

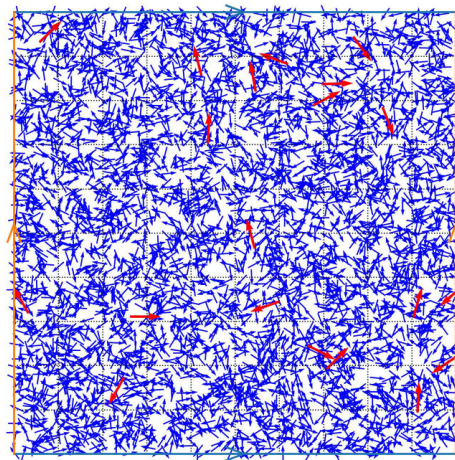
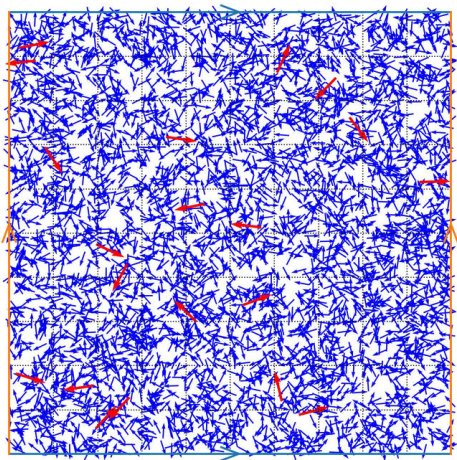
Hypo(coerciv-elliptic)ity in a model of aligning self-propelled particles

Amic Frouvelle – CEREMADE – Université Paris Dauphine PSL

Work in collaboration with Emeric Bouin (CEREMADE)

SPIKE Seminar,
IHP, March 10th 2026

Motivation : aligning self-propelled particles ^{1,2,3}



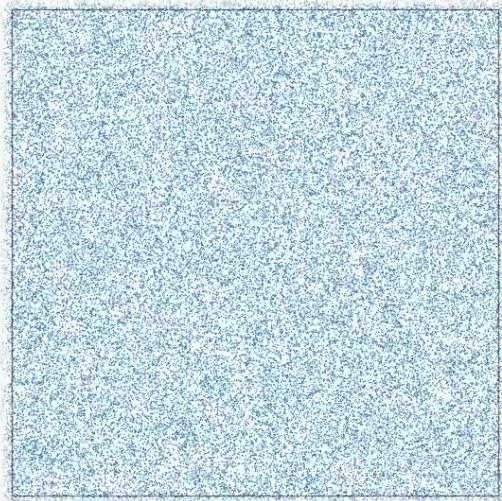
¹Vicsek *et al.*, *Phys. Rev. Lett.*, 1995 [VCBJ+95]

²Degond, Motsch, *M3AS*, 2008 [DM08]

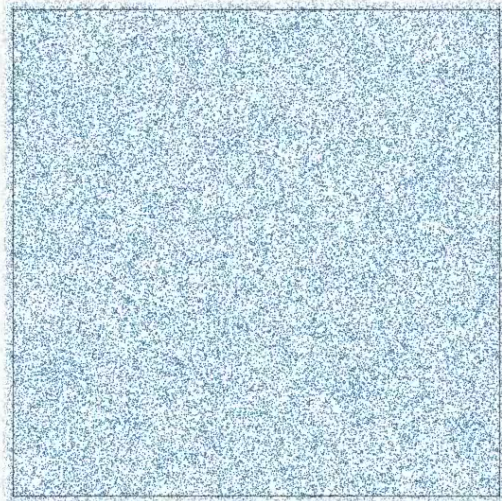
³Degond, F, Liu, *J. Nonlin. Sci.*, 2013 [DFL13]

Much more particles, larger domain (with GPU ⁴)

time=0.39



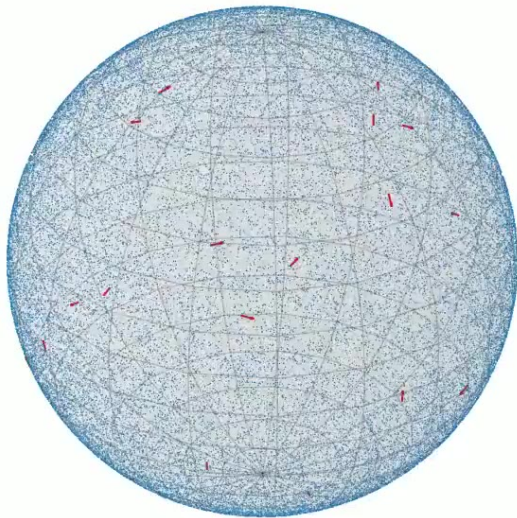
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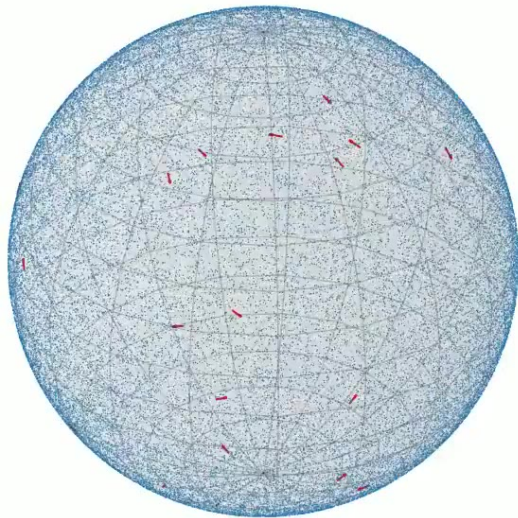
⁴Diez, F, Motsch, *in progress*

Other geometries ⁴

time=0.40

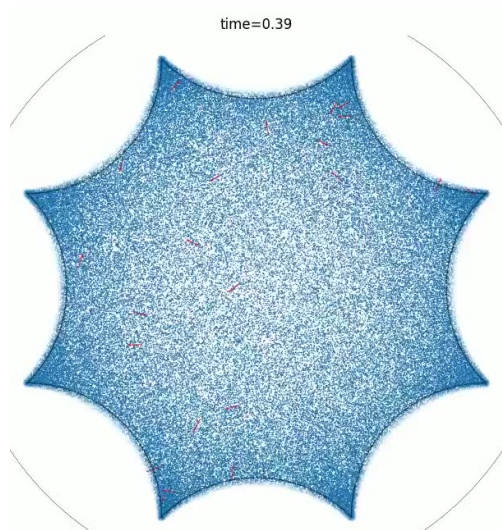
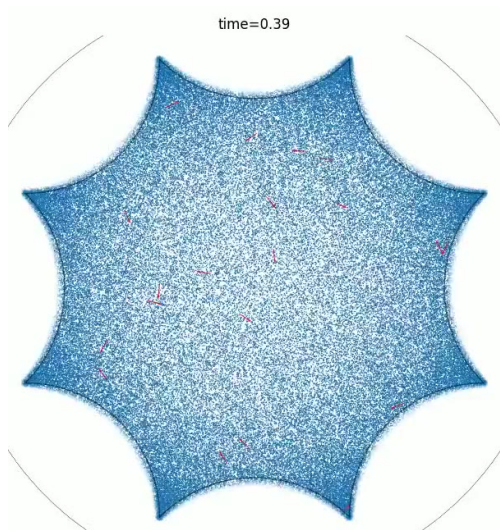


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⁴Diez, F, Motsch, *in progress*

Other (strange) geometries ⁴



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The self-propelled particles model

System of coupled SDEs

Particles at positions $X_k = \mathbb{R}^d$ (or a flat torus \mathbb{T}), speeds $V_k \in \mathbb{S}$ (unit sphere), $1 \leq k \leq N$.

$$\begin{cases} dX_k = c V_k dt \\ dV_k = - \sum_{j=1}^N \nu_{j,k} \frac{1}{2} \nabla_{V_j} \|V_j - V_k\|^2 dt + \sqrt{2\sigma} P_{V_k^\perp} \circ dB_{t,k}. \end{cases}$$

Careful : gradients on the sphere (for instance $\nabla_v(\mathbf{u} \cdot v) = P_{v^\perp} \mathbf{u}$), Laplace Beltrami, Stratonovich formulation for brownian motion on the sphere.

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Empirical distribution $f^N = \frac{\rho}{N} \sum_j \delta_{X_j} \otimes \delta_{V_j}$.

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where for a measure f , its first moment (in v) is denoted $\mathbb{J}[f] = \int_{\mathbb{S}} v f(v) dv$.

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Parameters : N, K, \mathbb{T}, ρ (hidden in f^N).

The mean-field or moderate interaction limit

Mean-field limit ⁵ : convergence of f^N to a density f , solution of a kinetic equation

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (P_{v^\perp} (K *_x \mathbb{J}[f])f) = \Delta_v f.$$

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6

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How to get rid of K ? Use $\frac{1}{\varepsilon_N^d} K(\frac{\cdot}{\varepsilon_N})$ instead, with $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$ (but $\varepsilon_N^d N \rightarrow \infty$).

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Moderate interaction limit expected if the limit kinetic equation is well posed ⁶

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→ Only remaining parameters : the shape of \mathbb{T} , and ρ (hidden in f).

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Theorem ⁷: (local in time) existence and uniqueness, initial condition f_0 in $L^\infty(\mathbb{R}^d \times \mathbb{S})$.

There exists a unique weak solution in $\mathcal{C}([0, T], L^\infty(\mathbb{R}^d \times \mathbb{S}))$ for all $T < \frac{\sqrt{d}}{(d-1)\|f_0\|_\infty}$. It is nonnegative and satisfies (maximum principle + finite speed propagation), for all t, x :

$$\|f(t, x, \cdot)\|_\infty \leq \|f_0\|_{\infty, B(x, t) \times \mathbb{S}} + (d-1) \int_0^t \|\mathbb{J}[f](s)\|_{\infty, B(x, s-t)} \|f(s)\|_{\infty, B(x, s-t) \times \mathbb{S}} ds.$$

⁵Bolley, Cañizo, Carrillo, *Appl. Math. Lett.* 2012 [BCC12]

⁶Chaintron, Diez, *Kinet. Relat. Models*, 2022 [CD22]

⁷adapted from : Briant, Meunier, *Kinet. Relat. Models*, 2023 [BM23]

Fokker–Planck formulation via von Mises distributions, free energy

Define the von Mises distribution $M_J(v) = \frac{e^{v \cdot J}}{\int_{\mathbb{S}} e^{v' \cdot J} dv'}$, then

$$\partial_t f = -\nabla_v \cdot (P_{v^\perp} \mathbb{J}[f] f) + \Delta_v f = \nabla_v \cdot \left(M_{\mathbb{J}[f]} \nabla_v \left(\frac{f}{M_{\mathbb{J}[f]}} \right) \right) = \nabla_v \cdot (f \nabla_v (\ln f - v \cdot \mathbb{J}[f])).$$

Dissipation of the free energy $\mathcal{F}[f] = \int_{\mathbb{S}} f \ln f - \frac{1}{2} |\mathbb{J}[f]|^2$: Fisher information (w.r.t $\rho M_{\mathbb{J}[f]}$).

$$\frac{d}{dt} \mathcal{F} = -\mathcal{D} = - \int_{\mathbb{S}} |\nabla_v (\ln f - v \cdot \mathbb{J}[f])|^2 f dv = -\mathcal{I}(f | \rho M_{\mathbb{J}[f]}).$$

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$\mathcal{D}[f] = 0 \Leftrightarrow$ critical point of \mathcal{F} under mass $\rho \Leftrightarrow f = \rho M_J$, with $\mathbb{J}[\rho M_J] = \rho \langle v \rangle_{M_J} = J$.

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The space-homogeneous setting : phase transition ⁸

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Compatibility equation : $J = \kappa \Omega$ with $\Omega \in \mathbb{S}$ and $\kappa = \rho c(\kappa)$ for $c(\kappa) = \frac{\int_0^\pi \cos \theta e^{\kappa \cos \theta} \sin^{d-2} \theta d\theta}{\int_0^\pi e^{\kappa \cos \theta} \sin^{d-2} \theta d\theta}$.

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Behaviour : $\frac{\kappa}{c(\kappa)} \nearrow +\infty$ as $\kappa \rightarrow +\infty$, and $\searrow \rho_c = d$ as $\kappa \rightarrow 0$.

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Our main goal : around isotropic state, $\rho < \rho_c = d$

Stability/instability for the space-homogeneous model

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Main result (spoiler) : it is stable ! In $H^{s,1}$ norm (s derivatives in x , one in v).

Assume that $s > 7/8$ if $d = 2$ or $s > \frac{d}{2} - \frac{1}{4}$ if $d \geq 3$. If $g_0 \in H^{s,0}(\mathbb{R}^d \times \mathbb{S})$ is small, the solution is global. There exists an energy, equivalent (for $t \geq 1$) to the $H^{s,1}(\mathbb{R}^d \times \mathbb{S})$ norm of g , that decays in time. On \mathbb{T} , the energy is exponentially decreasing.

On \mathbb{R}^d , if $\int_{\mathbb{R}^d} |x|^2 g_0^2 dx < +\infty$, the energy decreases at least as $t^{-\nu}$ (for some small ν).

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- Related work⁹ for BGK instead of Fokker-Planck : $\partial_t f + v \cdot \nabla_x f = \rho_f M_{\mathbb{J}[f]} - f$.

⁹Merino-Aceituno, Schmeiser, Winter *ArXiv* 2024 [MASW24]

Back to basics : one particle (Velocity Spherical Brownian Motion¹²)

A single self-propelled particle exploring around, no interaction.

$$X \in \mathbb{R}^d \text{ (or } \mathbb{T}), V \in \mathbb{S}, \begin{cases} dX = V dt \\ dV = \sqrt{2} P_{v^\perp} \circ dB_t. \end{cases}$$

Its law satisfies $\partial_t f + v \cdot \nabla_x f = \Delta_v f$.

10

11

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Looks simple, can't we do à la Villani¹⁰ ? Define an energy equivalent to the square of the H^1 norm (if $\beta < \alpha\gamma$) : $\mathcal{F} = \|f\|_2^2 + \alpha\|\nabla_v f\|_2^2 + 2\beta\langle\nabla_v f, \nabla_x f\rangle + \gamma\|\nabla_x f\|_2^2$, and then get

$$\begin{aligned} \frac{d\mathcal{F}}{dt} = & -2\|\nabla_v f\|_2^2 + 2\alpha\langle\nabla_v f, \nabla_v(\Delta_v f)\rangle + 2\beta\|P_{v^\perp} \nabla_x f\|_2^2 + 2\gamma\langle\nabla_x f, \nabla_x(\Delta_v f)\rangle \text{ (good terms)} \\ & -2\alpha\langle\nabla_v f, \nabla_x f\rangle - 2\beta[(d-1)\langle\nabla_v f, \nabla_x f\rangle - 2\langle\nabla_v f, \nabla_x(\Delta_v f)\rangle]. \text{ (bad terms)} \end{aligned}$$

Trouble or not ?

¹⁰Villani, 2009 [Vil09]

¹¹

¹²Baudoin, Tardif, *KRM* 2018 [BT18]

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Looks simple, can't we do à la Villani¹⁰ ? Define an energy equivalent to the square of the H^1 norm (if $\beta < \alpha\gamma$) : $\mathcal{F} = \|f\|_2^2 + \alpha\|\nabla_v f\|_2^2 + 2\beta\langle\nabla_v f, \nabla_x f\rangle + \gamma\|\nabla_x f\|_2^2$, and then get

$$\begin{aligned} \frac{d\mathcal{F}}{dt} = & -2\|\nabla_v f\|_2^2 + 2\alpha\langle\nabla_v f, \nabla_v(\Delta_v f)\rangle + 2\beta\|P_{v^\perp} \nabla_x f\|_2^2 + 2\gamma\langle\nabla_x f, \nabla_x(\Delta_v f)\rangle \text{ (good terms)} \\ & -2\alpha\langle\nabla_v f, \nabla_x f\rangle - 2\beta[(d-1)\langle\nabla_v f, \nabla_x f\rangle - 2\langle\nabla_v f, \nabla_x(\Delta_v f)\rangle]. \text{ (bad terms)} \end{aligned}$$

Trouble or not ? Good terms are indeed equivalent to H^2 norm (for a mean-zero function f), but that's not trivial to recover the missing $\|v \cdot \nabla_x f\|_2^2$.

¹⁰Villani, 2009 [Vil09]

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¹²Baudoin, Tardif, *KRM* 2018 [BT18]

Back to basics : one particle (Velocity Spherical Brownian Motion¹²)

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Furthermore, if we want a quantitative regularising estimate for short times à la Hérau¹¹, which corresponds to replace (α, β, γ) by $(\alpha t, \beta t^2, \gamma t^3)$, it does not work. Why ?

¹⁰Villani, 2009 [Vil09]

¹¹Hérau, *JFA* 2007 [Hé07]

¹²Baudoin, Tardif, *KRM* 2018 [BT18]

The trouble is the sphere — but there is a nice algebraic framework

We want to write our equation as $\partial_t f + \mathbb{T}f = A^2 f$.

Fancy decomposition of the Laplace-Beltrami on the sphere

Write $A_{i,j} = v \wedge \nabla_v$ (in coordinates where $v = \cos \theta w + \sin \theta (\cos \varphi_{i,j} e_i + \sin \varphi_{i,j} e_j)$ with $w \in \mathbb{S}, w \perp e_i, w \perp e_j$, it reads $A_{i,j} = \partial_{\varphi_{i,j}}$). Then, writing $A^2 = \sum_{i < j} A_{i,j}^2$:

- $A_{i,j}$ is antiselfadjoint on \mathbb{S} , commutes with A^2 , and $A_{i,j} v_k = \delta_{jk} v_i - \delta_{ik} v_j$.
- If $f, g \in C^1(\mathbb{S})$, then $\nabla_v f \cdot \nabla_v g = \sum_{i < j} A_{i,j} f A_{i,j} g$. Consequently $\Delta_v f = A^2 f$.

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Evolution of quadratic quantities and commutators

Write $T = v \cdot \nabla_x$. If X is a smooth differential operator and $Q_X = \int_{\mathbb{R}^d \times \mathbb{S}} f X f \, dx dv$, then $\frac{d}{dt} Q_X = Q_{\Phi(X)}$, where the operator $\Phi(X)$ goes as follows:

$$\Phi(X) = A^2 X + X A^2 + [T, X] = 2AXA + [A, X] + [T, X].$$

Villani's chain of commutators : start from $C_0 = A$ and then $C_{i+1} = [C_i, T]$, hoping to get all the missing "directions". Here it stops at $C_1 = [A, T] := S = v \wedge \nabla_x$, since then $[T, S] = 0$.

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→ Hörmander theory : commute as you can ! We get $[A, S] = -(d-1)T$. And we are happy since $T^2 + S^2 = \Delta_x$.

Theorem : a good H^1 energy for short-time estimates.

Set $\mathcal{F}_0 = Q_{\text{Id}}$, $\mathcal{F}_1 = \alpha\tau Q_{-A^2} + \beta\tau^2 Q_{-SA-AS} + \gamma\tau^3 Q_{-S^2} + \delta\tau^4 Q_{-T^2}$.

Then there exists coefficients $\alpha, \beta, \gamma, \delta$ such that $\mathcal{F}(t) = \mathcal{F}_0 + \mathcal{F}_1(t, f)$ is decreasing in time.

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Note that $\mathcal{F}_1(\tau, f) = \alpha\tau \|\nabla_v f\|_2^2 + 2\beta\tau^2 \langle \nabla_v f | \nabla_x f \rangle + \gamma\tau^3 \|P_{v^\perp} \nabla_x f\|_2^2 + \delta\tau^4 \|v \cdot \nabla_x f\|_2^2$,
and $\mathcal{F}_0 = \|\cdot\|_2^2$. We then get that the H^1 norm is controlled by $\frac{1}{t^2} \|f_0\|_2$ for short times (to compare with $t^{-\frac{3}{2}}$ for usual kinetic Fokker-Planck equations).

Good hypoellipticity functional à la Hérau

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Main tools

Writing $\frac{d}{dt} \mathcal{F}(\tau, f) = -2\mathcal{D}(\tau, f) + \mathcal{B}(\tau, f) \leq -(2 - \varepsilon)\mathcal{D}(\tau, f)$.

Getting $\partial_\tau \mathcal{F}_1(\tau, f(t)) \leq \varepsilon \mathcal{D}(\tau, f)$.

A lower bound for $\mathcal{D}(\tau, f)$ as a (squared) Sobolev seminorm, but with the good weights :

$$\mathcal{D}(\tau, f) \gtrsim \tau \|\Delta_v f\|_2^2 + \tau^2 \|P_{v^\perp} \nabla_x f\|_2^2 + \tau^3 \|(v \cdot \nabla_x) f\|_2^2.$$

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Trick for the last point: $STA - ATS = (d - 1)T^2 - S^2$, giving

$$\|v \cdot \nabla_x f\|_2^2 \leq \frac{1}{d-1} \|P_{v^\perp} \nabla_x f\|_2^2 + 2 \|P_{v^\perp} \nabla_x f\|_2 \|\nabla_v(v \cdot \nabla_x f)\|_2.$$

Theorem : a good H^1 energy for large time estimates.

Set $\mathcal{F}_0 = Q_{\text{Id}}$, $\mathcal{F}_1 = \alpha Q_{-A^2} + \beta Q_{-SA-AS} + \gamma Q_{-S^2} + \delta Q_{-T^2}$, for the same coefficients $\alpha, \beta, \gamma, \delta$. Then $\mathcal{F}(t) = \mathcal{F}_0 + \mathcal{F}_1(f)$ is decreasing in time.

Furthermore, on \mathbb{T} , this quantity is controlled by its dissipation if f has mean zero, and is equivalent to the H^1 norm of f . We therefore have existence of $C \geq 1$ and $\lambda > 0$ such that

$$\|f\|_{H^1} \leq C \|f_0\|_{H^1} e^{-\lambda t}.$$

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On \mathbb{R}^d , the L^1 norm is decreasing (constant if $f \geq 0$). Therefore we apply Nash inequality instead of Poincaré : $\|f\|_2^{2+\frac{4}{d}} \lesssim \|f\|_1^{\frac{4}{d}} \|\nabla_x f\|_2^2$. This leads to

$$\frac{d}{dt} \mathcal{F} \leq -C \mathcal{F}^{1+\frac{2}{d}}.$$

Improvements in view of the nonlinear equation

Remember $\|f\|_{H^1}^2 \lesssim \frac{1}{t^4} \|f_0\|_{L^2}^2$ (t small). But actually $Q_{-S^2} = \|Sf\|^2 = \|P_{v^\perp} \nabla_x f\|_2^2 \lesssim \frac{1}{t^3} \|f_0\|_{L^2}^2$.

Moments in v are a little bit better in short time

$$\|\nabla_x \mathbb{J}[f]\|_2^2 \lesssim \begin{cases} \|Sg\|_2^2 + \|Sg\|_2 \|\nabla_x g\|_2 & \text{if } d = 2, \\ \|Sg\|_2^2 + \|Sg\|_2^{2-\varepsilon} \|\nabla_x g\|_2^\varepsilon & \text{if } d = 3 \quad (\text{for all } \varepsilon \in (0, 1)), \\ \|Sg\|_2^2 & \text{if } d \geq 4. \end{cases}$$

This allows to obtain $\|\nabla_x \mathbb{J}[f]\|_2^2 \lesssim t^{-\frac{7}{2}}$ for $d = 2$ or $t^{-3+\frac{\varepsilon}{2}}$ for $d \geq 3$.

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Starting with $\int_{\mathbb{R}^d} |x|^2 f_0^2 < +\infty$ instead of L^1 on \mathbb{R}^d for long time

We have $\frac{d}{dt} \int_{\mathbb{R}^d} |x|^2 f^2 \leq \frac{1}{(d-1)^2} \mathcal{F}$.

Use Heisenberg instead of Nash : $\|f\|_2^4 \leq \frac{4}{d^2} \|\nabla_x f\|_2^2 \int |x|^2 f^2$.

There exists $\nu > 0$ such that $\mathcal{F} \leq \mathcal{F}_0 (1 + C_0 t)^{-\nu}$. Only C_0 depends on $\int_{\mathbb{R}^d} |x|^2 f_0^2 < +\infty$ (and of \mathcal{F}_0).

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The method is to write $U(t) = \int_{\mathbb{R}^d} |x|^2 f_0^2 + \mathcal{F}(0) + \int_0^t \mathcal{F}(s) ds$. We obtain a second order differential inequality : $U'' \leq -\eta \frac{(U')^2}{U}$, that can be solved exactly.

Back to our model, another nice algebraic view

A new operator (not differential)

We define $Uf = d v \wedge \mathbb{J}[f]$, meaning $U_{i,j}f = d(v_i e_j \cdot \mathbb{J}[f] - v_j e_i \cdot \mathbb{J}[f])$. We have

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Same quadratic quantities involved, new terms in the dissipation.

$$\frac{d}{dt} Q_X = Q_{\Phi(X)} + R_X,$$

where the non-linear term produces

$$R_X(f) = \frac{2}{d} \int_{\mathbb{R}^d \times \mathbb{S}} (AXf)(Uf) f v dx.$$

Back to our model, perturbation of a constant state

We write $f = \rho + g$, with g supposed small.

Higher order functional (main tool to prove our result).

Setting $\mathcal{F}_{s,0}(g) = \mathcal{F}_0((-\Delta_x)^{\frac{s}{2}}g)$ and $\mathcal{F}_{s,1}(\tau, g) = \mathcal{F}_1(\tau, (-\Delta_x)^{\frac{s}{2}}g)$, we write

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$$\mathcal{D}(\tau, g) = \mathcal{D}_0^\rho(g) + \mathcal{D}_{s,0}^\rho(g) + \mathcal{D}_1(\tau, g) + \mathcal{D}_{s,1}(\tau, g) + \delta\tau^4 \mathcal{D}_{s+1,1}(\tau, g),$$

where the dissipation term \mathcal{D}_1 is the same, but the dissipation term \mathcal{D}_0 has been modified into $\mathcal{D}_0^\rho(\cdot) = \mathcal{Q}_{-A^2 + \frac{\rho}{\gamma}U^2}$.

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where the dissipation term \mathcal{D}_1 is the same, but the dissipation term \mathcal{D}_0 has been modified into $\mathcal{D}_0^\rho(\cdot) = \mathcal{Q}_{-A^2 + \frac{\rho}{\gamma}U^2}$.

Control of the quadratic terms : exactly the same job !

Back to our model, perturbation of a constant state

We write $f = \rho + g$, with g supposed small.

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Control of the cubic terms : a little bit more painful. We write $m = s + \theta$ such that $m > \frac{d}{2}$ (so $H^m(\mathbb{R}^d)$ is an algebra). Then some terms (for $d \geq 3$) are controlled by $\tau^{-2\theta - \frac{1}{2}} \mathcal{F}^{\frac{3}{2}}$, the others essentially by $\tau^{\frac{1}{2} - 2\theta} \sqrt{\mathcal{F}} \mathcal{D}$. If $\theta < \frac{1}{4}$, we obtain differential inequalities that imply stability close to 0 for \mathcal{F} . Note that this procedure also work for proving local well-posedness in this space $H^{s,0}$.

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In long time, the procedure is similar. On \mathbb{R}^d , we do not have control of L^1 norm of the perturbation, therefore we use the other strategy with Heisenberg inequality.



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Thanks