Hypo(coerciv-elliptic)ity in a model of aligning self-propelled particles

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Work in collaboration with Emeric Bouin (CEREMADE)

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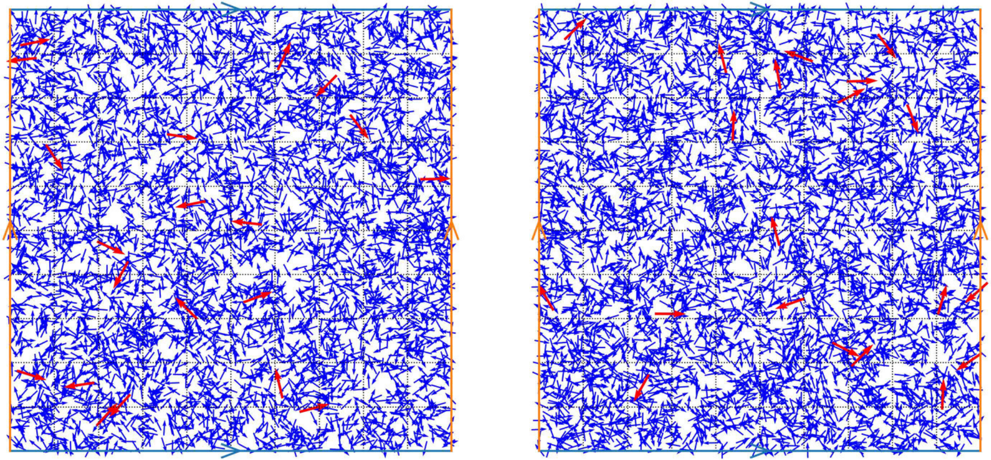
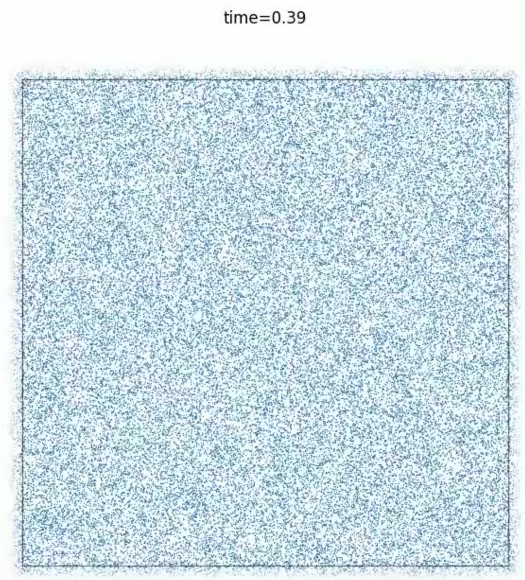
Poitiers, December 11th 2025

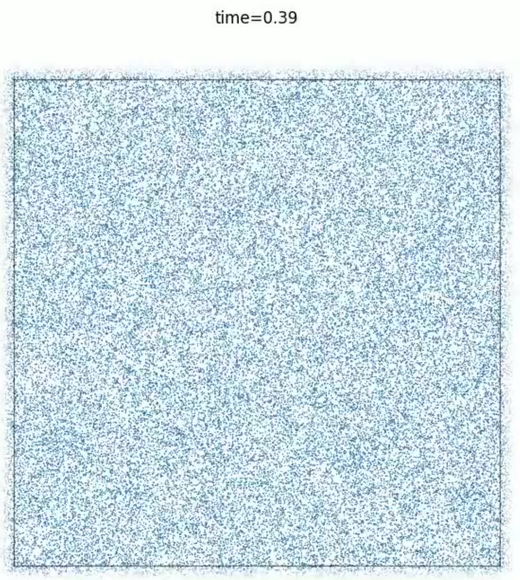
Motivation : aligning self-propelled particles [1,2,3]



[1] Vicsek *et al.*, *Phys. Rev. Lett.*, 1995 [VCBJ+95]
[2] Degond, Motsch, *M3AS*, 2008 [DM08]
[3] Degond, F, Liu, *J. Nonlin. Sci.*, 2013 [DFL13]

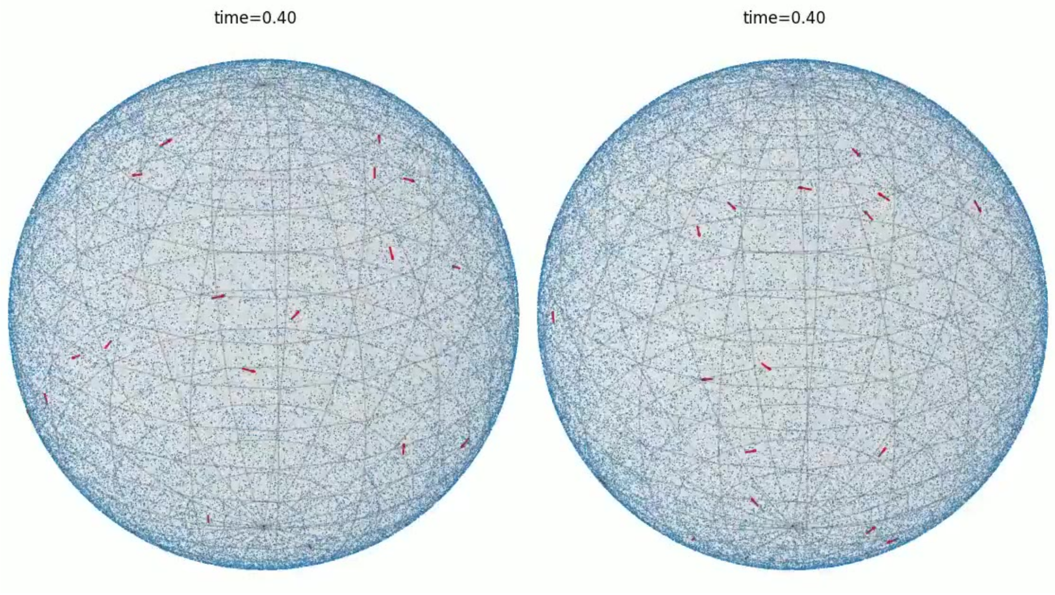
Much more particles, larger domain (with GPU [4])





[4] Diez, F, Motsch, *in progress*

Other geometries [4]



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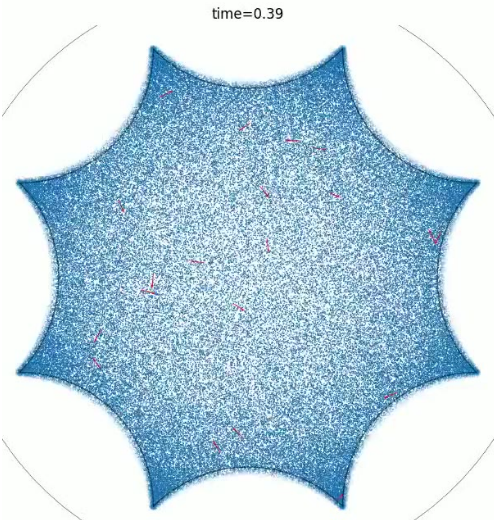
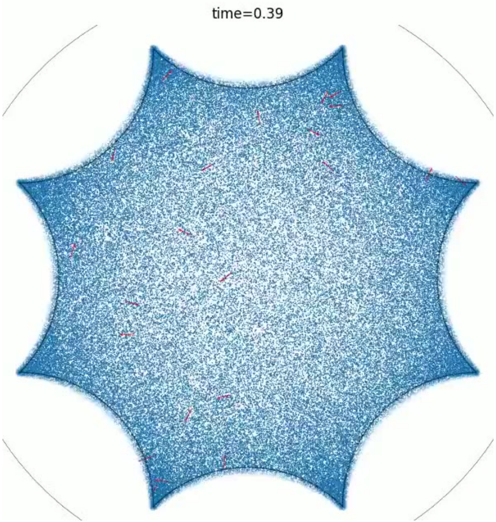
Other (strange) geometries [4]





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The self-propelled particles model

System of coupled SDEs

Particles at positions $X_k = \mathbb{R}^d$ (or a flat torus $\mathbb{T}$), speeds $V_k \in \mathbb{S}$ (unit sphere), $1 \leqslant k \leqslant N$.

$$\begin{cases} \mathrm{d}X_k = c\, V_k \mathrm{d}t \\ \mathrm{d}V_k = -\sum_{j=1}^{N} \nu_{j,k} \frac{1}{2} \nabla_{V_j} \|V_j - V_k\|^2 \mathrm{d}t + \sqrt{2\sigma} P_{V_k^\perp} \circ \mathrm{d}B_{t,k}. \end{cases}$$

Careful : gradients on the sphere (for instance $\nabla_v(\mathbf{u} \cdot v) = P_{v^\perp}\mathbf{u}$), Laplace Beltrami, Stratonovich formulation for brownian motion on the sphere.

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$$\mathrm{d}V_k = P_{V_k^\perp}\Big(\int_{\mathbb{T}\times\mathbb{S}} K(x - X_k) v \mathrm{d}f^N(x,v) \Big) \mathrm{d}t + \sqrt{2} P_{V_k^\perp} \circ \mathrm{d}B_{t,k}$$

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where for a measure f, its first moment (in v) is denoted $\mathbb{J}[f] = \int_{\mathbb{S}} v f(v) \mathrm{d}v$.

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Parameters : N, K, $\mathbb{T}$, ρ (hidden in f^N).

The mean-field or moderate interaction limit

Mean-field limit [5] : convergence of f^N to a density f, solution of a kinetic equation

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (P_{v^\perp}(K *_x \mathbb{J}[f])f) = \Delta_v f.$$

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How to get rid of K ? Use $\frac{1}{\varepsilon_N^d} K(\frac{\cdot}{\varepsilon_N})$ instead, with $\varepsilon_N \to 0$ as $N \to \infty$ (but $\varepsilon_N^d N \to \infty$).

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Moderate interaction limit expected if the limit kinetic equation is well posed [6]

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$\rightarrow$ Only remaining parameters : the shape of $\mathbb{T}$, and ρ (hidden in f).

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Theorem [7]: (local in time) existence and uniqueness, initial condition f_0 in $L^\infty(\mathbb{R}^d\times\mathbb{S})$.

There exists a unique weak solution in $\mathcal{C}([0,T],L^\infty(\mathbb{R}^d\times\mathbb{S}))$ for all $T<\frac{\sqrt{d}}{(d-1)\|f_0\|_\infty}$. It is nonnegative and satisfies (maximum principle + finite speed propagation), for all t,x:

$$\|f(t,x,\cdot)\|_\infty \leqslant \|f_0\|_{\infty,B(x,t)\times\mathbb{S}} + (d-1)\int_0^t \|\mathbb{J}[f](s)\|_{\infty,B(x,s-t)}\|f(s)\|_{\infty,B(x,s-t)\times\mathbb{S}}\,\mathrm{d}s.$$

[5] Bolley, Cañizo, Carrillo, *Appl. Math. Lett.* 2012 [BCC12]
[6] Chaintron, Diez, *Kinet. Relat. Models*, 2022 [CD22]
[7] adapted from : Briant, Meunier, *Kinet. Relat. Models*, 2023 [BM23]

The space-homegeneous setting : phase transition [8]

Fokker–Planck formulation via von Mises distributions, free energy

Define the von Mises distribution $M_J(v) = \frac{e^{v\cdot J}}{\int_{\mathbb{S}} e^{v'\cdot J}\mathrm{d}v'}$, then

$$\partial_t f = -\nabla_v \cdot (P_{v^\perp}\mathbb{J}[f]f) + \Delta_v f = \nabla_v \cdot \left(M_{\mathbb{J}[f]}\nabla_v(\frac{f}{M_{\mathbb{J}[f]}})\right) = \nabla_v \cdot (f\nabla_v(\ln f - v\cdot\mathbb{J}[f])).$$

Dissipation of the free energy $\mathcal{F}[f] = \int_{\mathbb{S}} f\ln f - \frac{1}{2}|\mathbb{J}[f]|^2$: Fisher information (w.r.t $\rho M_{\mathbb{J}[f]}$).

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} = -\mathcal{D} = -\int_{\mathbb{S}} |\nabla_v(\ln f - v\cdot\mathbb{J}[f])|^2 f\mathrm{d}v = -\mathcal{I}(f|\rho M_{\mathbb{J}[f]}).$$

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Criteria for steady states, compatibility equation.

$\mathcal{D}[f] = 0 \Leftrightarrow$ critical point of $\mathcal{F}$ under mass $\rho \Leftrightarrow f = \rho M_J$, with $\mathbb{J}[\rho M_J] = \rho\langle v\rangle_{M_J} = J$.

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Compatibility equation : $J = \kappa\Omega$ with $\Omega \in \mathbb{S}$ and $\kappa = \rho c(\kappa)$ for $c(\kappa) = \frac{\int_0^\pi \cos\theta e^{\kappa\cos\theta}\sin^{d-2}\theta\mathrm{d}\theta}{\int_0^\pi e^{\kappa\cos\theta}\sin^{d-2}\theta\mathrm{d}\theta}$.

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Behaviour : $\frac{\kappa}{c(\kappa)} \nearrow +\infty$ as $\kappa \to +\infty$, and $\searrow \rho_c = d$ as $\kappa \to 0$.

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Our main goal : around isotropic state, $\rho < \rho_c = d$

Stability/instability for the space-homogeneous model

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Main result (spoiler) : it is stable ! In $H^{s,1}$ norm (s derivatives in x, one in v).

Assume that $s > 3/8$ if $d = 2$ or $s > \frac{d}{2} - \frac{1}{4}$ if $d \geqslant 3$. If $g_0 \in H^{s,0}(\mathbb{R}^d \times \mathbb{S})$ is small, the solution is global. There exists an energy, equivalent (for $t \geqslant 1$) to the $H^{s,1}(\mathbb{R}^d \times \mathbb{S})$ norm of g, that decays in time. On $\mathbb{T}$, the energy is exponentially decreasing.
On $\mathbb{R}^d$, if $\int_{\mathbb{R}^d} |x|^2 g_0^2 \mathrm{d}x < +\infty$, the energy decreases at least as $t^{-\nu}$ (for some small ν).

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- Related work[9] for BGK instead of Fokker-Planck : $\partial_t f + v \cdot \nabla_x f = \rho_f M_{\mathbb{J}[f]} - f$.

[9] Merino-Aceituno, Schmeiser, Winter *ArXiv* 2024 [MASW24]

Back to basics : one particle (Velocity Spherical Brownian Motion[12])

A single self-propelled particle exploring around, no interaction.

$X \in \mathbb{R}^d$ (or $\mathbb{T}$), $V \in \mathbb{S}$, $\begin{cases} \mathrm{d}X = V\mathrm{d}t \\ \mathrm{d}V = \sqrt{2}\,P_{v^\perp} \circ \mathrm{d}B_t. \end{cases}$ Its law satisfies $\partial_t f + v \cdot \nabla_x f = \Delta_v f$.

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Looks simple, can't we do à la Villani[10] ? Define an energy equivalent to the square of the H^1 norm (if $\beta < \alpha\gamma$) : $\mathcal{F} = \|f\|_2^2 + \alpha\|\nabla_v f\|_2^2 + 2\beta\langle\nabla_v f, \nabla_x f\rangle + \gamma\|\nabla_x f\|_2^2$, and then get

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = -2\|\nabla_v f\|^2 + 2\alpha\langle\nabla_v f, \nabla_v(\Delta_v f)\rangle + 2\beta\|P_{v^\perp}\nabla_x f\|_2^2 + 2\gamma\langle\nabla_x f, \nabla_x(\Delta_v f)\rangle \text{ (good terms)}$$
$$-2\alpha\langle\nabla_v f, \nabla_x f\rangle - 2\beta[(d-1)\langle\nabla_v f, \nabla_x f\rangle - 2\langle\nabla_v f, \nabla_x(\Delta_v f)\rangle]. \text{ (bad terms)}$$

Trouble or not ?

[10] Villani, 2009 [Vil09]

[11]

[12] Baudoin, Tardif, *KRM* 2018 [BT18]

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$X \in \mathbb{R}^d$ (or $\mathbb{T}$), $V \in \mathbb{S}$, $\begin{cases} \mathrm{d}X = V\mathrm{d}t \\ \mathrm{d}V = \sqrt{2}\, P_{v^\perp} \circ \mathrm{d}B_t. \end{cases}$ Its law satisfies $\partial_t f + v \cdot \nabla_x f = \Delta_v f$.

Looks simple, can't we do à la Villani[10] ? Define an energy equivalent to the square of the H^1 norm (if $\beta < \alpha\gamma$) : $\mathcal{F} = \|f\|_2^2 + \alpha\|\nabla_v f\|_2^2 + 2\beta\langle\nabla_v f, \nabla_x f\rangle + \gamma\|\nabla_x f\|_2^2$, and then get

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = -2\|\nabla_v f\|^2 + 2\alpha\langle\nabla_v f, \nabla_v(\Delta_v f)\rangle + 2\beta\|P_{v^\perp}\nabla_x f\|_2^2 + 2\gamma\langle\nabla_x f, \nabla_x(\Delta_v f)\rangle \text{ (good terms)}$$
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Trouble or not ? Good terms are indeed equivalent to H^2 norm (for a mean-zero function f), but that's not trivial to recover the missing $\|v \cdot \nabla_x f\|_2^2$.

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Trouble or not ? Good terms are indeed equivalent to H^2 norm (for a mean-zero function f), but that's not trivial to recover the missing $\|v \cdot \nabla_x f\|_2^2$.
Furthermore, if we want a quantitative regularising estimate for short times à la Hérau[11], which corresponds to replace (α, β, γ) by $(\alpha t, \beta t^2, \gamma t^3)$, it does not work. Why ?

[10] Villani, 2009 [Vil09]
[11] Hérau, *JFA* 2007 [Hé07]
[12] Baudoin, Tardif, *KRM* 2018 [BT18]

The trouble is the sphere — but there is a nice algebraic framework

We want to write our equation as $\partial_t f + \mathsf{T} f = \mathsf{A}^2 f$.

Fancy decomposition of the Laplace-Beltrami on the sphere

Write $\mathsf{A}_{i,j} = v \wedge \nabla_v$ (in coordinates where $v = \cos\theta w + \sin\theta(\cos\varphi_{i,j} e_i + \sin\varphi_{i,j} e_j)$ with $w \in \mathbb{S}, w \perp e_i, w \perp e_j$, it reads $\mathsf{A}_{i,j} = \partial_{\varphi_{i,j}}$). Then, writing $\mathsf{A}^2 = \sum_{i<j} \mathsf{A}_{i,j}^2$:

- $\mathsf{A}_{i,j}$ is antiselfadjoint on $\mathbb{S}$, commutes with A^2, and $\mathsf{A}_{i,j} v_k = \delta_{jk} v_i - \delta_{ik} v_j$.
- If $f, g \in C^1(\mathbb{S})$, then $\nabla_v f \cdot \nabla_v g = \sum_{i<j} \mathsf{A}_{i,j} f \mathsf{A}_{i,j} g$. Consequently $\Delta_v f = \mathsf{A}^2 f$.

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Evolution of quadratic quantities and commutators

Write $\mathrm{T} = v \cdot \nabla_x$. If X is a smooth differential operator and $Q_{\mathrm{X}} = \int_{\mathbb{R}^d \times \mathbb{S}} f\, \mathrm{X} f \,\mathrm{d}x\mathrm{d}v$, then $\frac{\mathrm{d}}{\mathrm{d}t} Q_{\mathrm{X}} = Q_{\Phi(\mathrm{X})}$, where the operator $\Phi(\mathrm{X})$ goes as follows:

$$\Phi(\mathrm{X}) = \mathrm{A}^2\mathrm{X} + \mathrm{X}\mathrm{A}^2 + [\mathrm{T}, \mathrm{X}] = 2\mathrm{AXA} + [\mathrm{A}[\mathrm{A}, \mathrm{X}]] + [\mathrm{T}, \mathrm{X}].$$

Villani's chain of commutators : start from $\mathrm{C}_0 = A$ and then $\mathrm{C}_{i+1} = [\mathrm{C}_i, \mathrm{T}]$, hoping to get all the missing "directions". Here it stops at $\mathrm{C}_1 = [\mathrm{A}, \mathrm{T}] := \mathrm{S} = v \wedge \nabla_x$, since then $[\mathrm{T}, \mathrm{S}] = 0$.

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$\rightarrow$ Hörmander theory : commute as you can ! We get $[\mathrm{A}, \mathrm{S}] = -(d-1)\mathrm{T}$. And we are happy since $\mathrm{T}^2 + \mathrm{S}^2 = \Delta_x$.

Good hypoellipticity functional à la Hérau

Theorem : a good H^1 energy for short-time estimates.

Set $\mathcal{F}_0 = Q_{\mathsf{Id}}$, $\mathcal{F}_1 = \alpha\tau Q_{-\mathsf{A}^2} + \beta\tau^2 Q_{-\mathsf{SA}-\mathsf{AS}} + \gamma\tau^3 Q_{-\mathsf{S}^2} + \delta\tau^4 Q_{-\mathsf{T}^2}$.
Then there exists coefficients $\alpha, \beta, \gamma, \delta$ such that $\mathcal{F}(t) = \mathcal{F}_0 + \mathcal{F}_1(t, f)$ is decreasing in time.

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Main tools

Writing $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\tau, f) = -2\mathcal{D}(\tau, f) + \mathcal{B}(\tau, f) \leqslant -(2-\varepsilon)\mathcal{D}(\tau, f)$.
Getting $\partial_\tau\mathcal{F}_1(\tau, f(t)) \leqslant \varepsilon\mathcal{D}(\tau, f)$.
A lower bound for $\mathcal{D}(\tau, f)$ as a (squared) Sobolev seminorm, but with the good weights :

$$\mathcal{D}(\tau, f) \gtrsim \tau\|\Delta_v f\|_2^2 + \tau^2\|P_{v^\perp}\nabla_x f\|_2^2 + \tau^3\|(v\cdot\nabla_x) f\|_2^2.$$

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Trick for the last point: $\mathrm{STA} - \mathrm{ATS} = (d-1)\mathrm{T}^2 - \mathrm{S}^2$, giving

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Then $\mathcal{F}(t) = \mathcal{F}_0 + \mathcal{F}_1(f)$ is decreasing in time.
Furthermore, on $\mathbb{T}$, this quantity is controlled by its dissipation if f has mean zero, and is equivalent to the H^1 norm of f. We therefore have existence of $C \geqslant 1$ and $\lambda > 0$ such that

$$\|f\|_{H^1} \leqslant C\|f_0\|_{H^1} e^{-\lambda t}.$$

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Hypocoercivity for large times

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On $\mathbb{R}^d$, the L^1 norm is decreasing (constant if $f \geqslant 0$). Therefore we apply Nash inequality instead of Poincaré : $\|f\|_2^{2+\frac{4}{d}} \lesssim \|f\|_1^{\frac{4}{d}} \|\nabla_x f\|_2^2$. This leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} \leqslant -C\mathcal{F}^{1+\frac{2}{d}}.$$

Improvements in view of the nonlinear equation

Remember $\|f\|_{H^1}^2 \leqslant \frac{1}{t^4}\|f_0\|_{L^2}$. But actually $Q_{-\mathbb{S}^2} = \|Sf\|^2 = \|P_{v^\perp}\nabla_x f\|_2^2 \lesssim \frac{1}{t^3}\|f_0\|_{L^2}$.

Moments in v are a little bit better in short time

$$\|\nabla_x \mathbb{J}[f]\|_2^2 \lesssim \begin{cases} \|Sg\|_2^2 + \|Sg\|_2\|\nabla_x g\|_2 & \text{if } d = 2, \\ \|Sg\|_2^2 + \|Sg\|_2^{2-\varepsilon}\|\nabla_x g\|_2^{\varepsilon} & \text{if } d = 3 \quad (\text{for all } \varepsilon \in (0,1)), \\ \|Sg\|_2^2 & \text{if } d \geqslant 4. \end{cases}$$

This allows to obtain $\|\nabla_x \mathbb{J}[f]\|_2^2 \lesssim t^{-\frac{7}{2}}$ for $d = 2$ or $t^{-3+\varepsilon}$ for $d \geqslant 3$.

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Starting with $\int_{\mathbb{R}^d} |x|^2 f_0^2 < +\infty$ instead of L^1 on $\mathbb{R}^d$ for long time

We have $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |x|^2 f^2 \leqslant \frac{1}{(d-1)^2}\mathcal{F}$.

Use Heisenberg instead of Nash : $\|f\|_2^4 \leqslant \frac{4}{d^2}\|\nabla_x f\|_2^2 \int |x|^2 f^2$.

There exists $\nu > 0$ such that $\mathcal{F} \leqslant \mathcal{F}_0(1 + C_0 t)^{-\nu}$. Only C_0 depends on $\int_{\mathbb{R}^d} |x|^2 f_0^2 < +\infty$ (and of $\mathcal{F}_0$).

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The method is to write $U(t) = \int_{\mathbb{R}^d} |x|^2 f_0^2 + \mathcal{F}_0 + \int_0^t \mathcal{F}(s)\mathrm{d}s$. We obtain a second order differential inequality : $U'' \leqslant -\eta \frac{(U')^2}{U}$, that can be solved exactly.

Back to our model, another nice algebraic view

A new operator (not differential)

We define $\mathrm{U}f = d\, v \wedge \mathbb{J}[f]$, meaning $\mathrm{U}_{i,j} f = d(v_i\, e_j \cdot \mathbb{J}[f] - v_j\, e_i \cdot \mathbb{J}[f])$. We have

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We then get

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A new operator (not differential)

We define $\mathrm{U}f = d\, v \wedge \mathbb{J}[f]$, meaning $\mathrm{U}_{i,j}f = d(v_i\, e_j \cdot \mathbb{J}[f] - v_j\, e_i \cdot \mathbb{J}[f])$. We have

- $\mathrm{U}^2 f = -(d-1)d\, v \cdot \mathbb{J}[f]$
- $\mathrm{UA} = \mathrm{AU} = \mathrm{U}^2$, and therefore $\mathrm{A}^2 - \mathrm{U}^2 = (\mathrm{A} - \mathrm{U})^2$.
- $\mathrm{AU}^2 = \mathrm{U}^2\mathrm{A} = \mathrm{A}^2\mathrm{U} = \mathrm{UA}^2 = \mathrm{U}^3 = -(d-1)\mathrm{U}$.
- $\mathbb{J}[f] \cdot \nabla_{\mathrm{v}} f = \frac{1}{d} \sum_{i<j} \mathrm{U}_{i,j} f\, \mathrm{A}_{i,j} f$.

We then get

$$\partial_t f + \mathrm{T}f + \frac{1}{d}(\mathrm{A}(f\mathrm{U}f)) = \mathrm{A}^2 f.$$

Same quadratic quantities involved, new terms in the dissipation.

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_{\mathrm{X}} = Q_{\Phi(\mathrm{X})} + R_{\mathrm{X}},$$

where the non-linear term produces

$$R_{\mathrm{X}}(f) = \frac{2}{d} \int_{\mathbb{R}^d \times \mathbb{S}} (\mathrm{AX}f)\,(\mathrm{U}f)\, f\, v \mathrm{d}x.$$

Back to our model, perturbation of a constant state

We write $f = \rho + g$, with g supposed small.

Higher order functional (main tool to prove our result).

Setting $\mathcal{F}_{s,0}(g) = \mathcal{F}_0((-\Delta_x)^{\frac{s}{2}} g)$ and $\mathcal{F}_{s,1}(\tau, g) = \mathcal{F}_1(\tau, (-\Delta_x)^{\frac{s}{2}} g)$, we write

$$\mathcal{F}(\tau, g) = \mathcal{F}_0(g) + \mathcal{F}_{s,0}(g) + \mathcal{F}_1(\tau, g) + \mathcal{F}_{s,1}(\tau, g) + \delta\tau^4 \mathcal{F}_{s+1,1}(\tau, g).$$

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where the dissipation term $\mathcal{D}_1$ is the same, but the dissipation term $\mathcal{D}_0$ has been modified into $\mathcal{D}_0^{\rho}(\cdot) = Q_{-\mathsf{A}^2 + \frac{\rho}{d}\mathsf{U}^2}$.

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Control of the cubic terms : a little bit more painful. We write $m = s + \theta$ such that $m > \frac{d}{2}$ (so $H^m(\mathbb{R}^d)$ is an algebra). Then some terms (for $d \geqslant 3$) are controlled by $\tau^{-2\theta - \frac{1}{2}} \mathcal{F}^{\frac{3}{2}}$, the others essentially by $\tau^{\frac{1}{2} - 2\theta} \sqrt{\mathcal{F}} \mathcal{D}$. If $\theta < \frac{1}{4}$, we obtain differential inequalities that imply stability close to 0 for $\mathcal{F}$. Note that this procedure also work for proving local well-posedness in this space $H^{s,0}$.

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In long time, the procedure is similar. On $\mathbb{R}^d$, we do not have control of L^1 norm of the perturbation, therefore we use the other strategy with Heisenberg inequality.

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Thanks