Careful : gradients on the sphere (for instance $\nabla_v(\mathbf{u} \cdot v) = P_{v^\perp}\mathbf{u}$), Laplace Beltrami, Stratonovich formulation for brownian motion on the sphere.
Change scales : $c = \sigma = 1$. Assumption : $\nu_{j,k} = \frac{\rho}{N} K(X_j - X_k)$, $\int_{\mathbb{R}^d} K(y)\mathrm{d}y = 1$.

Empirical distribution $f^N = \frac{\rho}{N} \sum_j \delta_{X_j} \otimes \delta_{V_j}$.

$$\mathrm{d}V_k = P_{V_k^\perp} K *_x J_{f^N} \mathrm{d}t + \sqrt{2} P_{V_k^\perp} \circ \mathrm{d}B_{t,k},$$

where for a measure f, its first moment (in v) is denoted $J_f = \int_{\mathbb{S}} v f(v) \mathrm{d}v$.

Parameters : N, K, $\mathbb{T}$, ρ (hidden in f^N).

The mean-field or moderate interaction limit

Mean-field limit [4] : convergence of f^N to a density f, solution of a kinetic equation

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (P_{v^\perp}(K *_x J_f) f) = \Delta_v f.$$

[4] Bolley, Cañizo, Carrillo, *Appl. Math. Lett.* 2012 [BCC12]

5

The mean-field or moderate interaction limit

Mean-field limit [4] : convergence of f^N to a density f, solution of a kinetic equation

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (P_{v^\perp}(K *_x J_f) f) = \Delta_v f.$$

How to get rid of K ? Use $\frac{1}{\varepsilon_N^d} K(\frac{\cdot}{\varepsilon_N})$ instead, with $\varepsilon_N \to 0$ as $N \to \infty$ (but $\varepsilon_N^d N \to \infty$).

[4] Bolley, Cañizo, Carrillo, *Appl. Math. Lett.* 2012 [BCC12]

5

The mean-field or moderate interaction limit

Mean-field limit [4] : convergence of f^N to a density f, solution of a kinetic equation

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (P_{v^\perp}(K *_x J_f) f) = \Delta_v f.$$

How to get rid of K ? Use $\frac{1}{\varepsilon_N^d} K(\frac{\cdot}{\varepsilon_N})$ instead, with $\varepsilon_N \to 0$ as $N \to \infty$ (but $\varepsilon_N^d N \to \infty$).

Moderate interaction limit expected if the limit kinetic equation is well posed [5]

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (P_{v^\perp} J_f f) = \Delta_v f.$$

$\to$ Only remaining parameters : the shape of $\mathbb{T}$, and ρ (hidden in f).

[4] Bolley, Cañizo, Carrillo, *Appl. Math. Lett.* 2012 [BCC12]
[5] Chaintron, Diez, *Kinet. Relat. Models*, 2022 [CD22]

The mean-field or moderate interaction limit

Mean-field limit [4] : convergence of f^N to a density f, solution of a kinetic equation

$$\partial_t f + v\cdot\nabla_x f + \nabla_v\cdot(P_{v^\perp}(K *_x J_f)f) = \Delta_v f.$$

How to get rid of K ? Use $\frac{1}{\varepsilon_N^d}K(\frac{\cdot}{\varepsilon_N})$ instead, with $\varepsilon_N \to 0$ as $N\to\infty$ (but $\varepsilon_N^d N \to \infty$).

Moderate interaction limit expected if the limit kinetic equation is well posed [5]

$$\partial_t f + v\cdot\nabla_x f + \nabla_v\cdot(P_{v^\perp} J_f f) = \Delta_v f.$$

$\to$ Only remaining parameters : the shape of $\mathbb{T}$, and ρ (hidden in f).

Theorem : (local in time) existence and uniqueness, initial condition f_0 in $L^\infty(\mathbb{R}^d\times\mathbb{S})$.

There exists a unique weak solution in $\mathcal{C}([0,T],L^\infty(\mathbb{R}^d\times\mathbb{S}))$ for all $T<\frac{1}{(d-1)\|f_0\|_\infty}$. It is nonnegative and satisfies the following estimate (maximum principle):

$$\forall t\in[0,T],\qquad \|f(t)\|_\infty \leqslant \|f_0\|_\infty + (d-1)\int_0^t \|J_f(s)\|_\infty\|f(s)\|_\infty\,\mathrm{d}s.$$

[4] Bolley, Cañizo, Carrillo, *Appl. Math. Lett.* 2012 [BCC12]
[5] Chaintron, Diez, *Kinet. Relat. Models*, 2022 [CD22]

The space-homegeneous setting : phase transition [6]

Fokker–Planck formulation via von Mises distributions, free energy

Define the von Mises distribution $M_J(v) = \frac{e^{v\cdot J}}{\int_{\mathbb{S}} e^{v'\cdot J} \mathrm{d}v'}$, then

$$\partial_t f = -\nabla_v \cdot (P_{v^\perp} J_f f) + \Delta_v f = \nabla_v \cdot \big(M_{J_f} \nabla_v (\frac{f}{M_{J_f}})\big) = \nabla_v \cdot (f \nabla_v (\ln f - v \cdot J_f)).$$

Dissipation of the free energy $\mathcal{F}[f] = \int_{\mathbb{S}} f \ln f - \frac{1}{2}|J_f|^2$: Fisher information (w.r.t ρM_{J_f}).

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} = -\mathcal{D} = -\int_{\mathbb{S}} |\nabla_v (\ln f - v \cdot J_f)|^2 f \mathrm{d}v = -\mathcal{I}(f|\rho M_{J_f}).$$

[6] F, Liu *SIAM J. Math. Anal.*, 2012 [FL12]

Fokker–Planck formulation via von Mises distributions, free energy

Define the von Mises distribution $M_J(v) = \frac{e^{v\cdot J}}{\int_{\mathbb{S}} e^{v'\cdot J}\mathrm{d}v'}$, then

$$\partial_t f = -\nabla_v \cdot (P_{v^\perp} J_f f) + \Delta_v f = \nabla_v \cdot \big(M_{J_f} \nabla_v (\frac{f}{M_{J_f}})\big) = \nabla_v \cdot (f \nabla_v (\ln f - v \cdot J_f)).$$

Dissipation of the free energy $\mathcal{F}[f] = \int_{\mathbb{S}} f \ln f - \frac{1}{2}|J_f|^2$: Fisher information (w.r.t ρM_{J_f}).

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} = -\mathcal{D} = -\int_{\mathbb{S}} |\nabla_v (\ln f - v \cdot J_f)|^2 f \,\mathrm{d}v = -\mathcal{I}(f|\rho M_{J_f}).$$

Criteria for steady states, compatibility equation.

$\mathcal{D}[f] = 0 \Leftrightarrow$ critical point of $\mathcal{F}$ under mass $\rho \Leftrightarrow f = \rho M_J$, with $J_{\rho M_J} = \rho \langle v \rangle_{M_J} = J$.

[6] F, Liu *SIAM J. Math. Anal.*, 2012 [FL12]

Fokker–Planck formulation via von Mises distributions, free energy

Define the von Mises distribution $M_J(v) = \frac{e^{v\cdot J}}{\int_{\mathbb{S}} e^{v'\cdot J} \mathrm{d}v'}$, then

$$\partial_t f = -\nabla_v \cdot (P_{v^\perp} J_f f) + \Delta_v f = \nabla_v \cdot \big(M_{J_f} \nabla_v (\frac{f}{M_{J_f}})\big) = \nabla_v \cdot (f \nabla_v (\ln f - v\cdot J_f)).$$

Dissipation of the free energy $\mathcal{F}[f] = \int_{\mathbb{S}} f \ln f - \frac{1}{2}|J_f|^2$: Fisher information (w.r.t ρM_{J_f}).

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} = -\mathcal{D} = -\int_{\mathbb{S}} |\nabla_v (\ln f - v\cdot J_f)|^2 f \,\mathrm{d}v = -\mathcal{I}(f|\rho M_{J_f}).$$

Criteria for steady states, compatibility equation.

$\mathcal{D}[f] = 0 \Leftrightarrow$ critical point of $\mathcal{F}$ under mass $\rho \Leftrightarrow f = \rho M_J$, with $J_{\rho M_J} = \rho \langle v\rangle_{M_J} = J$.

Compatibility equation : $J = \kappa\Omega$ with $\Omega \in \mathbb{S}$ and $\kappa = \rho c(\kappa)$ for $c(\kappa) = \frac{\int_0^\pi \cos\theta e^{\kappa\cos\theta} \sin^{d-2}\theta \mathrm{d}\theta}{\int_0^\pi e^{\kappa\cos\theta}\sin^{d-2}\theta \mathrm{d}\theta}$.

[6] F, Liu *SIAM J. Math. Anal.*, 2012 [FL12]

Fokker–Planck formulation via von Mises distributions, free energy

Define the von Mises distribution $M_J(v) = \frac{e^{v\cdot J}}{\int_{\mathbb{S}} e^{v'\cdot J}\mathrm{d}v'}$, then

$$\partial_t f = -\nabla_v\cdot(P_{v^\perp}J_f f) + \Delta_v f = \nabla_v\cdot\big(M_{J_f}\nabla_v(\frac{f}{M_{J_f}})\big) = \nabla_v\cdot(f\nabla_v(\ln f - v\cdot J_f)).$$

Dissipation of the free energy $\mathcal{F}[f] = \int_{\mathbb{S}} f\ln f - \frac{1}{2}|J_f|^2$: Fisher information (w.r.t ρM_{J_f}).

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} = -\mathcal{D} = -\int_{\mathbb{S}}|\nabla_v(\ln f - v\cdot J_f)|^2 f\,\mathrm{d}v = -\mathcal{I}(f|\rho M_{J_f}).$$

Criteria for steady states, compatibility equation.

$\mathcal{D}[f] = 0 \Leftrightarrow$ critical point of $\mathcal{F}$ under mass $\rho \Leftrightarrow f = \rho M_J$, with $J_{\rho M_J} = \rho\langle v\rangle_{M_J} = J$.

Compatibility equation : $J = \kappa\Omega$ with $\Omega\in\mathbb{S}$ and $\kappa = \rho c(\kappa)$ for $c(\kappa) = \frac{\int_0^\pi \cos\theta e^{\kappa\cos\theta}\sin^{d-2}\theta\mathrm{d}\theta}{\int_0^\pi e^{\kappa\cos\theta}\sin^{d-2}\theta\mathrm{d}\theta}$.

Behaviour : $\frac{\kappa}{c(\kappa)} \nearrow +\infty$ as $\kappa\to+\infty$, and $\searrow \rho_c = d$ as $\kappa\to 0$.

[6] F, Liu *SIAM J. Math. Anal.*, 2012 [FL12]

Our main goal : around isotropic state, $\rho < \rho_c = d$

Stability/instability for the space-homogeneous model

- $\rho \leqslant \rho_c$: only solution $\kappa = 0$. Isotropic state, stable (exponentially if $\rho < \rho_c$).

7

Our main goal : around isotropic state, $\rho < \rho_c = d$

Stability/instability for the space-homogeneous model

- $\rho \leqslant \rho_c$: only solution $\kappa = 0$. Isotropic state, stable (exponentially if $\rho < \rho_c$).
- $\rho > \rho_c$: either $\kappa = 0$ (isotropic state, unstable), or a solution $\kappa(\rho) > 0$. If $J_{f^0} \neq 0$, exponential convergence of f to $\rho M_{\kappa(\rho)\Omega_\infty}$ for some $\Omega_\infty \in \mathbb{S}$.

7

Our main goal : around isotropic state, $\rho < \rho_c = d$

Stability/instability for the space-homogeneous model

- $\rho \leqslant \rho_c$: only solution $\kappa = 0$. Isotropic state, stable (exponentially if $\rho < \rho_c$).
- $\rho > \rho_c$: either $\kappa = 0$ (isotropic state, unstable), or a solution $\kappa(\rho) > 0$. If $J_{f^0} \neq 0$, exponential convergence of f to $\rho M_{\kappa(\rho)\Omega_\infty}$ for some $\Omega_\infty \in \mathbb{S}$.

- For the inhomogeneous model, the homogeneous steady states are the same. Can we say something about their stability ?

7

Our main goal : around isotropic state, $\rho < \rho_c = d$

Stability/instability for the space-homogeneous model

- $\rho \leqslant \rho_c$: only solution $\kappa = 0$. Isotropic state, stable (exponentially if $\rho < \rho_c$).
- $\rho > \rho_c$: either $\kappa = 0$ (isotropic state, unstable), or a solution $\kappa(\rho) > 0$. If $J_{f^0} \neq 0$, exponential convergence of f to $\rho M_{\kappa(\rho)\Omega_\infty}$ for some $\Omega_\infty \in \mathbb{S}$.

- For the inhomogeneous model, the homogeneous steady states are the same. Can we say something about their stability ?
- Interplay between transport and an operator relaxing in v only : hypocoercivity approach.

7

Our main goal : around isotropic state, $\rho < \rho_c = d$

Stability/instability for the space-homogeneous model

- $\rho \leqslant \rho_c$: only solution $\kappa = 0$. Isotropic state, stable (exponentially if $\rho < \rho_c$).
- $\rho > \rho_c$: either $\kappa = 0$ (isotropic state, unstable), or a solution $\kappa(\rho) > 0$. If $J_{f^0} \neq 0$, exponential convergence of f to $\rho M_{\kappa(\rho)\Omega_\infty}$ for some $\Omega_\infty \in \mathbb{S}$.

- For the inhomogeneous model, the homogeneous steady states are the same. Can we say something about their stability ?
- Interplay between transport and an operator relaxing in v only : hypocoercivity approach.
- We concentrate on $\rho < \rho_c$, write $f = \rho + g$, with g small (of zero average if on $\mathbb{T}$) :

$$\partial_t g + v \cdot \nabla_x g - (\rho + g)(d-1) v \cdot J_g + J_g \cdot \nabla_v g = \Delta_v g.$$

7

Our main goal : around isotropic state, $\rho < \rho_c = d$

Stability/instability for the space-homogeneous model

- $\rho \leqslant \rho_c$: only solution $\kappa = 0$. Isotropic state, stable (exponentially if $\rho < \rho_c$).
- $\rho > \rho_c$: either $\kappa = 0$ (isotropic state, unstable), or a solution $\kappa(\rho) > 0$. If $J_{f^0} \neq 0$, exponential convergence of f to $\rho M_{\kappa(\rho)\Omega_\infty}$ for some $\Omega_\infty \in \mathbb{S}$.

- For the inhomogeneous model, the homogeneous steady states are the same. Can we say something about their stability ?
- Interplay between transport and an operator relaxing in v only : hypocoercivity approach.
- We concentrate on $\rho < \rho_c$, write $f = \rho + g$, with g small (of zero average if on $\mathbb{T}$) :

$$\partial_t g + v \cdot \nabla_x g - (\rho + g)(d-1)v \cdot J_g + J_g \cdot \nabla_v g = \Delta_v g.$$

Main result (spoiler) : it is stable ! In $H^{s,1}$ norm (s derivatives in x, one in v).

Assume that $s \geqslant d$ if d is odd or $s \geqslant d+1$ if d is even. If $g_0 \in L^\infty(\mathbb{R}^d \times \mathbb{S}) \cap H^{s,1}(\mathbb{R}^d \times \mathbb{S})$ is small, the solution is global. There exists an energy, equivalent to the $H^{s,1}(\mathbb{R}^d \times \mathbb{S})$ norm of g, that decays in time. On $\mathbb{T}$, the energy is exponentially decreasing.

7

Our main goal : around isotropic state, $\rho < \rho_c = d$

Stability/instability for the space-homogeneous model

- $\rho \leqslant \rho_c$: only solution $\kappa = 0$. Isotropic state, stable (exponentially if $\rho < \rho_c$).
- $\rho > \rho_c$: either $\kappa = 0$ (isotropic state, unstable), or a solution $\kappa(\rho) > 0$. If $J_{f^0} \neq 0$, exponential convergence of f to $\rho M_{\kappa(\rho)\Omega_\infty}$ for some $\Omega_\infty \in \mathbb{S}$.

- For the inhomogeneous model, the homogeneous steady states are the same. Can we say something about their stability ?
- Interplay between transport and an operator relaxing in v only : hypocoercivity approach.
- We concentrate on $\rho < \rho_c$, write $f = \rho + g$, with g small (of zero average if on $\mathbb{T}$) :

$$\partial_t g + v \cdot \nabla_x g - (\rho + g)(d-1) v \cdot J_g + J_g \cdot \nabla_v g = \Delta_v g.$$

Main result (spoiler) : it is stable ! In $H^{s,1}$ norm (s derivatives in x, one in v).

Assume that $s \geqslant d$ if d is odd or $s \geqslant d+1$ if d is even. If $g_0 \in L^\infty(\mathbb{R}^d \times \mathbb{S}) \cap H^{s,1}(\mathbb{R}^d \times \mathbb{S})$ is small, the solution is global. There exists an energy, equivalent to the $H^{s,1}(\mathbb{R}^d \times \mathbb{S})$ norm of g, that decays in time. On $\mathbb{T}$, the energy is exponentially decreasing.

- Related work[7] for BGK instead of Fokker-Planck : $\partial_t f + v \cdot \nabla_x f = \rho_f M_{J_f} - f$.

[7] Merino-Aceituno, Schmeiser, Winter *ArXiv* 2024 [MASW24]

Back to basics : one particle (Velocity Spherical Brownian Motion[10])

A single self-propelled particle exploring around, no interaction.

$$X \in \mathbb{R}^d \text{ (or } \mathbb{T}), \; V \in \mathbb{S}, \begin{cases} \mathrm{d}X = V\mathrm{d}t \\ \mathrm{d}V = P_{v^\perp} \circ \mathrm{d}B_t. \end{cases}$$

Its law satisfies $\partial_t f + v \cdot \nabla_x f = \Delta_v f$.

8

9

[10] Baudoin, Tardif, *KRM* 2018 [BT18]

A single self-propelled particle exploring around, no interaction.

$X \in \mathbb{R}^d$ (or $\mathbb{T}$), $V \in \mathbb{S}$, $\begin{cases} \mathrm{d}X = V\mathrm{d}t \\ \mathrm{d}V = P_{v^\perp} \circ \mathrm{d}B_t. \end{cases}$ Its law satisfies $\partial_t f + v \cdot \nabla_x f = \Delta_v f$.

Looks simple, can't we do à la Villani[8] ? Define an energy equivalent to the square of the H^1 norm (if $\beta < \alpha\gamma$) : $\mathcal{F} = \|f\|_2^2 + \alpha\|\nabla_v f\|_2^2 + 2\beta\langle\nabla_v f, \nabla_x f\rangle + \gamma\|\nabla_x f\|_2^2$, and then get

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = -2\|\nabla_v f\|^2 + 2\alpha\langle\nabla_v f, \nabla_v(\Delta_v f)\rangle + 2\beta\|P_{v^\perp}\nabla_x f\|_2^2 + 2\gamma\langle\nabla_x f, \nabla_x(\Delta_v f)\rangle \text{ (good terms)}$$
$$-2\alpha\langle\nabla_v f, \nabla_x f\rangle - 2\beta[(d-1)\langle\nabla_v f, \nabla_x f\rangle - 2\langle\nabla_v f, \nabla_x(\Delta_v f)\rangle]. \text{ (bad terms)}$$

Trouble or not ?

[8] Villani, 2009 [Vil09]

[9]

[10] Baudoin, Tardif, *KRM* 2018 [BT18]

Back to basics : one particle (Velocity Spherical Brownian Motion[10])

A single self-propelled particle exploring around, no interaction.

$X \in \mathbb{R}^d$ (or $\mathbb{T}$), $V \in \mathbb{S}$, $\begin{cases} \mathrm{d}X = V\mathrm{d}t \\ \mathrm{d}V = P_{v^\perp} \circ \mathrm{d}B_t. \end{cases}$ Its law satisfies $\partial_t f + v \cdot \nabla_x f = \Delta_v f$.

Looks simple, can't we do à la Villani[8] ? Define an energy equivalent to the square of the H^1 norm (if $\beta < \alpha\gamma$) : $\mathcal{F} = \|f\|_2^2 + \alpha\|\nabla_v f\|_2^2 + 2\beta\langle\nabla_v f, \nabla_x f\rangle + \gamma\|\nabla_x f\|_2^2$, and then get

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = -2\|\nabla_v f\|^2 + 2\alpha\langle\nabla_v f, \nabla_v(\Delta_v f)\rangle + 2\beta\|P_{v^\perp}\nabla_x f\|_2^2 + 2\gamma\langle\nabla_x f, \nabla_x(\Delta_v f)\rangle \text{ (good terms)}$$
$$-2\alpha\langle\nabla_v f, \nabla_x f\rangle - 2\beta[(d-1)\langle\nabla_v f, \nabla_x f\rangle - 2\langle\nabla_v f, \nabla_x(\Delta_v f)\rangle]. \text{ (bad terms)}$$

Trouble or not ? Good terms are indeed equivalent to H^2 norm (for a mean-zero function f), but that's not trivial to recover the missing $\|v \cdot \nabla_x f\|_2^2$.

[8] Villani, 2009 [Vil09]

[9]

[10] Baudoin, Tardif, *KRM* 2018 [BT18]

Back to basics : one particle (Velocity Spherical Brownian Motion[10])

A single self-propelled particle exploring around, no interaction.

$X \in \mathbb{R}^d$ (or $\mathbb{T}$), $V \in \mathbb{S}$, $\begin{cases} \mathrm{d}X = V\mathrm{d}t \\ \mathrm{d}V = P_{v^\perp} \circ \mathrm{d}B_t. \end{cases}$ Its law satisfies $\partial_t f + v \cdot \nabla_x f = \Delta_v f$.

Looks simple, can't we do à la Villani[8] ? Define an energy equivalent to the square of the H^1 norm (if $\beta < \alpha\gamma$) : $\mathcal{F} = \|f\|_2^2 + \alpha\|\nabla_v f\|_2^2 + 2\beta\langle\nabla_v f, \nabla_x f\rangle + \gamma\|\nabla_x f\|_2^2$, and then get

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = -2\|\nabla_v f\|^2 + 2\alpha\langle\nabla_v f, \nabla_v(\Delta_v f)\rangle + 2\beta\|P_{v^\perp}\nabla_x f\|_2^2 + 2\gamma\langle\nabla_x f, \nabla_x(\Delta_v f)\rangle \text{ (good terms)}$$
$$-2\alpha\langle\nabla_v f, \nabla_x f\rangle - 2\beta[(d-1)\langle\nabla_v f, \nabla_x f\rangle - 2\langle\nabla_v f, \nabla_x(\Delta_v f)\rangle]. \text{ (bad terms)}$$

Trouble or not ? Good terms are indeed equivalent to H^2 norm (for a mean-zero function f), but that's not trivial to recover the missing $\|v \cdot \nabla_x f\|_2^2$.
Furthermore, if we want a quantitative regularising estimate for short times à la Hérau[9] (that is (α, β, γ) replaced by $(\alpha t, \beta t^2, \gamma t^3)$) it does not work. Why ?

[8] Villani, 2009 [Vil09]
[9] Hérau, *JFA* 2007 [Hé07]
[10] Baudoin, Tardif, *KRM* 2018 [BT18]

The trouble is the sphere — but there is a nice algebraic framework

We want to write our equation as $\partial_t f + \mathrm{T} f = \mathrm{A}^2 f$.

Fancy decomposition of the Laplace-Beltrami on the sphere

Write $\mathrm{A}_{i,j} = [e_i \cdot \nabla_v, e_j \cdot \nabla_v]$ (in coordinates where $v = \cos\theta w + \sin\theta(\cos\varphi_{i,j} e_i + \sin\varphi_{i,j} e_j)$ with $w \in \mathbb{S}, w \perp e_i, w \perp e_j$, it reads $\mathrm{A}_{i,j} = \partial_{\varphi_{i,j}}$). Then, writing $\mathrm{A}^2 = \sum_{i<j} A_{i,j}^2$:

- $\mathrm{A}_{i,j}$ is antiselfadjoint on $\mathbb{S}$, and commutes with A^2, and $\mathrm{A}_{i,j} v_k = \delta_{jk} v_i - \delta_{ik} v_j$.
- If $f, g \in C^1(\mathbb{S})$, then $\nabla_v f \cdot \nabla_v g = \sum_{i<j} \mathrm{A_{i,j}} f \mathrm{A_{i,j}} g$. Consequently $\Delta_v f = \mathrm{A}^2 f$.

The trouble is the sphere — but there is a nice algebraic framework

We want to write our equation as $\partial_t f + \mathrm{T} f = \mathrm{A}^2 f$.

Fancy decomposition of the Laplace-Beltrami on the sphere

Write $\mathrm{A}_{i,j} = [e_i \cdot \nabla_v, e_j \cdot \nabla_v]$ (in coordinates where $v = \cos\theta w + \sin\theta(\cos\varphi_{i,j} e_i + \sin\varphi_{i,j} e_j)$ with $w \in \mathbb{S}$, $w \perp e_i$, $w \perp e_j$, it reads $\mathrm{A}_{i,j} = \partial_{\varphi_{i,j}}$). Then, writing $\mathrm{A}^2 = \sum_{i<j} A_{i,j}^2$:

- $\mathrm{A}_{i,j}$ is antiselfadjoint on $\mathbb{S}$, and commutes with A^2, and $\mathrm{A}_{i,j} v_k = \delta_{jk} v_i - \delta_{ik} v_j$.
- If $f, g \in C^1(\mathbb{S})$, then $\nabla_v f \cdot \nabla_v g = \sum_{i<j} \mathrm{A_{i,j}} f \mathrm{A_{i,j}} g$. Consequently $\Delta_v f = \mathrm{A}^2 f$.

Evolution of quadratic quantities and commutators

Write $\mathrm{T} = v \cdot \nabla_x$. If X is a smooth differential operator and $Q_\mathrm{X} = \int_{\mathbb{R}^d \times \mathbb{S}} f\, \mathrm{X} f \,\mathrm{d}x\mathrm{d}v$, then $\frac{\mathrm{d}}{\mathrm{d}t} Q_\mathrm{X} = Q_{\Phi(\mathrm{X})}$, where the operator $\Phi(\mathrm{X})$ goes as follows:

$$\Phi(\mathrm{X}) = \mathrm{A}^2\mathrm{X} + \mathrm{X}\mathrm{A}^2 + [\mathrm{T}, \mathrm{X}] = 2\mathrm{A}\mathrm{X}\mathrm{A} + [\mathrm{A}[\mathrm{A}, \mathrm{X}]] + [\mathrm{T}, \mathrm{X}].$$

Villani's chain of commutators : start from $\mathrm{C}_0 = A$ and then $\mathrm{C}_{i+1} = [\mathrm{T}, C_i]$, hoping to get all the missing “directions”. Here it stops at $C_1 = [\mathrm{T}, \mathrm{A}] := \mathrm{S} = v \wedge \nabla_x$, since then $[\mathrm{T}, \mathrm{S}] = 0$.

The trouble is the sphere — but there is a nice algebraic framework

We want to write our equation as $\partial_t f + \mathrm{T} f = \mathrm{A}^2 f$.

Fancy decomposition of the Laplace-Beltrami on the sphere

Write $\mathrm{A}_{i,j} = [e_i \cdot \nabla_v, e_j \cdot \nabla_v]$ (in coordinates where $v = \cos\theta w + \sin\theta(\cos\varphi_{i,j} e_i + \sin\varphi_{i,j} e_j)$ with $w \in \mathbb{S}$, $w \perp e_i$, $w \perp e_j$, it reads $\mathrm{A}_{i,j} = \partial_{\varphi_{i,j}}$). Then, writing $\mathrm{A}^2 = \sum_{i<j} A_{i,j}^2$:

- $\mathrm{A}_{i,j}$ is antiselfadjoint on $\mathbb{S}$, and commutes with A^2, and $\mathrm{A}_{i,j} v_k = \delta_{jk} v_i - \delta_{ik} v_j$.
- If $f, g \in C^1(\mathbb{S})$, then $\nabla_v f \cdot \nabla_v g = \sum_{i<j} \mathrm{A_{i,j}} f \mathrm{A_{i,j}} g$. Consequently $\Delta_v f = \mathrm{A}^2 f$.

Evolution of quadratic quantities and commutators

Write $\mathrm{T} = v \cdot \nabla_x$. If X is a smooth differential operator and $Q_{\mathrm{X}} = \int_{\mathbb{R}^d \times \mathbb{S}} f \, \mathrm{X} f \, \mathrm{d}x \mathrm{d}v$, then $\frac{\mathrm{d}}{\mathrm{d}t} Q_{\mathrm{X}} = Q_{\Phi(\mathrm{X})}$, where the operator $\Phi(\mathrm{X})$ goes as follows:

$$\Phi(\mathrm{X}) = \mathrm{A}^2\mathrm{X} + \mathrm{X}\mathrm{A}^2 + [\mathrm{T}, \mathrm{X}] = 2\mathrm{A}\mathrm{X}\mathrm{A} + [\mathrm{A}[\mathrm{A}, \mathrm{X}]] + [\mathrm{T}, \mathrm{X}].$$

Villani's chain of commutators : start from $\mathrm{C}_0 = A$ and then $\mathrm{C}_{i+1} = [\mathrm{T}, C_i]$, hoping to get all the missing "directions". Here it stops at $C_1 = [\mathrm{T}, \mathrm{A}] := \mathrm{S} = v \wedge \nabla_x$, since then $[\mathrm{T}, \mathrm{S}] = 0$. $\to$ Hörmander theory : commute as you can ! We get $[\mathrm{A}, \mathrm{S}] = (d-1)\mathrm{T}$. And we are happy since $\mathrm{T}^2 + \mathrm{S}^2 = \Delta_x$.

Good hypoellipticity functional à la Hérau

Weights of operators

We will always take operators X composed thanks to A (weight $\frac{1}{2}$), S (weight $\frac{3}{2}$) and Δ_x (weight 4). Weights of compositions are the sum of weights.

11

Good hypoellipticity functional à la Hérau

Weights of operators

We will always take operators X composed thanks to A (weight $\frac{1}{2}$), S (weight $\frac{3}{2}$) and Δ_x (weight 4). Weights of compositions are the sum of weights.

Theorem : a good H^1 energy for short-time estimates.

Set $\mathcal{F}_0 = Q_{\mathrm{Id}}$, $\mathcal{F}_1(\tau,\cdot) = \alpha\tau Q_{-\mathrm{A}^2} + \beta\tau^2 Q_{\mathrm{SA+AS}} + \gamma\tau^3 Q_{-\mathrm{S}^2} + \delta\tau^4 Q_{-\Delta_x}$.
Then there exists coefficients α, β, γ such that $\mathcal{F}(t) = \mathcal{F}_0 + \mathcal{F}_1(\min(t,1), f)$ is decreasing in time. Furthermore, on $\mathbb{T}$, this quantity is equivalent to the H^1 norm of f if f has mean zero, and is controlled by its dissipation at positive time, leading to an exponential decay.

11

Good hypoellipticity functional à la Hérau

Weights of operators

We will always take operators X composed thanks to A (weight $\frac{1}{2}$), S (weight $\frac{3}{2}$) and Δ_x (weight 4). Weights of compositions are the sum of weights.

Theorem : a good H^1 energy for short-time estimates.

Set $\mathcal{F}_0 = Q_{\mathrm{Id}}$, $\mathcal{F}_1(\tau, \cdot) = \alpha\tau Q_{-\mathrm{A}^2} + \beta\tau^2 Q_{\mathrm{SA}+\mathrm{AS}} + \gamma\tau^3 Q_{-\mathrm{S}^2} + \delta\tau^4 Q_{-\Delta_x}$.
Then there exists coefficients α, β, γ such that $\mathcal{F}(t) = \mathcal{F}_0 + \mathcal{F}_1(\min(t,1), f)$ is decreasing in time. Furthermore, on $\mathbb{T}$, this quantity is equivalent to the H^1 norm of f if f has mean zero, and is controlled by its dissipation at positive time, leading to an exponential decay.

Remark : $\mathcal{F}_0 = \|f\|_2^2$, and $\mathcal{F}_1(\tau, f) = \alpha\tau\|\nabla_v\|_2^2 + 2\beta\tau^2\langle\nabla_v|\nabla_x\rangle + \gamma\tau^3\|P_{v^\perp}\nabla_x\|_2^2 + \delta\tau^4\|\nabla_x\|_2^2$.
We then get that the H^1 norm is controlled by $\frac{1}{t^2}\|f_0\|_2^2$ for short times (to compare with $\frac{1}{t^{\frac{3}{2}}}$ for usual kinetic Fokker-Planck equations).

11

Good hypoellipticity functional à la Hérau

Weights of operators

We will always take operators X composed thanks to A (weight $\frac{1}{2}$), S (weight $\frac{3}{2}$) and Δ_x (weight 4). Weights of compositions are the sum of weights.

Theorem : a good H^1 energy for short-time estimates.

Set $\mathcal{F}_0 = Q_{\mathrm{Id}}$, $\mathcal{F}_1(\tau,\cdot) = \alpha\tau Q_{-\mathrm{A}^2} + \beta\tau^2 Q_{\mathrm{SA}+\mathrm{AS}} + \gamma\tau^3 Q_{-\mathrm{S}^2} + \delta\tau^4 Q_{-\Delta_x}$.
Then there exists coefficients α,β,γ such that $\mathcal{F}(t) = \mathcal{F}_0 + \mathcal{F}_1(\min(t,1), f)$ is decreasing in time. Furthermore, on $\mathbb{T}$, this quantity is equivalent to the H^1 norm of f if f has mean zero, and is controlled by its dissipation at positive time, leading to an exponential decay.

Remark : $\mathcal{F}_0 = \|f\|_2^2$, and $\mathcal{F}_1(\tau, f) = \alpha\tau\|\nabla_v\|_2^2 + 2\beta\tau^2\langle\nabla_v|\nabla_x\rangle + \gamma\tau^3\|P_{v^\perp}\nabla_x\|_2^2 + \delta\tau^4\|\nabla_x\|_2^2$.
We then get that the H^1 norm is controlled by $\frac{1}{t^2}\|f_0\|_2^2$ for short times (to compare with $\frac{1}{t^{\frac{3}{2}}}$ for usual kinetic Fokker-Planck equations).
Higher order (in x only) : $\mathcal{F}_k(\tau,\cdot) = \nu_k\,\tau^{4(k-1)} \sum_{|m|=k-1} \binom{k-1}{m} \mathcal{F}_1(\tau, \partial_x^m)$.

[11] Coti-Zelati, Dietert, Gérard-Varet, *Annals of PDE*, 2023 [CZDGV23]

Good hypoellipticity functional à la Hérau

Weights of operators

We will always take operators X composed thanks to A (weight $\frac{1}{2}$), S (weight $\frac{3}{2}$) and Δ_x (weight 4). Weights of compositions are the sum of weights.

Theorem : a good H^1 energy for short-time estimates.

Set $\mathcal{F}_0 = Q_{\mathrm{Id}}$, $\mathcal{F}_1(\tau,\cdot) = \alpha\tau Q_{-\mathrm{A}^2} + \beta\tau^2 Q_{\mathrm{SA}+\mathrm{AS}} + \gamma\tau^3 Q_{-\mathrm{S}^2} + \delta\tau^4 Q_{-\Delta_x}$.
Then there exists coefficients α,β,γ such that $\mathcal{F}(t) = \mathcal{F}_0 + \mathcal{F}_1(\min(t,1), f)$ is decreasing in time. Furthermore, on $\mathbb{T}$, this quantity is equivalent to the H^1 norm of f if f has mean zero, and is controlled by its dissipation at positive time, leading to an exponential decay.

Remark : $\mathcal{F}_0 = \|f\|_2^2$, and $\mathcal{F}_1(\tau, f) = \alpha\tau\|\nabla_v\|_2^2 + 2\beta\tau^2\langle\nabla_v|\nabla_x\rangle + \gamma\tau^3\|P_{v^\perp}\nabla_x\|_2^2 + \delta\tau^4\|\nabla_x\|_2^2$.
We then get that the H^1 norm is controlled by $\frac{1}{t^2}\|f_0\|_2^2$ for short times (to compare with $\frac{1}{t^{\frac{3}{2}}}$ for usual kinetic Fokker-Planck equations).
Higher order (in x only) : $\mathcal{F}_k(\tau,\cdot) = \nu_k\,\tau^{4(k-1)} \sum_{|m|=k-1} \binom{k-1}{m}\mathcal{F}_1(\tau, \partial_x^m)$.

In the case of the torus in space (and $d = 3$), see also the recent work on the model of Saintillan–Shelley model[11].

[11] Coti-Zelati, Dietert, Gérard-Varet, *Annals of PDE*, 2023 [CZDGV23]

Back to our model, another nice algebraic view

Our equation on the perturbation g ($f = \rho + g$) :

$$\partial_t g + v \cdot \nabla_x g - (\rho + g)(d-1)v \cdot J_g + J_g \cdot \nabla_v g = \Delta_v g.$$

Back to our model, another nice algebraic view

Our equation on the perturbation g ($f = \rho + g$) :

$$\partial_t g + v \cdot \nabla_x g - (\rho + g)(d-1) v \cdot J_g + J_g \cdot \nabla_v g = \Delta_v g.$$

A new operator (not differential)

We define $\mathrm{U}_{i,j} g = d(v_i\, e_j \cdot J_g - v_j\, e_i \cdot J_g)$ (or in condensed form $\mathrm{U} g = dv \wedge J_g$). We have

Back to our model, another nice algebraic view

Our equation on the perturbation g ($f = \rho + g$) :

$$\partial_t g + v \cdot \nabla_x g - (\rho + g)(d-1) v \cdot J_g + J_g \cdot \nabla_v g = \Delta_v g.$$

A new operator (not differential)

We define $\mathrm{U}_{i,j} g = d(v_i\, e_j \cdot J_g - v_j\, e_i \cdot J_g)$ (or in condensed form $\mathrm{U}g = dv \wedge J_g$). We have

- $\mathrm{U}^2 g = -(d-1) dv \cdot J_g$

Back to our model, another nice algebraic view

Our equation on the perturbation g ($f = \rho + g$) :

$$\partial_t g + v \cdot \nabla_x g - (\rho + g)(d-1) v \cdot J_g + J_g \cdot \nabla_v g = \Delta_v g.$$

A new operator (not differential)

We define $\mathrm{U}_{i,j} g = d(v_i\, e_j \cdot J_g - v_j\, e_i \cdot J_g)$ (or in condensed form $\mathrm{U} g = dv \wedge J_g$). We have

- $\mathrm{U}^2 g = -(d-1) dv \cdot J_g$
- $\mathrm{UA} = \mathrm{AU} = \mathrm{U}^2$, and therefore $\mathrm{A}^2 - \mathrm{U}^2 = (\mathrm{A} - \mathrm{U})^2$.

Back to our model, another nice algebraic view

Our equation on the perturbation g ($f = \rho + g$) :

$$\partial_t g + v \cdot \nabla_x g - (\rho + g)(d-1) v \cdot J_g + J_g \cdot \nabla_v g = \Delta_v g.$$

A new operator (not differential)

We define $\mathrm{U}_{i,j} g = d(v_i\, e_j \cdot J_g - v_j\, e_i \cdot J_g)$ (or in condensed form $\mathrm{U} g = dv \wedge J_g$). We have

- $\mathrm{U}^2 g = -(d-1) dv \cdot J_g$
- $\mathrm{UA} = \mathrm{AU} = \mathrm{U}^2$, and therefore $\mathrm{A}^2 - \mathrm{U}^2 = (\mathrm{A} - \mathrm{U})^2$.
- $\mathrm{AU}^2 = \mathrm{U}^2\mathrm{A} = \mathrm{A}^2\mathrm{U} = \mathrm{UA}^2 = \mathrm{U}^3 = -(d-1)\mathrm{U}$.

Back to our model, another nice algebraic view

Our equation on the perturbation g ($f = \rho + g$) :

$$\partial_t g + v \cdot \nabla_x g - (\rho + g)(d-1) v \cdot J_g + J_g \cdot \nabla_v g = \Delta_v g.$$

A new operator (not differential)

We define $\mathrm{U}_{i,j} g = d(v_i\, e_j \cdot J_g - v_j\, e_i \cdot J_g)$ (or in condensed form $\mathrm{U}g = dv \wedge J_g$). We have

- $\mathrm{U}^2 g = -(d-1) dv \cdot J_g$
- $\mathrm{UA} = \mathrm{AU} = \mathrm{U}^2$, and therefore $\mathrm{A}^2 - \mathrm{U}^2 = (\mathrm{A} - \mathrm{U})^2$.
- $\mathrm{AU}^2 = \mathrm{U}^2\mathrm{A} = \mathrm{A}^2\mathrm{U} = \mathrm{UA}^2 = \mathrm{U}^3 = -(d-1)\mathrm{U}$.
- $J_g \cdot \nabla_{\mathrm{v}} g = \frac{1}{d} \sum_{i<j} \mathrm{U}_{i,j} g \mathrm{A}_{i,j} g$.

Back to our model, another nice algebraic view

Our equation on the perturbation g $(f = \rho + g)$:

$$\partial_t g + v \cdot \nabla_x g - (\rho + g)(d-1) v \cdot J_g + J_g \cdot \nabla_v g = \Delta_v g.$$

A new operator (not differential)

We define $\mathrm{U}_{i,j} g = d(v_i\, e_j \cdot J_g - v_j\, e_i \cdot J_g)$ (or in condensed form $\mathrm{U}g = dv \wedge J_g$). We have

- $\mathrm{U}^2 g = -(d-1) dv \cdot J_g$
- $\mathrm{UA} = \mathrm{AU} = \mathrm{U}^2$, and therefore $\mathrm{A}^2 - \mathrm{U}^2 = (\mathrm{A} - \mathrm{U})^2$.
- $\mathrm{AU}^2 = \mathrm{U}^2\mathrm{A} = \mathrm{A}^2\mathrm{U} = \mathrm{UA}^2 = \mathrm{U}^3 = -(d-1)\mathrm{U}$.
- $J_g \cdot \nabla_v g = \frac{1}{d} \sum_{i<j} \mathrm{U}_{i,j} g \mathrm{A}_{i,j} g$.

We then get

$$\partial_t g + \mathrm{T}g = \mathrm{A}^2 g - \frac{\rho + g}{d} \mathrm{U}^2 g - \frac{1}{d} \mathrm{U}g\, \mathrm{A}g = \left(\mathrm{A}^2 - \frac{\rho}{d}\mathrm{U}^2\right) g - \frac{1}{d}(\mathrm{A}(g\mathrm{U}g)).$$

To simplify notations, we note $\mathrm{L} = \mathrm{A} - \left(1 - \sqrt{1 - \frac{\rho}{d}}\right)\mathrm{U}$, so that $\mathrm{L}^2 = \mathrm{A}^2 - \frac{\rho}{d}\mathrm{U}^2$.

Back to our model, nonlinear part

$$\partial_t g + \mathsf{T} g = \mathsf{L}^2 g - \frac{1}{d}(\mathsf{A}(g \mathsf{U} g)).$$

Same functional $\mathcal{F}(\tau, g(t, \cdot))$, new terms in the dissipation.

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_{\mathsf{X}} = Q_{\Phi^\rho(\mathsf{X})} + R_{\mathsf{X}}, \text{ where this time}$$

$$\Phi^\rho(\mathsf{X}) = \mathsf{A}^2\mathsf{X} + \mathsf{X}\mathsf{A}^2 - \frac{\rho}{d}(\mathsf{U}^2\mathsf{X} + \mathsf{X}\mathsf{U}^2) + [\mathsf{T}, \mathsf{X}] = \mathsf{L}^2\mathsf{X} + \mathsf{X}\mathsf{L}^2 + [\mathsf{T}, \mathsf{X}],$$

and where the non-linear term produces

$$R_{\mathsf{X}}(g) = \frac{1}{d} \int_{\mathbb{R}^d \times \mathbb{S}} (\mathsf{A}\mathsf{X} g)\,(\mathsf{U} g)\, g\, v \mathrm{d}x.$$

Back to our model, nonlinear part

$$\partial_t g + \mathsf{T} g = \mathsf{L}^2 g - \frac{1}{d}(\mathsf{A}(g \mathsf{U} g)).$$

Same functional $\mathcal{F}(\tau, g(t,\cdot))$, new terms in the dissipation.

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_{\mathsf{X}} = Q_{\Phi^\rho(\mathsf{X})} + R_{\mathsf{X}}, \text{ where this time}$$

$$\Phi^\rho(\mathsf{X}) = \mathsf{A}^2\mathsf{X} + \mathsf{X}\mathsf{A}^2 - \frac{\rho}{d}(\mathsf{U}^2\mathsf{X} + \mathsf{X}\mathsf{U}^2) + [\mathsf{T},\mathsf{X}] = \mathsf{L}^2\mathsf{X} + \mathsf{X}\mathsf{L}^2 + [\mathsf{T},\mathsf{X}],$$

and where the non-linear term produces

$$R_{\mathsf{X}}(g) = \frac{1}{d}\int_{\mathbb{R}^d\times\mathbb{S}} (\mathsf{A}\mathsf{X}g)\,(\mathsf{U}g)\,g\,v\mathrm{d}x.$$

Control of the quadratic terms : exactly the same job !

Back to our model, nonlinear part

$$\partial_t g + \mathrm{T} g = \mathrm{L}^2 g - \frac{1}{d}(\mathrm{A}(g\mathrm{U}g)).$$

Same functional $\mathcal{F}(\tau, g(t,\cdot))$, new terms in the dissipation.

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_{\mathrm{X}} = Q_{\Phi^\rho(\mathrm{X})} + R_{\mathrm{X}}, \text{ where this time}$$

$$\Phi^\rho(\mathrm{X}) = \mathrm{A}^2\mathrm{X} + \mathrm{X}\mathrm{A}^2 - \frac{\rho}{d}(\mathrm{U}^2\mathrm{X} + \mathrm{X}\mathrm{U}^2) + [\mathrm{T},\mathrm{X}] = \mathrm{L}^2\mathrm{X} + \mathrm{X}\mathrm{L}^2 + [\mathrm{T},\mathrm{X}],$$

and where the non-linear term produces

$$R_{\mathrm{X}}(g) = \frac{1}{d}\int_{\mathbb{R}^d\times\mathbb{S}} (\mathrm{A}\mathrm{X}g)\,(\mathrm{U}g)\,g\,v\mathrm{d}x.$$

Control of the quadratic terms : exactly the same job !

Control of the cubic terms : a little bit more painful. We control them by $\sqrt{\mathcal{F}\mathcal{D}}$. This time, no short-time regularity, we need really high order norms, but only with one derivative in v. This allows to get the nonlinear stability.

Back to our model, nonlinear part

$$\partial_t g + \mathsf{T} g = \mathsf{L}^2 g - \frac{1}{d}(\mathsf{A}(g\mathsf{U}g)).$$

Same functional $\mathcal{F}(\tau, g(t,\cdot))$, new terms in the dissipation.

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_{\mathsf{X}} = Q_{\Phi^\rho(\mathsf{X})} + R_{\mathsf{X}}, \text{ where this time}$$

$$\Phi^\rho(\mathsf{X}) = \mathsf{A}^2\mathsf{X} + \mathsf{X}\mathsf{A}^2 - \frac{\rho}{d}(\mathsf{U}^2\mathsf{X} + \mathsf{X}\mathsf{U}^2) + [\mathsf{T},\mathsf{X}] = \mathsf{L}^2\mathsf{X} + \mathsf{X}\mathsf{L}^2 + [\mathsf{T},\mathsf{X}],$$

and where the non-linear term produces

$$R_{\mathsf{X}}(g) = \frac{1}{d}\int_{\mathbb{R}^d\times\mathbb{S}} (\mathsf{A}\mathsf{X}g)\,(\mathsf{U}g)\,g\,v\mathrm{d}x.$$

Control of the quadratic terms : exactly the same job !

Control of the cubic terms : a little bit more painful. We control them by $\sqrt{\mathcal{F}}\mathcal{D}$. This time, no short-time regularity, we need really high order norms, but only with one derivative in v. This allows to get the nonlinear stability.

Special case : if f only depends on one space variable. Then, by small time regularity, we have stability in $L^2 \cap L^\infty$.

Bibliography I

François Bolley, José A. Cañizo, and José A. Carrillo.
Mean-field limit for the stochastic Vicsek model.
Appl. Math. Lett., 3(25):339–343, 2012.

Fabrice Baudoin and Camille Tardif.
Hypocoercive estimates on foliations and velocity spherical Brownian motion.
Kinet. Relat. Models, 11(1):1–23, 2018.

Louis-Pierre Chaintron and Antoine Diez.
Propagation of chaos: a review of models, methods and applications. II: Applications.
Kinet. Relat. Models, 15(6):1017–1173, 2022.

Michele Coti Zelati, Helge Dietert, and David Gérard-Varet.
Orientation Mixing in Active Suspensions.
Annals of PDE, 9(2):20, October 2023.

Pierre Degond, Amic Frouvelle, and Jian-Guo Liu.
Macroscopic limits and phase transition in a system of self-propelled particles.
J. Nonlinear Sci., 23(3):427–456, 2013.

Bibliography II

Pierre Degond and Sébastien Motsch.
Continuum limit of self-driven particles with orientation interaction.
Math. Models Methods Appl. Sci., 18:1193–1215, 2008.

Amic Frouvelle and Jian-Guo Liu.
Dynamics in a kinetic model of oriented particles with phase transition.
SIAM J. Math. Anal., 44(2):791–826, 2012.

Frédéric Hérau.
Short and long time behavior of the Fokker-Planck equation in a confining potential and applications.
J. Funct. Anal., 244(1):95–118, 2007.

Sara Merino-Aceituno, Christian Schmeiser, and Raphael Winter.
Stability of equilibria of the spatially inhomogeneous Vicsek-BGK equation across a bifurcation, 2024.

Tamás Vicsek, András Czirók, Eshel Ben-Jacob, Inon Cohen, and Ofer Shochet.
Novel type of phase transition in a system of self-driven particles.
Phys. Rev. Lett., 75(6):1226–1229, 1995.

Cédric Villani.
Hypocoercivity, volume 950.
Providence, RI: American Mathematical Society (AMS), 2009.