

Kinetic models for self-organized dynamics of alignment II : from kinetic to macroscopic models

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Monokinetic solutions for Vlasov equation

Vlasov equation : alignment without noise

We consider $K \geq 0$, $\int_{\mathbb{R}^2} K(x)dx = 1$, and the evolution equation

$$\partial_t f + \tau(\theta) \cdot \nabla_x f + \nu \partial_\theta (\partial_\theta (\tau(\theta) \cdot J_f^K) f) = 0$$

where $J_f = \int_{\mathbb{R}} \tau(\theta) df(\theta)$, and $J_f^K = K * J_f$.

Are there monokinetic solutions $f = \rho(x, t) \delta_{\psi(x, t)}(\theta)$?

Exercise : the Euler equations with nonlocal alignment.

On a test function φ , obtain

$$\int_{\mathbb{R}^2} [(i)] \varphi(x, \psi) dx + \int_{\mathbb{R}^2} [(ii)] \partial_\theta \varphi(x, \psi) dx = 0,$$

$$\text{with } \begin{cases} (i) = \partial_t \rho + \nabla_x \cdot (\rho \tau(\psi)) = 0 \\ (ii) = \rho \partial_t \psi + \rho \tau(\psi) \cdot \nabla_x \psi + \nu \tau^\perp(\psi) \cdot K * (\rho \tau(\psi)) = 0. \end{cases}$$

Gradient, divergence and Laplace-Beltrami on \mathbb{S}

We still consider $K \geq 0$, $\int_{\mathbb{R}^2} K(x)dx = 1$, and the evolution equation

$$\partial_t f + v \cdot \nabla_x f + \nu \nabla_v \cdot (\nabla_v (v \cdot J_f^K) f) = 0$$

where $J_f = \langle v \rangle_f = \int_{\mathbb{S}} v df(v)$, and $J_f^K = K * J_f$.

Euler-alignment system for monokinetic solutions $f = \rho(x)\delta_u(v)$:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0 \\ \rho(\partial_t u + u \cdot \nabla_x u) + \nu P_{u^\perp}(K * (\rho u)) = 0. \end{cases}$$

The unit norm of u is preserved.

Nonlocal to local ? We could obtain the pressureless dynamics on \mathbb{S} :

$$\partial_t u + u \cdot \nabla_x u = 0.$$

Vlasov-Fokker-Planck equation, velocity on \mathbb{S}

Density $f(x, v, t)$ (not a probability anymore), dimensionless

We still consider $K \geq 0$, $\int_{\mathbb{R}^2} K(x) dx = 1$, and the evolution equation

$$\underbrace{\partial_t f + v \cdot \nabla_x f}_{\text{transport}} + \underbrace{\nabla_v \cdot (\nabla_v (v \cdot J_f^K) f)}_{\text{alignment}} = \underbrace{\Delta_v f}_{\text{diffusion}}$$

where $J_f = \langle v \rangle_f = \int_{\mathbb{S}} v df(v)$, and $J_f^K = K * J_f$.

We also denote $\rho_f = \langle 1 \rangle_f = \int_{\mathbb{S}} df(v)$, which scales as old $\frac{\nu}{\sigma}$.
Fokker-Planck formulation

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (e^{v \cdot J_f^K} \nabla_v (e^{-v \cdot J_f^K} f)).$$

The right-hand side takes in account the space variable, if $K \neq \delta_0 \dots$

Definitions: Fisher–von Mises distribution

$$M_{\kappa\Omega}(v) = \frac{e^{\kappa v \cdot \Omega}}{\int_{\mathbb{S}} e^{\kappa w \cdot \Omega} dw}.$$

Orientation $\Omega \in \mathbb{S}$, concentration $\kappa \geq 0$.

Order parameter: $c(\kappa) = |J_{M_{\kappa\Omega}}| = \frac{\int_0^\pi \cos \theta e^{\kappa \cos \theta} \sin^{d-2} \theta d\theta}{\int_0^\pi e^{\kappa \cos \theta} \sin^{d-2} \theta d\theta}$.

For $\kappa_f = |J_f|$ and $\Omega_f = \frac{J_f}{|J_f|}$, we get:

$$Q(f) = \nabla_v \cdot \left[M_{\kappa_f \Omega_f} \nabla_v \left(\frac{f}{M_{\kappa_f \Omega_f}} \right) \right].$$

Local equilibria: $f_{eq} = \rho M_{\kappa\Omega}$, for some $\Omega \in \mathbb{S}$.

Compatibility condition: $\kappa = \kappa_{f_{eq}} = |J_{f_{eq}}| = \rho |J_{\kappa\Omega}| = \rho c(\kappa)$.

Study of the (homogeneous) steady states

Proposition : solutions to the compatibility condition $\rho c(\kappa) = \kappa$

The function $\kappa \mapsto \frac{c(\kappa)}{\kappa}$ is decreasing, its limit is $\frac{1}{d}$ when $\kappa \rightarrow 0$.

- $\rho \leq d$, only one solution: $\kappa = 0$. Uniform equilibrium.
- $\rho > d$, uniform equilibrium for $\kappa = 0$ + unique solution $\kappa(\rho) > 0$.
Manifold of equilibria:

$$\{\rho M_{\kappa(\rho)\Omega}, \Omega \in \mathbb{S}\}.$$

Convergence to the equilibrium for $\partial_t f = Q(f)$.

Theorem (AF, J.-G. Liu)[FL12]

- If $\rho_{f_0} < d$, exponential convergence to the uniform distribution $f \rightarrow \rho_{f_0}$.
- If $\rho_{f_0} > d$ and $J_{f_0} \neq 0$, there exists $\Omega_\infty \in \mathbb{S}$ such that f converges exponentially to $\rho_{f_0} M_{\kappa(\rho)\Omega_\infty}$.

Ideas of the proofs, tools used

- Decay of the free energy $\mathcal{F}(f) = \int_{\mathbb{S}} f \ln f - \frac{1}{2} |J_f|^2$
- Instantaneous regularity, compactness \Rightarrow LaSalle Principle
- Use of the spherical harmonics to derive a new conservation relation:

$$\frac{1}{2} \frac{d}{dt} \|f - 1\|_{\tilde{H}^{-\frac{d-1}{2}}}^2 = -\|f - 1\|_{\tilde{H}^{-\frac{d-3}{2}}}^2 + \frac{\rho_{f_0}}{(d-2)!} |J[f]|^2,$$

viewed as the dissipation of a “new entropy” when $\rho_{f_0} < d$

- Expansion of \mathcal{F} and its dissipation term around a “moving equilibrium” $M_{\kappa\Omega(t)}$ when $\rho_{f_0} > d$:

$$f = (1 + \alpha v \cdot \Omega(t) + g) M_{\kappa(\tau)\Omega(t)},$$

with exponential decay of α and g , which then gives the convergence of $\Omega(t)$ to Ω_{∞}

- When $\rho = d$, we obtain a decay rate of $\frac{C}{\sqrt{t}}$.

We define $f^\varepsilon(x, v, t) = f\left(\frac{x}{\varepsilon}, v, \frac{t}{\varepsilon}\right)$. If the kernel is symmetric, we get $J_{f^\varepsilon}^{K^\varepsilon} = J_f^K\left(\frac{x}{\varepsilon}\right) = J_{f^\varepsilon}(x) + \mathcal{O}(\varepsilon^2)$.

Rescaled equation

$$\varepsilon(\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon) = Q(f^\varepsilon) + \mathcal{O}(\varepsilon^2).$$

Notice we do $N \rightarrow \infty$, then $\varepsilon \rightarrow 0$. . . The $\mathcal{O}(\varepsilon^2)$ can be 0 if we take the “moderate interaction”. Other option : scale K with other power of ε . The $\mathcal{O}(\varepsilon^2)$ can become $\mathcal{O}(\varepsilon^\alpha)$. . .

When ε is small, we expect $Q(f^\varepsilon)$ to be small, and then f^ε to be close to a steady state. Which one ? Think stability.

- When $\rho^\varepsilon (= \rho_{f^\varepsilon}) > d$, the von Mises are stable for the homogeneous, they are good candidates.
- When $\rho^\varepsilon < d$, the uniform equilibrium on the sphere is stable.

Region where $\rho^\varepsilon(x, t) - d \gg \varepsilon$

Starting point: when $\varepsilon \rightarrow 0$, we suppose f^ε converges to $\rho M_{\kappa(\rho)}\Omega$. If we know that the convergence is strong and the functions smooth, can we get evolution equations ?

Equation on ρ : conservation of mass (integration of the kinetic equation against a constant).

$$\partial_t \rho^\varepsilon + \nabla_x \cdot J^{\varepsilon, K_\varepsilon} = 0$$

In the limit $\varepsilon \rightarrow 0$, we get

$$\partial_t \rho + \nabla_x \cdot (\rho c(\kappa(\rho))\Omega) = 0$$

Evolution of Ω ? No more conservation relation. . .

$$\int_{\mathbb{S}} Q(f^\varepsilon) \psi(v) dv \neq 0 \text{ in general } (\psi \text{ non constant}).$$

Idea: integrate against $\psi_{\rho^\varepsilon, \Omega^\varepsilon}(v)$ instead.

Generalized collisional invariants

Linearized operator: $Q(f) = L_{\kappa_f \Omega_f}(f)$, with

$$L_{\kappa \Omega}(f) = -\Delta_v f + \kappa \nabla_v \cdot (P_{v^\perp} \Omega f) = -\nabla_v \cdot \left[M_{\kappa \Omega} \nabla_v \left(\frac{f}{M_{\kappa \Omega}} \right) \right],$$

Definition: GCIs associated to κ and Ω

$$\mathcal{C}_{\kappa \Omega} = \left\{ \psi \mid \int_{v \in \mathbb{S}} L_{\kappa \Omega}(f) \psi \, dv = 0, \forall f \text{ such that } J_f \parallel \Omega \right\}.$$

In particular, for any generalized collisional invariant $\psi \in \mathcal{C}_{\kappa \Omega}$:

$$\forall f \text{ such that } \Omega_f = \Omega \text{ and } \kappa_f = \kappa, \int_{v \in \mathbb{S}} Q(f) \psi \, dv = 0.$$

Proposition

$$\psi \in \mathcal{C}_{\kappa \Omega} \Leftrightarrow \psi = C + h_\kappa(v \cdot \Omega) A \cdot v, A \in \mathbb{R}^d, A \perp \Omega.$$

The macroscopic model

$$A \cdot \int_{v \in \mathbb{S}} Q(f^\varepsilon) h_{\kappa f^\varepsilon}(v \cdot \Omega_{f^\varepsilon}) v \, dv = 0 \text{ for all } A \in \mathbb{R}^d \text{ s.t. } A \cdot \Omega_{f^\varepsilon} = 0$$

Equivalently, defining $\vec{\psi}_{\kappa, \Omega} = h_{\kappa}(v \cdot \Omega) P_{\Omega^\perp} v$, we get

$$\int_{v \in \mathbb{S}} Q(f^\varepsilon) \vec{\psi}_{\kappa f^\varepsilon, \Omega_{f^\varepsilon}} \, dv = 0$$

Theorem (P. Degond, AF, J.-G. Liu) [DFL13]

When $\varepsilon \rightarrow 0$, the (formal) limit of f^ε is $f^0 = \rho(x, t) M_{\kappa(\rho)\Omega(x,t)}$ and the functions ρ, Ω satisfy the system

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho c \Omega) = 0, \\ \rho (\partial_t \Omega + \tilde{c}(\Omega \cdot \nabla_x) \Omega) + \lambda P_{\Omega^\perp} \nabla_x \rho = 0, \end{cases}$$

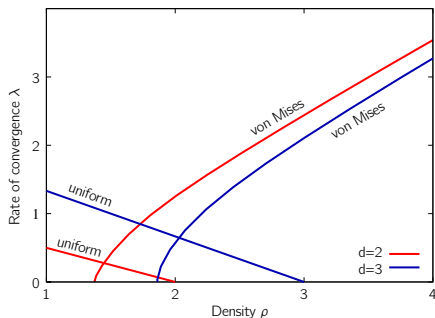
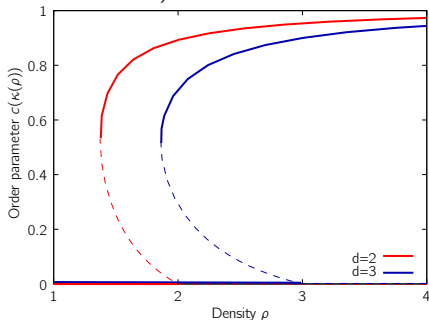
with $\tilde{c} = \langle \cos \theta \rangle_{M_\kappa}$, and $\lambda = \frac{\rho - d - \kappa \tilde{c}}{\kappa(\rho - d - \kappa c)}$.

Problem : $\lambda < 0$, the system is not hyperbolic. . .

Other types of phase transition, hysteresis [DFL15]

- Change force : instead of proportional to J_f^K , make it proportional to $\nu(|J_f^K|, \rho_f^K) \frac{J_f^K}{|J_f^K|}$. Noise $\sigma(|J_f^K|, \rho_f^K)$.
- New compatibility equation $\rho c(\kappa) = \alpha$, $\kappa = \frac{\nu(j, \rho)}{\sigma(j, \rho)}$.
- The solution may not be monotone in ρ or κ . More than one branch of equilibria.

Exemple: $\nu(|J|) = |J|$, $\sigma(|J|) = \frac{1}{1+|J|}$ (extrinsic noise, 1st order phase transition).



Why is the macroscopic model a good candidate ?

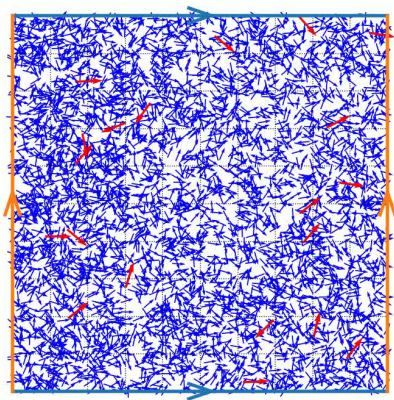
Changes on the model, with the same ingredients : alignment and noise, all lead to the same model.

Self-Organized Hydrodynamics (SOH) system

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho c \Omega) = 0, \\ \rho (\partial_t \Omega + \tilde{c}(\Omega \cdot \nabla_x) \Omega) + \lambda P_{\Omega^\perp} \nabla_x \rho = 0, \end{cases}$$

- [DM08] : first derivation, $c, \tilde{c}, \lambda > 0$ constants. Hyperbolic. Comes from $\nu(|J|) = c$ (singular, no phase transition).
- [DM11] : comes from third order particle dynamics (Persistent Turning Walker model), with alignment.
- [Fro12] : K not isotropic, coefficients depend on ρ . Drastic changes in \tilde{c} . May lose hyperbolicity.
- [DLMP13] : incorporates repulsion, different scale for K , viscosity added in the macroscopic model.
- [DM16] : BGK version (alignment of particles by velocity jumps).

Why is it a bad candidate ? (remember the bands)



- Nonclassical shocks (not conservative). Bad behavior due to lack of hyperbolicity ? Not clear. Back to the kinetics for that. . .
- [MN11] : Comparison with relaxation model (no more constraint $|\Omega| = 1$), for which SOH is the limit in the strong relaxation regime. Splitting scheme for numerics.

Chapman–Enskog expansion (P. Degond, AF, J.-G. Liu) [DFL15]

When $\varepsilon \rightarrow 0$, a first order correction is (formally) given by

$$f^\varepsilon(x, v, t) = \rho^\varepsilon(x, t) - \varepsilon \frac{\rho_c v \cdot \nabla_x \rho^\varepsilon(x, t)}{(d-1)\sigma_0(\rho_c - \rho^\varepsilon(x, t))},$$

And the density $\rho^\varepsilon(x, t)$ satisfies the following (nonlinear) diffusion equation:

$$\partial_t \rho^\varepsilon = \frac{\varepsilon}{(d-1)d\sigma_0} \nabla_x \cdot \left(\frac{\rho_c}{\rho_c - \rho^\varepsilon} \nabla_x \rho^\varepsilon \right).$$

Work in progress:

- Convergence to a uniform steady state (inhomogeneous perturbation). Hypocoercivity.
- Understanding the behavior at the boundary. Look for kinetic fronts connecting two regions. Good behavior for “escaping”, bad for “invading”...



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