# Kinetic models for self-organized dynamics of alignment

# I : from particles to kinetic equations

Amic Frouvelle – CEREMADE – Université Paris Dauphine

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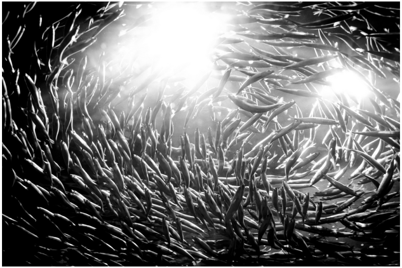
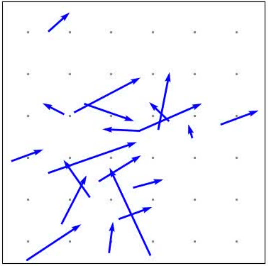


# First example: Cucker–Smale Model

$N$ particles, at positions $x_i \in \mathbb{R}^d$, aligning their velocities $v_i \in \mathbb{R}^d$.

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = v_i, \qquad \frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{1}{N}\sum_{j=1}^{N} K(x_j - x_i)(v_j - v_i).$$



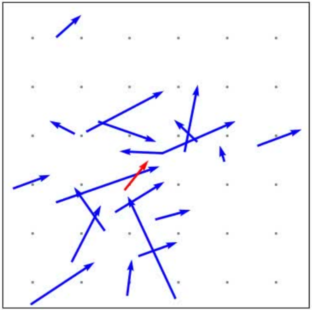
**Theorem : Cucker-Smale 2007 [CS07, HL09]**

**Flocking** if $\int_0^{+\infty} K(r)\mathrm{d}r = \infty$.
For any initial condition, there exists $C, \lambda, M > 0$ and $v_\infty \in \mathbb{R}^d$ with $\begin{cases} |v_i(t) - v_\infty| \leqslant Ce^{-\lambda t} \\ |x_i(t) - x_j(t)| \leqslant M \end{cases}$
(for all $i, j$ and $t \geqslant 0$).

# First example: Cucker–Smale Model

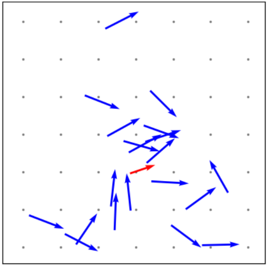
**Exercise : a strong observation kernel is needed**

For 2 particles, with $x_1 = x_2 = 0$ initially, prove that there exists some "observation kernel" $K > 0$ such that $|x_1(t) - x_2(t)| \to +\infty$.
Idea : reverse engineering (start by given trajectories $x_1(t)$ and $x_2(t)$ and find $K$).



$N$ particles, at positions $x_i \in \mathbb{R}^2$, with speed 1 and angle $\theta_i \in \mathbb{R}$.

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix} = \tau(\theta_i), \qquad \frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \frac{1}{N}\sum_{j=1}^{N} K(x_j - x_i)\sin(\theta_j - \theta_i).$$



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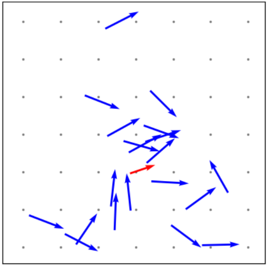

**Exercise : well-posed ODE in $\mathbb{R}^{3N}$**

This is an ODE of the form $\frac{\mathrm{d}Z}{\mathrm{d}t} = B(Z)$. If $K$ is bounded and Lipschitz, then $B$ is globally Lipschitz and Cauchy-Lipschitz theorem applies.

# Empirical measure, indistinguishable particles

**Definition : Empirical measure** $\mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i} \otimes \delta_{\theta_i}$

$$\forall \varphi \in C^b(\mathbb{R}^2 \times \mathbb{R}), \langle \varphi \rangle_{\mu_t^N} = \int_{\mathbb{R}^2\times\mathbb{R}} \varphi \, \mathrm{d}\mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \varphi(x_i(t), \theta_i(t))$$

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Exchangeability : $\mu_t^N$ invariant under label permutations, all the particles $z_i = (x_i, \theta_i)$ follow the (non-autonomous) flow

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \tau(\theta) \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} = b[\mu_t^N](x,\theta), \end{cases}$$

where we have set $b[f](x,\theta) = \int_{\mathbb{R}^2\times\mathbb{R}} K(x'-x)\sin(\theta'-\theta)\,\mathrm{d}f(x',\theta')$.

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**Exercise : close equation for $\mu_t^N$ !**

Prove that $\mu_t^N$ is a weak solution of the following kinetic equation :

$$\partial_t \mu_t^N + \tau(\theta)\cdot\nabla_x \mu_t^N + \partial_\theta(b[\mu_t^N]\mu_t^N) = 0.$$

# Vlasov equation

Useful notation : $J_f = \int_{\mathbb{R}} \tau(\theta) \mathrm{d}f(\theta)$, and $J_f^K = K * J_f$. We then get $b[f](x,\theta) = \partial_\theta(\tau(\theta)\cdot J_f^K)$.

**Theorem : global well-posedness**

If $K$ is bounded and Lipschitz, then the Vlasov equation

$$\partial_t f + \tau(\theta)\cdot\nabla_x f + \partial_\theta(\partial_\theta(\tau(\theta)\cdot J_f^K)f) = 0$$

is well-posed in $C([0,T],\mathcal{P}_1(\mathbb{R}^2\times\mathbb{R}))$.

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- Solve the linear (nonautonomous) equation $\partial_t f + \tau(\theta) \cdot \nabla_x f + \partial_\theta(\partial_\theta(\tau(\theta) \cdot J(x,t)) f) = 0$ by characteristics : $f = \Phi(J)$.

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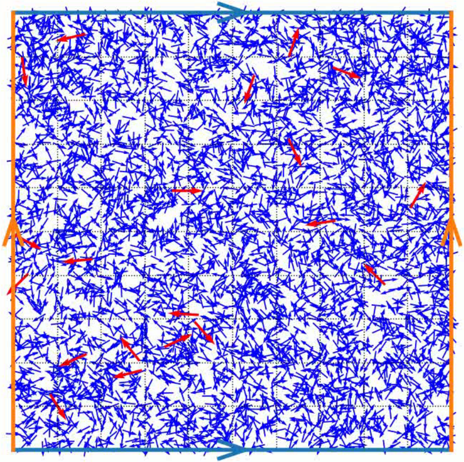
- Fixed point in $C([0,T], \mathcal{P}_1(\mathbb{R}^2 \times \mathbb{R}))$ of $f \mapsto \Phi(J_f^K)$. Links between the Lipschitz character of $K$ and contraction in Wasserstein distance.

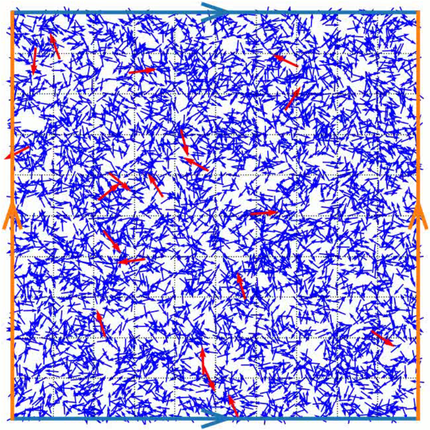
# Artificial confinement

If $K$ is periodic (think with small compact support + periodization), then everything works in $C([0,T],\mathcal{P}(\mathbb{R}^2/\Gamma\times\mathbb{R}/2\pi\mathbb{Z}))$.

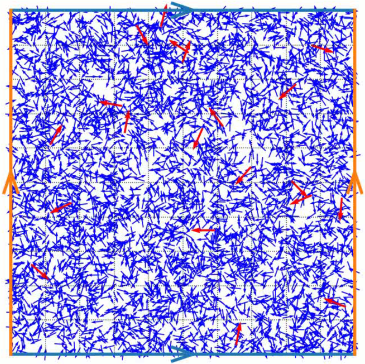


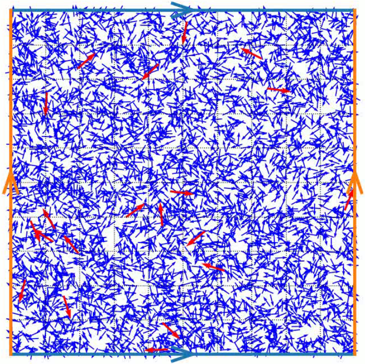
# Same thing, with noise

$$\mathrm{d}x_i = \tau(\theta_i)\,\mathrm{d}t, \quad \mathrm{d}\theta_i = \frac{1}{N}\sum_{j=1}^{N} K(x_j - x_i)\sin(\theta_j - \theta_i)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_t^i.$$



# More noise !





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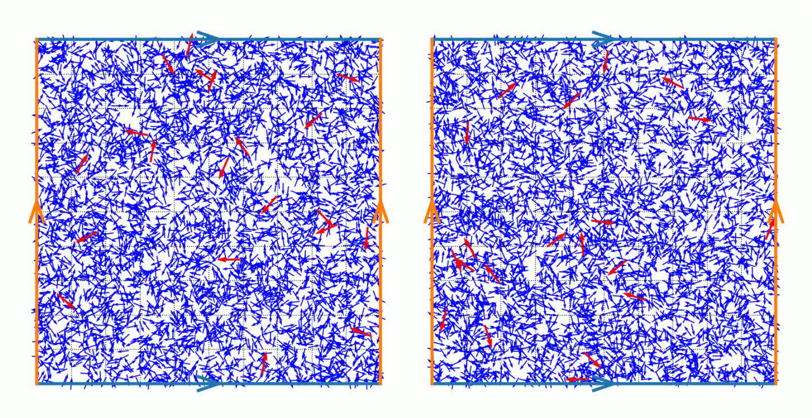

Now $\mu_t^N$ is a probabilistic object (realisations of the brownian motions).
How to get a kinetic equation for that ?

$\rightsquigarrow$ Law of large numbers... $\langle\varphi\rangle_{\mu_t^N} = \dfrac{1}{N}\sum_{i=1}^{N}\varphi(x_i(t)) \overset{?}{\to} \langle\varphi\rangle_{\mu_t}$.

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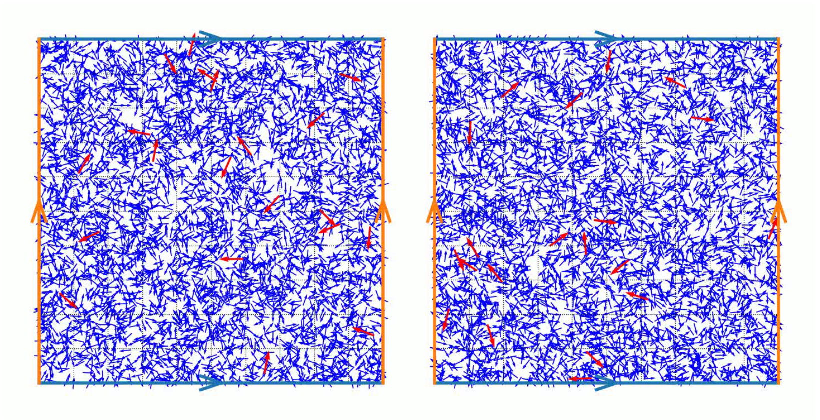

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Main ref.: A.-S. Sznitman, "Topics in propagation of chaos" [Szn91].

# Existence theory for SDEs

Rewrite the system in integral formulation, and with $b[\mu_t^N]$ (exchangeability).

$$x_i(t) = x_i^0 + \int_0^t \tau(\theta_i(s))\,\mathrm{d}s,$$

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## Law of the 3$N$-dimensional process (linear), Ito formula

Probability density $f^N(x_1,\theta_1,\dots)$ of finding particles around $(x_i,\theta_i)$ :

$$\partial_t f^N + \sum_{i=1}^N [\tau(\theta_i)\cdot\nabla_{x_i} f^N + \partial_{\theta_i}(b[\mu^N](x_i,\theta_i) f^N)] = \sigma \sum_{i=1}^N \partial^2_{\theta_i\theta_i} f^N,$$

where $b[\mu^N](x,\theta) = \dfrac{1}{N}\sum_{j=1}^N K(x_j - x)\sin(\theta_j - \theta)$.

# Marginals, as in BBGKY hierarchy ?

**Exercise : equation of the first marginal**

If $f^{N,1}(x,\theta) = \int f^N(x,\theta,x_2,\theta_2\dots)\mathrm{d}x_2\mathrm{d}\theta_2\dots\mathrm{d}x_n\mathrm{d}\theta_n$
and $f^{N,2}(x,\theta,x',\theta') = \dots$, then

$$\partial_t f^{N,1} + \tau(\theta)\cdot\nabla_x f^{N,1} + \tfrac{N-1}{N}\partial_\theta(b[f^{N,2}(x,\theta,\cdot,\cdot)](x,\theta)) = \sigma\partial^2_{\theta\theta} f^{N,1}.$$

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If $f^{N,1}\to f$ and $f^{N,2}\to f\otimes f$ (asymptotic indep. of two particles), then

$$\partial_t f + \tau(\theta)\cdot\nabla_x f + \partial_\theta(b[f]f) = \sigma\partial^2_{\theta\theta}f.$$

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This is very formal, but we recover the (nonlinear) Vlasov equation with a diffusion. Can we rigorously derive this equation ?
Notice that it can be written (since $b[f] = \partial_\theta(J^K_f\cdot\tau(\theta))$) as

$$\partial_t f + \tau(\theta)\cdot\nabla_x f = \sigma\partial_\theta\left(e^{\frac{1}{\sigma}J^K_f\cdot\tau(\theta)}\partial_\theta\left(\frac{f}{e^{\frac{1}{\sigma}J^K_f\cdot\tau(\theta)}}\right)\right).$$

The right-hand side Fokker-Planck term is not only acting on the angular variable $\theta$, but not so far…

# The nonlinear coupling processes

We look at processes which are the natural limits of $(x_i, \theta_i)$ as $N \to \infty$.

**Theorem [Szn91]**

There exist a unique solution, pathwise and in law, to

$$\bar{x}(t) = x^0 + \int_0^t \tau(\bar{\theta}(s))\, \mathrm{d}s,$$

$$\bar{\theta}(t) = \theta^0 + \int_0^t b[\bar{\mu}_s](\bar{x}(s), \bar{\theta}(s))\, \mathrm{d}s + \sqrt{2\sigma}\, B_t,$$

whee $\bar{\mu}_t$ is the law of the process $\bar{x}(t), \bar{\theta}(t)$ : $\mathbb{E}[\varphi(\bar{x}(t), \bar{\theta}(t))] = \langle \varphi \rangle_{\bar{\mu}_t}$.

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- No more index $i$, no more $N$, only one particle interacting with its own law.
- Fixed point problem for the law $\bar{\mu}_t$.
- PDE satisfied by $\bar{\mu}_t$ ? by Itō formula, we get

$$\partial_t \bar{\mu}_t + \tau(\theta) \cdot \nabla_x \bar{\mu}_t + \partial_\theta (b[\bar{\mu}_t]\bar{\mu}_t) = \sigma \partial^2_{\theta\theta} \bar{\mu}_t.$$

# Distance between processes

We therefore introduce $(\bar{x}_i, \bar{\theta}_i)$, the nonlinear coupling processes associated to the same initial conditions $x_i^0, \theta_i^0$ and Brownian motion $B_t^i$. Their law is always $\bar{\mu}_t$ (if the initial condition are iid), and they are independent. They satisfy

$$x_i - \bar{x}_i(t) = \int_0^t [\tau(\theta_i) - \tau(\bar{\theta}_i)](s))\,\mathrm{d}s,$$

$$\theta_i - \bar{\theta}_i(t) = \int_0^t \left(b[\mu_s^N](x_i(s), \theta_i(s)) - b[\bar{\mu}_s](\bar{x}_i(s), \bar{\theta}_i(s))\right)\,\mathrm{d}s,$$

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For convenience : $z_t^i = (x_i(t), \theta_i(t))$, $\bar{z}_t^i = (\bar{x}_i(t), \bar{\theta}_i(t))$, and the new empirical distribution $\bar{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\bar{z}_t^i}$. We obtain

$$\begin{aligned}|\theta_i - \bar{\theta}_i(t)| \leqslant \int_0^t |b[\mu_s^N](z_s^i) - b[\mu_s^N](\bar{z}_s^i)|\mathrm{d}s \quad &(\text{use } b[f](\cdot) \text{ Lipschitz})\\ + \int_0^t |b[\mu_s^N](\bar{z}_s^i) - b[\bar{\mu}_s^N](\bar{z}_s^i)|\mathrm{d}s \quad &(\text{use } b[\cdot](z) \text{ Lipschitz})\\ + \int_0^t |b[\bar{\mu}_s^N](\bar{z}_s^i) - b[\mu_s](\bar{z}_s^i)|\mathrm{d}s \quad &(\text{LLN for indep. processes}).\end{aligned}$$

# Distance between processes (continued)

We obtain, after using exchangeability and taking expectation

$$\mathbb{E}[|z_t^1 - \bar{z}_t^1|] \leqslant C \int_0^t \mathbb{E}[|z_s^1 - \bar{z}_s^1|]\mathrm{d}s + \int_0^t \mathbb{E}[|b[\bar{\mu}_s^N](\bar{z}_s^1) - b[\mu_s](\bar{z}_s^1)|]\mathrm{d}s.$$

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Recall: $b[\bar{\mu}_s^N](\bar{z}_s^1) = \frac{1}{N}\sum_{i=1}^{N} \beta(\bar{z}_s^i, \bar{z}_s^1)$ and $b[\mu_s](\bar{z}_s^1) = \int \beta(z, \bar{z}_s^1)\mathrm{d}\mu_s(z)$, with $\beta(z, z') = K(x - x')\sin(\theta - \theta')$, so $\beta$ is bounded and Lipschitz.

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Recall: $b[\bar{\mu}_s^N](\bar{z}_s^1) = \frac{1}{N}\sum_{i=1}^N \beta(\bar{z}_s^i, \bar{z}_s^1)$ and $b[\mu_s](\bar{z}_s^1) = \int \beta(z, \bar{z}_s^1)\mathrm{d}\mu_s(z)$, with $\beta(z, z') = K(x - x')\sin(\theta - \theta')$, so $\beta$ is bounded and Lipschitz.
Therefore $\mathbb{E}[|b[\bar{\mu}_s^N](\bar{z}_s^1) - b[\mu_s](\bar{z}_s^1)|] = \mathbb{E}\Big[\Big|\frac{1}{N}\sum_{i=1}^N Y_i\Big|\Big]$, where the random variables $Y_i = \beta(\bar{z}_s^i, \bar{z}_s^1) - \int \beta(z, \bar{z}_s^1)\mathrm{d}\mu_s(z)$ are centered, bounded, pairwise independent. We get $\mathbb{E}\Big[\Big|\frac{1}{N}\sum_{i=1}^N Y_i\Big|\Big]^2 \leqslant \frac{1}{N^2}\mathbb{E}\Big[\Big(\sum_{i=1}^N Y_i\Big)^2\Big] \leqslant \dfrac{C}{N}$.

# Distance between processes (continued)

We obtain, after using exchangeability and taking expectation

$$\mathbb{E}[|z_t^1 - \bar{z}_t^1|] \leqslant C\int_0^t \mathbb{E}[|z_s^1 - \bar{z}_s^1|]\mathrm{d}s + \int_0^t \mathbb{E}[|b[\bar{\mu}_s^N](\bar{z}_s^1) - b[\mu_s](\bar{z}_s^1)|]\mathrm{d}s.$$

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**Final estimation (with Gronwall)**

$$\sup_{t\in[0,T]} W_1(f^{N,1}(t), \bar{\mu}_t) \leqslant \sup_{t\in[0,T]} \mathbb{E}[|z_t^1 - \bar{z}_t^1|] \leqslant \frac{C(T)}{\sqrt{N}}.$$

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We want "local" interactions...

In simulations $K(x) = K_0(R - |x|)_+$, with $R$ small : in average, a particle (among 4000) interacts with only 90. The strength $K_0$ has to be big, so bad Lipschitz constants... Scalings ? The goal is to increase the size of the periodic box, without changing $K$.

# Moderate interaction [Oel85] (Méléard, Jourdain,. . . )

**Interaction between particles within distance $R_N = \frac{1}{N^{\beta/2}}$.**

Take $K_N = \frac{1}{R_N^2} K(\frac{x}{R_N})$, interaction with around $N^{1-\beta}$ particles :

$$\mathrm{d}x_i = \tau(\theta_i)\,\mathrm{d}t, \quad \mathrm{d}\theta_i = \frac{1}{N^{1-\beta}} \sum_{j=1}^{N} K(N^{\frac{\beta}{2}}(x_j - x_i)) \sin(\theta_j - \theta_i)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_t^i.$$

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Non-linear process I :

$$\mathrm{d}\bar{x}^N = \tau(\bar{\theta}^N)\,\mathrm{d}t, \quad \bar{\mu}_t^N = \mathrm{law}(\bar{x}^N(t), \bar{\theta}^N(t))$$

$$\bar{\theta} = \int N^{\beta} K(N^{\frac{\beta}{2}}(x - \bar{x}^N)) \sin(\theta - \bar{\theta}^N)\mathrm{d}\bar{\mu}_t^N(x,\theta)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_t.$$

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PDE for the evolution of the law $\bar{\mu}_t^N$ :

$$\partial_t \bar{\mu}_t + \tau(\theta)\cdot\nabla_x \bar{\mu}_t + \partial_\theta(b^N[\bar{\mu}_t]\bar{\mu}_t) = \sigma \partial^2_{\theta\theta}\bar{\mu}_t,$$

where $b[f](x,\theta) = \int_{\mathbb{R}^2\times\mathbb{R}} N^{\beta} K(N^{\frac{\beta}{2}}(x' - x)) \sin(\theta' - \theta) \mathrm{d}f(x',\theta')$.

# Moderate interaction (continued)

Non-linear process II :

$$\mathrm{d}\bar{x} = \tau(\bar{\theta})\,\mathrm{d}t, \quad \bar{\mu}_t = \mathrm{law}(\bar{x}(t), \bar{\theta}(t))$$

$$\bar{\theta} = \nu \int_{\mathbb{R}} \sin(\theta - \bar{\theta})\mathrm{d}\bar{\mu}_t(\bar{x}, \theta)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_t, \quad \nu = \int_{\mathbb{R}^2} K(x)\mathrm{d}x.$$

PDE for $\bar{\mu}_t$ : $\partial_t \bar{\mu}_t + \tau(\theta) \cdot \nabla_x \bar{\mu}_t + \partial_\theta(\bar{b}[\bar{\mu}_t]\bar{\mu}_t) = \sigma \partial^2_{\theta\theta} \bar{\mu}_t$,
where $\bar{b}[f](x, \theta) = \int_{\mathbb{R}} \sin(\theta' - \theta)\mathrm{d}f(x, \theta') = \partial_\theta(\tau(\theta) \cdot J_f(x))$ is an operator acting only on the angle variable $\theta$.

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## Candidate : Vlasov-Fokker-Planck, local in $x$

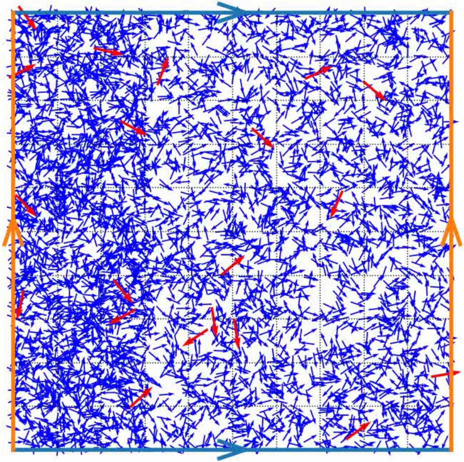
$$\partial_t f + \tau(\theta)\cdot\nabla_x f = \sigma\partial_\theta\left(e^{\frac{\nu}{\sigma}J_f\cdot\tau(\theta)}\partial_\theta\left(\frac{f}{e^{\frac{\nu}{\sigma}J_f\cdot\tau(\theta)}}\right)\right).$$

Higher dimensions : $\tau(\theta) \rightsquigarrow v \in \mathbb{S}^{d-1}$, $J_f = \int_{\mathbb{S}^{d-1}} v\mathrm{d}f(v)$.

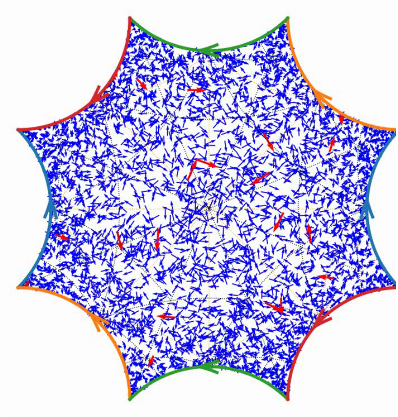
$$\partial_t f + v\cdot\nabla_x f = \sigma\nabla_v\cdot\left(e^{\frac{\nu}{\sigma}J_f\cdot v}\nabla_v\left(\frac{f}{e^{\frac{\nu}{\sigma}J_f\cdot v}}\right)\right).$$

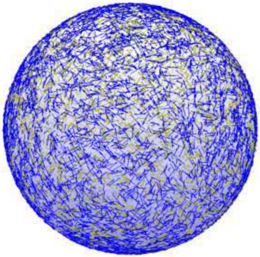
Problems : kinetic version + relies on the well-posedness of the PDE.

# The travelling bands



# More geometries (in progress, with S. Motsch)





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