Macroscopic limits of a system of self-propelled particles with phase transition

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Goal: macroscopic description of some animal societies

- Local interactions without leader
- Emergence of macroscopic structures

Images Benson Kua (flickr) and

Modeling of interacting self-propelled particles

- Vicsek *et al.* (1995).
  Discrete in time (interval Δt), alignment only, synchronous reorientation.

$$\begin{array}{c}\text{New}\\ \text{direction}\end{array} = \begin{array}{c}\text{Mean direction of neighboring}\\ \text{particles at previous step}\end{array} + \text{Noise}$$

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  Time-continuous version: relaxation (with constant rate ν) towards the local mean direction.
  Hydrodynamic limit without phase transition phenomenon.

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- Model presented here: making ν proportional to the local mean momentum.

Original discrete model

$\Delta t = 0,07$

Time-continuous version

$\nu = 15$

Outline

1. Time-continuous Vicsek model with phase transition
   - Presentation of the model
   - Kinetic model – Hydrodynamic scaling
   - The phase transition

Individual dynamics

Particles at positions: $X_1, \ldots, X_N$ in $\mathbb{R}^n$.
Orientations $\omega_1, \ldots, \omega_N$ in $\mathbb{S}$ (unit sphere).

$$\begin{cases} \frac{\mathrm{d}X_k}{\mathrm{d}t} = \omega_k \\ \frac{\mathrm{d}\omega_k}{\mathrm{d}t} = \nu P_{\omega_k^\perp}(\bar{\omega}_k - \omega_k) \end{cases}$$

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Target direction:

$$\bar{\omega}_k = \frac{J_k}{|J_k|}, \quad J_k = \frac{1}{N}\sum_{j=1}^{N} K(|X_j - X_k|)\,\omega_j.$$

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Setting $\nu = |J_k|\,\nu_0$, no more singularity (binary interactions):

$$\begin{cases} \mathrm{d}X_k = \omega_k \mathrm{d}t \\ \mathrm{d}\omega_k = \nu_0(\mathrm{Id} - \omega_k \otimes \omega_k)\,J_k \mathrm{d}t + \sqrt{2d}(\mathrm{Id} - \omega_k \otimes \omega_k) \circ \mathrm{d}B_t^k \end{cases}$$

Kinetic description

Theorem (Bolley, Cañizo, Carrillo, 2011)

Probability density function $f(x, \omega, t)$, as $N \to \infty$:

$$\partial_t f + \omega \cdot \nabla_x f + \nu_0 \nabla_\omega \cdot ((\mathrm{Id} - \omega \otimes \omega)\mathcal{J}_f f) = d\Delta_\omega f$$

$$\mathcal{J}_f(x, \omega, t) = \int_{y \in \mathbb{R}^n,\, v \in \mathbb{S}} K(|y - x|)\, v\, f(y, v, t)\, \mathrm{d}y\, \mathrm{d}v\,.$$

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Tool : coupling process + estimations.

$$\begin{cases} \mathrm{d}\bar{X}_k = \bar{\omega}_k \mathrm{d}t \\ \mathrm{d}\bar{\omega}_k = \nu_0(\mathrm{Id} - \bar{\omega}_k \otimes \bar{\omega}_k)\, \mathcal{J}_{f_t^N}\, \mathrm{d}t + \sqrt{2d}(\mathrm{Id} - \bar{\omega}_k \otimes \bar{\omega}_k) \circ \mathrm{d}B_t^k \\ f_t^N = \mathrm{law}(\bar{X}_1, \bar{\omega}_1) = \mathrm{law}(\bar{X}_k, \bar{\omega}_k) \end{cases}$$

Hydrodynamic scaling

Scaling, with $\varepsilon \ll 1$ (and $K_0 = \int_{\mathbb{R}^n} K(x)\mathrm{d}x$):

$$f^\varepsilon(x,\omega,t) = \nu_0 K_0 f(\tfrac{1}{d\varepsilon}x, \omega, \tfrac{1}{d\varepsilon}t).$$

Mean-field reduced and rescaled equation:

$$\varepsilon(\partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon) = Q(f^\varepsilon) + O(\varepsilon^2),$$

with an effect of localization in space:

$$Q(f) = -\nabla_\omega \cdot ((\mathrm{Id} - \omega \otimes \omega) J_f\, f) + \Delta_\omega f,$$
$$J_f(x,t) = \int_{\mathbb{S}} f(x,\omega,t)\,\omega\,\mathrm{d}\omega.$$

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Since $(\mathrm{Id} - \omega \otimes \omega)J = \nabla_\omega(J \cdot \omega)$, we get

$$Q(f) = \nabla_\omega \cdot (e^{\omega \cdot J_f} \nabla_\omega (e^{-\omega \cdot J_f} f)).$$

Local equilibria

Definitions: Fisher–von Mises distribution

$$M_{\kappa\Omega}(\omega) = \frac{e^{\kappa\,\omega\cdot\Omega}}{\int_{\mathbb{S}} e^{\kappa\,v\cdot\Omega}\,\mathrm{d}v}.$$

Orientation $\Omega \in \mathbb{S}$, concentration $\kappa \geqslant 0$.

Order parameter: $c(\kappa) = |J_{M_{\kappa\Omega}}| = \dfrac{\int_0^\pi \cos\theta\, e^{\kappa\cos\theta} \sin^{n-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{\kappa\cos\theta} \sin^{n-2}\theta\,\mathrm{d}\theta}$.

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For $J_f = \kappa_f \Omega_f$, we can write Q under the form:

$$Q(f) = \nabla_\omega \cdot \left[M_{\kappa_f\Omega_f} \nabla_\omega \left(\frac{f}{M_{\kappa_f\Omega_f}} \right) \right].$$

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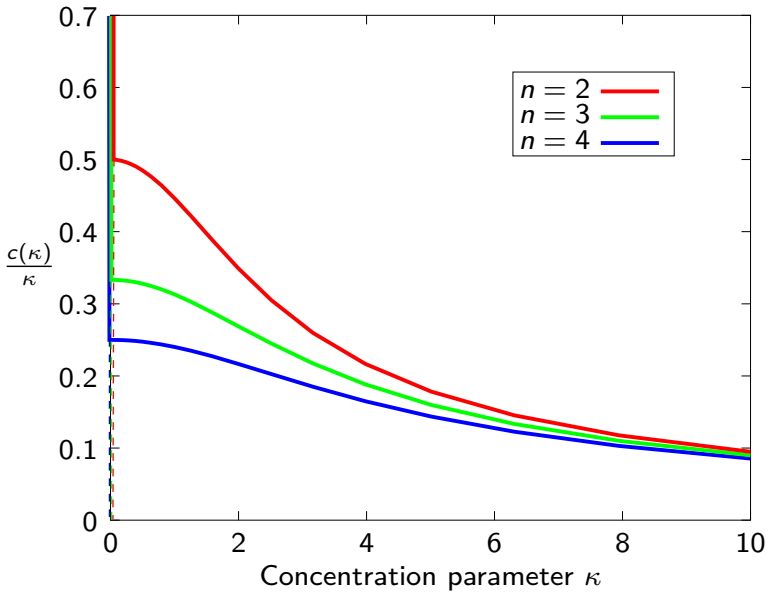
Local equilibria: $f_{eq} = \rho M_{\kappa\Omega}$, for some $\Omega \in \mathbb{S}$.

Compatibility condition: $\kappa = \kappa_{f_{eq}} = |J_{f_{eq}}| = \rho |J_{\kappa\Omega}| = \rho c(\kappa)$.

Solutions to the compatibility condition $\rho c(\kappa) = \kappa$

Proposition

The function $\kappa \mapsto \frac{c(\kappa)}{\kappa}$ is decreasing, its limit is $\frac{1}{n}$ when $\kappa \to 0$.



- $\rho \leqslant n$, only one solution: $\kappa = 0$. Uniform equilibrium.
- $\rho > n$, uniform equilibrium for $\kappa = 0$.
  Unique solution $\kappa(\rho) > 0$.
  Manifold of equilibria:
  $$\{\rho M_{\kappa(\rho)\Omega},\ \Omega \in \mathbb{S}\}.$$

Homogeneous case: convergence to the equilibrium

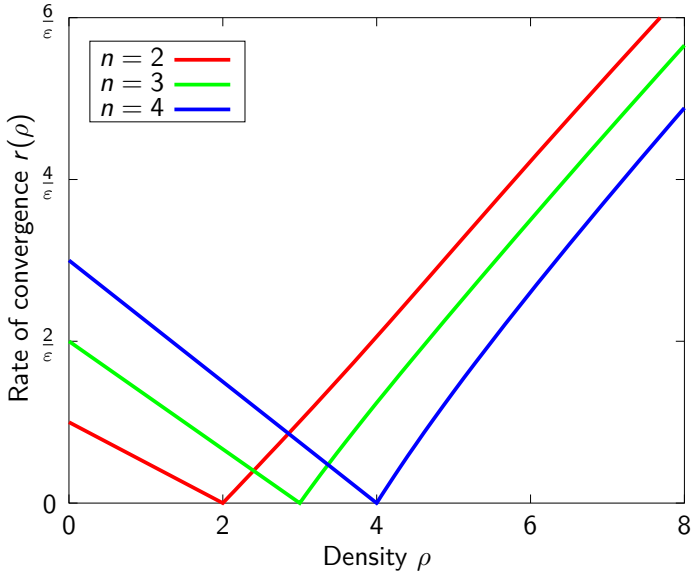
Spatial homogeneous case: the equation becomes

$$\varepsilon \partial_t f = -\nabla_\omega \cdot ((\mathrm{Id} - \omega \otimes \omega) J_f f) + \Delta_\omega f,$$

also called Doi equation (with dipolar potential).

Theorem (AF, J.-G. Liu)

- If $\rho < n$, exponential convergence to the uniform distribution $f \to \rho_{f_0}$.
- If $\rho > n$ and $J_{f_0} \neq 0$, there exists $\Omega_\infty \in \mathbb{S}$ such that f converges exponentially to $\rho_{f_0} M_{\kappa(\rho)\Omega_\infty}$.



Subcritical case
$\nu = 1.9$

Supercritical case
$\nu = 2.3$

Formal derivation of macroscopic models

Theorem (P. Degond, AF, J.-G. Liu)

- Region where $\rho^\varepsilon(x,t) - n \gg \varepsilon$: hydrodynamic model.
  As $\varepsilon \to 0$, the limit of f^ε is $f^0 = \rho(x,t) M_{\kappa(\rho)\Omega(x,t)}$ (formally), and the functions ρ, Ω satisfy the system:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, c\, \Omega) = 0, \\ \rho\, (\partial_t \Omega + \widetilde{c}\, (\Omega \cdot \nabla_x)\Omega) + \lambda\, (\mathrm{Id} - \Omega \otimes \Omega) \nabla_x \rho = 0. \end{cases}$$

- Region where $\rho^\varepsilon(x,t) - n \gg \varepsilon$: diffusion model.
  First order correction in ε for ρ^ε:

$$\partial_t \rho^\varepsilon = \frac{\varepsilon}{n-1} \nabla_x \cdot \left(\frac{1}{n - \rho^\varepsilon} \nabla_x \rho^\varepsilon \right).$$

Thanks!