Jumping in the middle, from alignment models to sexual population dynamics

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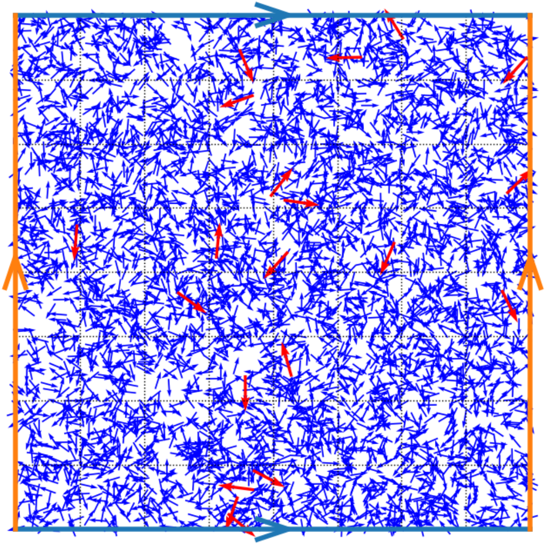
Based on works with Pierre Degond, Gaël Raoul, Cécile Taing

ABPDE 5 – Lille, June 9th 2023

Motivation : self-propelled particles, BDG model[1]



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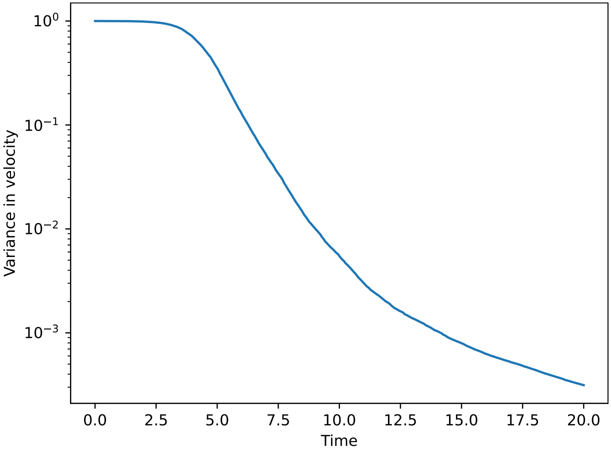

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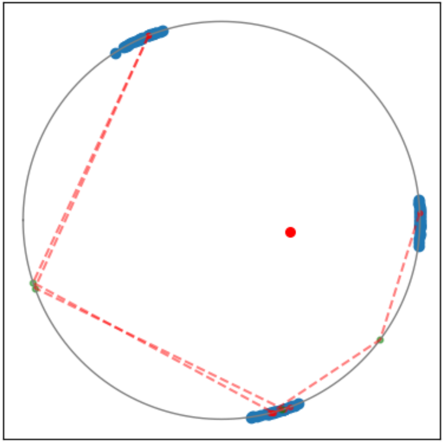
Homogeneous version : jump process on the circle



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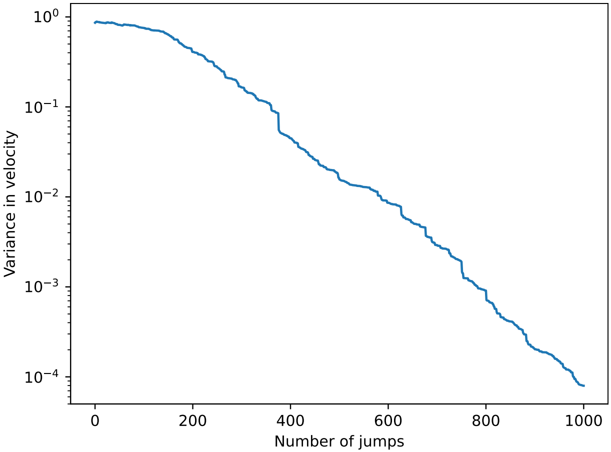
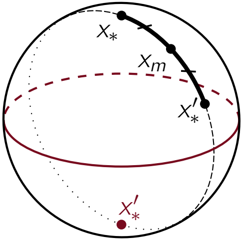


Kinetic level : midpoint model on the sphere[2]

Collision probability kernel $K(\cdot, x_*, x_*') = \delta_{x_m}$, where $x_m = \frac{x_*+x_*'}{\|x_*+x_*'\|}$:



Time evolution of a probability measure f

$$\partial_t f(t,x) = \int_{\mathbb{S}\times\mathbb{S}} K(x, x_*, x_*')\, f(t,x_*)\mathrm{d}x_* f(t,x_*')\mathrm{d}x_* - f(t,x).$$

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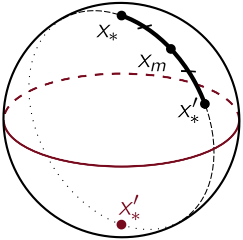

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Flat version in $\mathbb{R}^d$: sticky particles, economy-related models[3,4,5].

- Conservation of center of mass $\bar{x} = \int_{\mathbb{R}^d} x f(x)\mathrm{d}x$.
- Linear ODE for 2nd moment $m_2 = \int_{\mathbb{R}^d} |x-\bar{x}|^2 f(x)\mathrm{d}x$: $\frac{\mathrm{d}}{\mathrm{d}t} m_2 = -\frac{m_2}{2}$.
- Exponential convergence towards a Dirac mass : $W_2(f,\delta_{\bar{x}}) = W_2(f_0,\delta_{\bar{x}})e^{-\frac{t}{4}}$.

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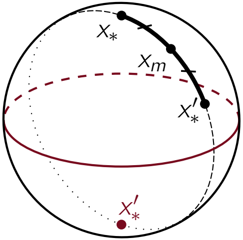

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How to get rid of $\bar{x}$? The energy $E(f) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 f(x)\mathrm{d}x\, f(y)\mathrm{d}y$ is actually $2m_2$!

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How to get rid of the non-conservation of the center of mass ?

- Energy: $E(f) = \int_{\mathbb{S}\times\mathbb{S}} d(x,y)^2\, f(x)\mathrm{d}x\, f(y)\mathrm{d}y$.

Useful Lemma : link “Energy – Wasserstein” (+ Markov inequalities around $\bar{x}$)

For $f \in \mathcal{P}(\mathbb{S})$, there exists $\bar{x} \in \mathbb{S}$ / for all $x \in \mathbb{S}$: $W_2(f,\delta_{\bar{x}})^2 \leqslant E(f) \leqslant 4\, W_2(f,\delta_x)^2$.
Furthermore, for all $\kappa > 0$ (and $\ell \in \{0,1\}$) : $\int_{\{x\in\mathbb{S};\, d(x,\bar{x})\geqslant\kappa\}} |d(x,\bar{x})|^{\ell} f(x)\mathrm{d}x \leqslant \frac{1}{\kappa^{2-\ell}} E(f)$.

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- Evolution of the energy : $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(f) = \int_{\mathbb{S}\times\mathbb{S}\times\mathbb{S}} \alpha(x_*, x_*', y)\, \mathrm{d}f(x_*)\, \mathrm{d}f(x_*')\, \mathrm{d}f(y)$.

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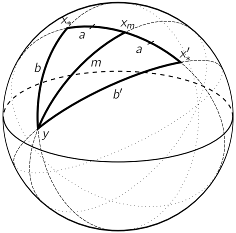

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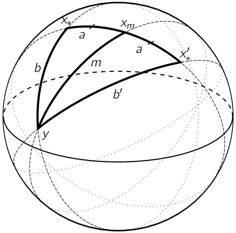

- Configuration of Apollonius: $\alpha(x_*,x_*',y) = m^2 - \frac{b^2+b'^2}{2}$.
- Global and local (if distances all less than κ) estimates :

$$\alpha(x_*,x_*',y) \leqslant -\frac{1}{4}d(x_*,x_*')^2 + \begin{cases} 2\,d(x_*,x_*')\min\big(d(x_*,y),\ d(x_*',y)\big) \\ C_1\,\kappa^2\,d(x_*,x_*')^2. \end{cases}$$

Nonlinear stability of Dirac masses

- Cutting the triple integral following a κ-neighborhood of $\bar{x}$:

$$\frac{1}{2}\frac{d}{dt}E(f) \leqslant -\frac{1}{4}E(f) + \underbrace{C\,\kappa^2\,E(f)}_{\text{Local lemma}} + \underbrace{12\frac{E(f)^{\frac{3}{2}}}{\kappa} + 24\frac{E(f)^2}{\kappa^2}}_{\substack{\text{Global lemma + Markov} \\ \text{(and Cauchy-Schwarz)}}}.$$

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Theorem: local stability of Dirac masses [DFR14]

There exists $C_1 > 0$ and $\eta > 0$ such that for all solution $f \in C(\mathbb{R}_+, \mathcal{P}(\mathbb{S}))$ with initial condition f_0 satisfying $W_2(f_0, \delta_{x_0}) < \eta$ for a $x_0 \in \mathbb{S}$, there exists $x_\infty \in \mathbb{S}$ such that

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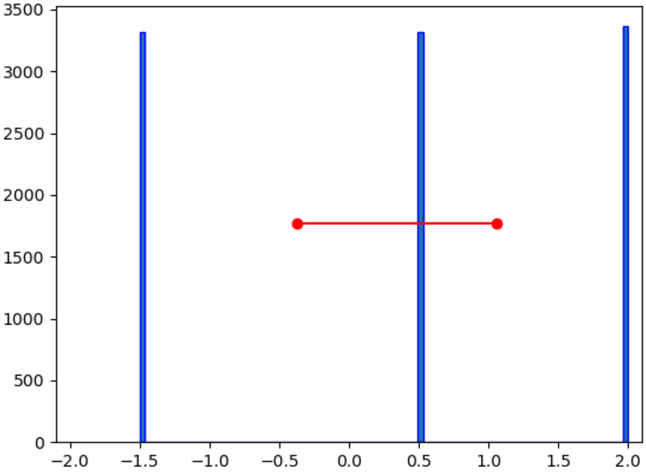
Open question : does any solution converge either to a Dirac mass or to the isotropic state (maybe on a lower-dimensional sphere) ?
Extensions : other manifolds, or when not exactly jumping in the middle (still contracting).

Back to a model on the line : population dynamics

- At rate N, reproduction : parents with traits x, x_*, newborn with trait $\frac{x+x_*}{2}$ replacing randomly another individual.
- At rate $m(x)$ an individual with trait x dies, (replaced by a random duplicate).



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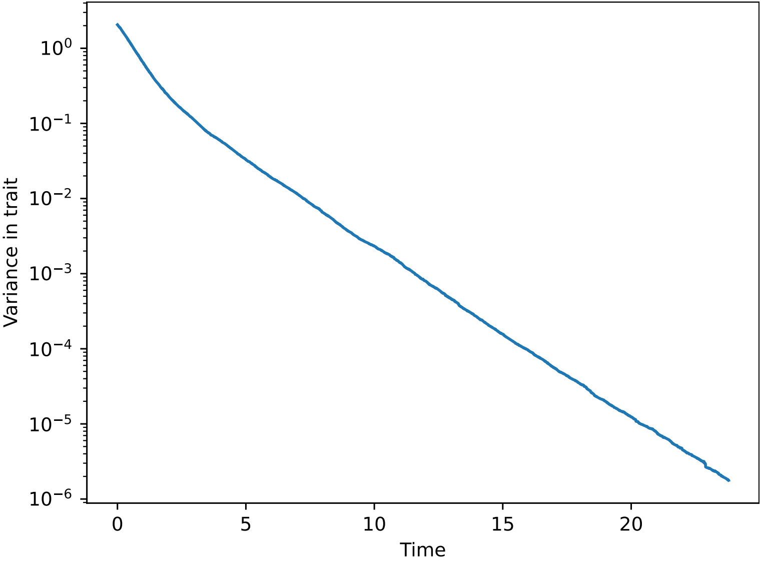
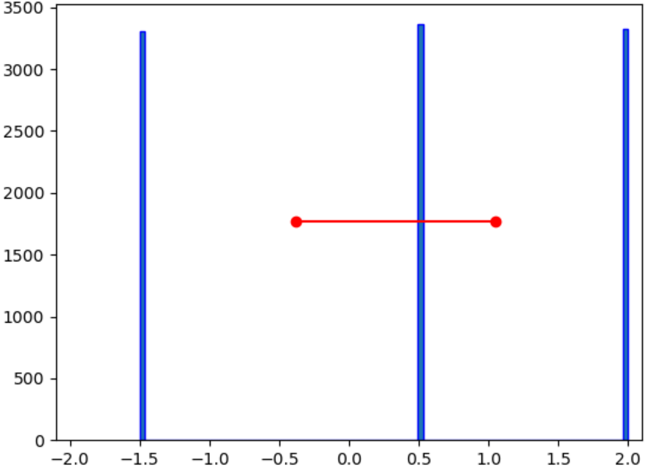


The self-similar shape of the distribution



A measure-valued evolution equation (Fisher infinitesimal model)

A sexual population model[7] (with Gaussian variability ε)

Population density $f(t,x)$ structured by trait $x \in \mathbb{R}$.

$$\begin{cases} \partial_t f(t,x) = B_\varepsilon[f(t,\cdot)](x) - m(x)f(t,x), \\ B_\varepsilon[f](x) := \iint_{\mathbb{R}^2} G_{\frac{z_1+z_2}{2},\varepsilon}(x) f(z_1) \frac{f(z_2)}{\int_{\mathbb{R}} f(z')\,\mathrm{d}z'}\,\mathrm{d}z_1\,\mathrm{d}z_2. \end{cases}$$

Our work[6] : no variability ($\varepsilon = 0$). Midpoint model with non-constant mortality (no mass conservation, neither center of mass).

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Existence and uniqueness of measure-valued solution [FT23]

If m is measurable and bounded below, unique weak solution to $\partial_t f = B_0[f] - mf$:

$$\int_A f(t,x)\mathrm{d}x = \int_A e^{-m(x)t} f^0(x)\mathrm{d}x + \int_0^t \int_A e^{-m(x)(t-s)} B_0[f(s,\cdot)](x)\mathrm{d}x\,\mathrm{d}s.$$

Based on a fixed point in total variation distance (B_0 is 3-Lipschitz).

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The constant mortality rate $m = 1$

- If $m = 1$ ($\varepsilon = 0$), known results[2].
- Conservation of mass and center of mass. We already saw $m_2(t) = m_2(0)e^{-\frac{1}{2}t}$: exponential convergence towards a Dirac mass.
- Up to rescaling, $f \rightsquigarrow \gamma$ is centered with unit mass and variance.
- Fourier particularly useful : $\widehat{B_0[\gamma]}(\xi) = \widehat{\gamma}(\frac{\xi}{2})^2$. Use of Fourier-adapted distance (for $s \in (2,3)$):

$$d_s(\gamma_1, \gamma_2) = \sup_{\xi \neq 0} \frac{|\widehat{\gamma_1}(\xi) - \widehat{\gamma_2}(\xi)|}{|\xi|^s}.$$

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$$d_s(\gamma_1, \gamma_2) = \sup_{\xi \neq 0} \frac{|\widehat{\gamma_1}(\xi) - \widehat{\gamma_2}(\xi)|}{|\xi|^s}.$$

The rescaled equation is contracting (B_0 is 2^{1-s}-Lipschitz for d_s, only $\frac{1}{2}$-Lipschitz for W_2)

$$\partial_t \widehat{\gamma} = \widehat{\gamma}(t, \tfrac{\xi}{2})^2 - \widehat{\gamma}(t,\xi) + \frac{1}{4}\xi \partial_\xi \widehat{\gamma}(t,\xi).$$

For $\lambda_s = 1 - \frac{s}{4} - 2^{1-s} (> 0)$, and two solutions : $d_s(\gamma_1(t), \gamma_2(t)) \leqslant d_s(\gamma_1(0), \gamma_2(0))e^{-\lambda_s t}$.

Actually an explicit (unique) steady-state $\gamma_\infty(x) = \frac{2}{\pi(1+x^2)^2}$ (heavy-tailed). Thus convergence towards this self-similar profile !

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Non-constant m : local stability of Dirac masses

Probability density $g = \frac{1}{\rho} f$, center of mass $\bar{x}(t)$.
Key quantity (positive if not too much mortality) : $\eta(\bar{x}) = \inf_{\mathbb{R}} m + \frac{1}{2} - m(x)$.

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Evolution of centered moments M_2 and M_4 of g

Assumption : m does not grow faster than quadratic + locally Lipschitz.
If $\eta(\bar{x}_0) > \delta_0 > 0$, and M_4^0 small, then :

$$\eta(\bar{x}(t)) \geqslant \delta_0, \quad M_2(t) \leqslant M_2^0 e^{-\delta_0 t}, \quad M_4(t) \leqslant M_4^0 e^{-\delta_0 t}.$$

Furthermore $\bar{x}(t)$ converges exponentially fast towards some $\bar{x}_\infty \in \mathbb{R}$. Thus if g is close to a Dirac mass $\bar{x}_0$ (for W_4), then it converges towards a Dirac mass located at some point $\bar{x}_0$.

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Unfortunately, not precise enough to obtain convergence towards a self-similar profile...

$$\frac{\mathrm{d}}{\mathrm{d}t} M_{2k} \leqslant \left(-1 + \tfrac{1}{2^{2k-1}} + \varepsilon \right) M_{2k} + \left(\tfrac{1}{2^{2k}} \sum_{\ell=2}^{2k-2} \binom{2k}{\ell} \right) M_2 M_{2k-2} + \begin{cases} C_\varepsilon M_{2k+2} \\ (\frac{1}{2} - \delta_0) M_{2k} \end{cases} .$$

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$$\frac{\mathrm{d}}{\mathrm{d}t} M_{2k} \leqslant \big(-1 + \tfrac{1}{2^{2k-1}} + \varepsilon\big) M_{2k} + \Big(\tfrac{1}{2^{2k}} \sum_{\ell=2}^{2k-2} \tbinom{2k}{\ell}\Big) M_2 M_{2k-2} + \begin{cases} C_\varepsilon M_{2k+2} \\ (\frac{1}{2} - \delta_0) M_{2k} \end{cases} .$$

Improvements on rates : if a high enough moment is initially finite, then M_{2k} has a rate nearly $\min(1 - \frac{1}{2^{2k-1}}, \frac{1}{2} + \delta_0)$.

Why we need more precise estimates on the moments

Perturbed equation : $\partial_t\widehat{\gamma}(t,\xi) = \widehat{\gamma}(t,\frac{\xi}{2})^2 - \widehat{\gamma}(t,\xi) + \frac{1}{4}\xi\partial_\xi\widehat{\gamma}(t,\xi) + R(t,\xi)$.

Proposition : control on the remainder R gives convergence to self-similarity.

Fix $s \in (2,3)$, set $\lambda_s = 1 - \frac{s}{4} - 2^{1-s}$ (positive by concavity).

Suppose there exist $K > 0$ and $c > 0$ with $|R(t,\xi)| \leqslant |\xi|^s K e^{-ct}$, $\forall \xi \in \mathbb{R}$.

Then, for all time t, $d_s(\gamma,\gamma_\infty)(t) \leqslant d_s(\gamma_0,\gamma_\infty)e^{-\lambda_s t} + K\frac{e^{-ct}-e^{-\lambda_s t}}{\lambda_s - c}$.

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Proposition : lower bound on M_2 if the profile is “shrinked”.

If $\frac{M_{2k_0}}{M_2}$ is initially small (with $\frac{1}{2^{2k_0-1}} < \delta_0$), then it decays with rate $(\delta_0 - \frac{1}{2^{2k_0-1}})$.

Furthermore, in that case, $M_2(t) \geqslant C_{k_0} M_2^0\, e^{-\frac{t}{2}}$ (with $C_{k_0} \to 1$ as $\frac{M_{2k_0}^0}{M_2^0} \to 0$).

Consequently, for all $k \geqslant 2$ (with $k \leqslant k_0$), $\frac{M_{2k}}{M_2}$ decays with rate $\min(\frac{1}{2} - \frac{1}{2^{2k-1}}, \delta_0 - \frac{1}{2^{2k_0-1}})$.

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Finally : exponential convergence towards the self-similar profile (same as for $m = 1$) in distance d_s, for $s \in]2,3[$ when $\delta_0 \geqslant \frac{1}{4}$, and for $s \in]2, 2 + \frac{\delta_0}{\frac{1}{2}-\delta_0}[$ if $\delta_0 \leqslant \frac{1}{4}$.

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