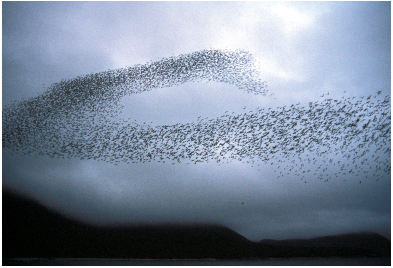
A model of alignment interaction for oriented particles with phase transition

Amic Frouvelle
Institut de Mathématiques de Toulouse

Joint work with Jian-Guo Liu (Duke Univ.) and Pierre Degond (IMT)

Goal: macroscopic description of some animal societies



- Local interactions without leader
- Emergence of macroscopic structures

Modeling of interacting self-propelled particles

- Vicsek *et al.* (1995).
  Discrete in time (interval Δt), alignment only, synchronous reorientation.

$$\begin{array}{c}\text{New}\\ \text{direction}\end{array} = \begin{array}{c}\text{Mean direction of neighboring}\\ \text{particles at previous step}\end{array} + \text{Noise}$$

  Simulations: phase transition phenomenon, emergence of coherent structures.

Modeling of interacting self-propelled particles

- Vicsek *et al.* (1995).
  Discrete in time (interval Δt), alignment only, synchronous reorientation.

  $$\begin{array}{c}\text{New}\\\text{direction}\end{array} = \begin{array}{c}\text{Mean direction of neighboring}\\\text{particles at previous step}\end{array} + \text{Noise}$$

  Simulations: phase transition phenomenon, emergence of coherent structures.
- Degond-Motsch (2008).
  Time-continuous version: relaxation (with constant rate ν) towards the local mean direction.
  Hydrodynamic limit without phase transition phenomenon.

Modeling of interacting self-propelled particles

- Vicsek *et al.* (1995).
  Discrete in time (interval Δt), alignment only, synchronous reorientation.

  $$\begin{matrix}\text{New} \\ \text{direction}\end{matrix} = \begin{matrix}\text{Mean direction of neighboring} \\ \text{particles at previous step}\end{matrix} + \text{Noise}$$

  Simulations: phase transition phenomenon, emergence of coherent structures.
- Degond-Motsch (2008).
  Time-continuous version: relaxation (with constant rate ν) towards the local mean direction.
  Hydrodynamic limit without phase transition phenomenon.
- Model presented here: making ν proportional to the local mean momentum.

Original discrete model
$\Delta t = 0,07$

Time-continuous version
$\nu = 15$

Outline

Individual dynamics

Particles at positions: $X_1, \dots, X_N$ in $\mathbb{R}^n$.
Orientations $\omega_1, \dots, \omega_N$ in $\mathbb{S}$ (unit sphere).

$$\begin{cases} \frac{\mathrm{d}X_k}{\mathrm{d}t} = \omega_k \\ \frac{\mathrm{d}\omega_k}{\mathrm{d}t} = \nu P_{\omega_k^\perp}(\bar{\omega}_k - \omega_k) \end{cases}$$

Individual dynamics

Particles at positions: $X_1, \dots, X_N$ in $\mathbb{R}^n$.
Orientations $\omega_1, \dots, \omega_N$ in $\mathbb{S}$ (unit sphere).

$$\begin{cases} \frac{\mathrm{d}X_k}{\mathrm{d}t} = \omega_k \\ \frac{\mathrm{d}\omega_k}{\mathrm{d}t} = \nu(\mathrm{Id} - \omega_k \otimes \omega_k)\,\bar{\omega}_k \qquad (= \nu \nabla_\omega(\omega \cdot \bar{\omega}_k)|_{\omega=\omega_k}) \end{cases}$$

Individual dynamics

Particles at positions: $X_1, \dots, X_N$ in $\mathbb{R}^n$.
Orientations $\omega_1, \dots, \omega_N$ in $\mathbb{S}$ (unit sphere).

$$\begin{cases} \mathrm{d}X_k = \omega_k \mathrm{d}t \\ \mathrm{d}\omega_k = \nu(\mathrm{Id} - \omega_k \otimes \omega_k)\,\bar{\omega}_k \mathrm{d}t + \sqrt{2d}\,\text{``}P_{\omega_k^\perp} \mathrm{d}B_t^k\text{''} \end{cases}$$

Individual dynamics

Particles at positions: $X_1, \ldots, X_N$ in $\mathbb{R}^n$.
Orientations $\omega_1, \ldots, \omega_N$ in $\mathbb{S}$ (unit sphere).

$$\begin{cases} \mathrm{d}X_k = \omega_k \mathrm{d}t \\ \mathrm{d}\omega_k = \nu(\mathrm{Id} - \omega_k \otimes \omega_k)\,\bar{\omega}_k \mathrm{d}t + \sqrt{2d}(\mathrm{Id} - \omega_k \otimes \omega_k) \circ \mathrm{d}B_t^k \end{cases}$$

Individual dynamics

Particles at positions: $X_1, \dots, X_N$ in $\mathbb{R}^n$.
Orientations $\omega_1, \dots, \omega_N$ in $\mathbb{S}$ (unit sphere).

$$\begin{cases} \mathrm{d}X_k = \omega_k \mathrm{d}t \\ \mathrm{d}\omega_k = \nu(\mathrm{Id} - \omega_k \otimes \omega_k)\,\bar{\omega}_k \mathrm{d}t + \sqrt{2d}(\mathrm{Id} - \omega_k \otimes \omega_k) \circ \mathrm{d}B_t^k \end{cases}$$

Target direction:

$$\bar{\omega}_k = \frac{J_k}{|J_k|}, \quad J_k = \frac{1}{N} \sum_{j=1}^{N} K(|X_j - X_k|)\,\omega_j.$$

Individual dynamics

Particles at positions: $X_1, \dots, X_N$ in $\mathbb{R}^n$.
Orientations $\omega_1, \dots, \omega_N$ in $\mathbb{S}$ (unit sphere).

$$\begin{cases} \mathrm{d}X_k = \omega_k \mathrm{d}t \\ \mathrm{d}\omega_k = \nu(\mathrm{Id} - \omega_k \otimes \omega_k)\,\bar{\omega}_k \mathrm{d}t + \sqrt{2d}(\mathrm{Id} - \omega_k \otimes \omega_k) \circ \mathrm{d}B_t^k \end{cases}$$

Target direction:

$$\bar{\omega}_k = \frac{J_k}{|J_k|}, \quad J_k = \frac{1}{N}\sum_{j=1}^{N} K(|X_j - X_k|)\,\omega_j.$$

Setting $\nu = |J_k|\,\nu_0$, no more singularity (binary interactions):

$$\begin{cases} \mathrm{d}X_k = \omega_k \mathrm{d}t \\ \mathrm{d}\omega_k = \nu_0(\mathrm{Id} - \omega_k \otimes \omega_k)\,J_k \mathrm{d}t + \sqrt{2d}(\mathrm{Id} - \omega_k \otimes \omega_k) \circ \mathrm{d}B_t^k \end{cases}$$

Kinetic description

Theorem (Bolley, Cañizo, Carrillo, 2011)

Probability density function $f(x, \omega, t)$, as $N \to \infty$:

$$\partial_t f + \omega \cdot \nabla_x f + \nu_0 \nabla_\omega \cdot ((\mathrm{Id} - \omega \otimes \omega) \mathcal{J}_f f) = d \Delta_\omega f$$

$$\mathcal{J}_f(x, \omega, t) = \int_{y \in \mathbb{R}^n,\, \upsilon \in \mathbb{S}} K(|y - x|)\, \upsilon\, f(y, \upsilon, t)\, \mathrm{d}y\, \mathrm{d}\upsilon\,.$$

Kinetic description

Theorem (Bolley, Cañizo, Carrillo, 2011)

Probability density function $f(x,\omega,t)$, as $N \to \infty$:

$$\partial_t f + \omega \cdot \nabla_x f + \nu_0 \nabla_\omega \cdot ((\mathrm{Id} - \omega \otimes \omega)\mathcal{J}_f\, f) = d\Delta_\omega f$$

$$\mathcal{J}_f(x,\omega,t) = \int_{y\in\mathbb{R}^n,\, \upsilon\in\mathbb{S}} K(|y-x|)\, \upsilon\, f(y,\upsilon,t)\, \mathrm{d}y\, \mathrm{d}\upsilon\,.$$

Tool : coupling process + estimations.

$$\begin{cases} \mathrm{d}\bar{X}_k = \bar{\omega}_k \mathrm{d}t \\ \mathrm{d}\bar{\omega}_k = \nu_0(\mathrm{Id} - \bar{\omega}_k \otimes \bar{\omega}_k)\, \mathcal{J}_{f_t^N}\, \mathrm{d}t + \sqrt{2d}(\mathrm{Id} - \bar{\omega}_k \otimes \bar{\omega}_k) \circ \mathrm{d}B_t^k \\ f_t^N = \mathrm{law}(\bar{X}_1, \bar{\omega}_1) = \mathrm{law}(\bar{X}_k, \bar{\omega}_k) \end{cases}$$

Hydrodynamic scaling

Scaling, with $\varepsilon \ll 1$ (and $K_0 = \int_{\mathbb{R}^n} K(x)\mathrm{d}x$):

$$f^\varepsilon(x,\omega,t) = \nu_0 K_0 f(\tfrac{1}{d\varepsilon}x,\omega,\tfrac{1}{d\varepsilon}t).$$

Mean-field rescaled and reduced equation:

$$\varepsilon(\partial_t f^\varepsilon + \omega\cdot\nabla_x f^\varepsilon) = Q(f^\varepsilon) + O(\varepsilon^2),$$

with

$$Q(f) = -\nabla_\omega\cdot((\mathrm{Id}-\omega\otimes\omega)J_f\, f) + \Delta_\omega f,$$
$$J_f(x,t) = \int_{\mathbb{S}} f(x,\omega,t)\,\omega\,\mathrm{d}\omega.$$

(localization in space)

Local equilibria

Fisher–von Mises distribution (orientation $\Omega \in \mathbb{S}$, concentration $\kappa \geqslant 0$):

$$M_{\kappa\Omega}(\omega) = \frac{e^{\kappa\,\omega\cdot\Omega}}{\int_{\mathbb{S}} e^{\kappa\,\upsilon\cdot\Omega}\,\mathrm{d}\upsilon}.$$

For $J_f = \kappa_f \Omega_f$, we get

$$Q(f) = \nabla_\omega \cdot \left[M_{\kappa_f\Omega_f} \nabla_\omega \left(\frac{f}{M_{\kappa_f\Omega_f}} \right) \right].$$

Local equilibria

Fisher–von Mises distribution (orientation $\Omega \in \mathbb{S}$, concentration $\kappa \geqslant 0$):

$$M_{\kappa\Omega}(\omega) = \frac{e^{\kappa\,\omega\cdot\Omega}}{\int_{\mathbb{S}} e^{\kappa\,\upsilon\cdot\Omega}\,\mathrm{d}\upsilon}.$$

For $J_f = \kappa_f \Omega_f$, we get

$$Q(f) = \nabla_\omega \cdot \left[M_{\kappa_f\Omega_f} \nabla_\omega \left(\frac{f}{M_{\kappa_f\Omega_f}} \right) \right].$$

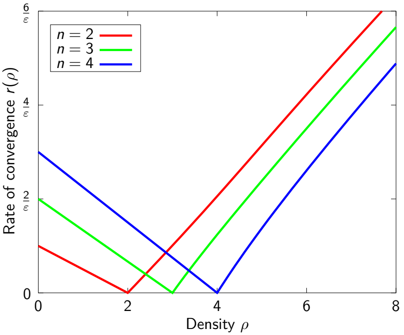
Local equilibria: $\rho M_{\kappa\Omega}$, for some $\Omega \in \mathbb{S}$.
Compatibility condition for $\kappa \geqslant 0$ and $\rho > 0$:

$$\rho c(\kappa) = \kappa, \text{ where } c(\kappa) = |J_{M_{\kappa\Omega}}| = \frac{\int_0^\pi \cos\theta\, e^{\kappa\cos\theta} \sin^{n-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{\kappa\cos\theta} \sin^{n-2}\theta\,\mathrm{d}\theta}.$$

Solutions to the compatibility condition, equilibria

Proposition

The function $\kappa \mapsto \frac{c(\kappa)}{\kappa}$ is decreasing, its limit is $\frac{1}{n}$ when $\kappa \to 0$.



- $\rho \leqslant n$, only one solution: $\kappa = 0$. Uniform equilibrium ("stable").
- $\rho > n$, uniform equilibrium ("unstable") for $\kappa = 0$. Unique solution $\kappa(\rho) > 0$. Manifold of equilibria ("stable"): $\rho M_{\kappa(\rho)\Omega}$, $\Omega \in \mathbb{S}$.

Subcritical case
$\nu = 1.9$

Supercritical case
$\nu = 2.3$

Stochastic model and its mean-field limit

Orientations ω_k only.

$$\begin{cases} \mathrm{d}\omega_k = (\mathrm{Id} - \omega_k \otimes \omega_k) J_k \,\mathrm{d}t + \sqrt{2\tau}\,(\mathrm{Id} - \omega_k \otimes \omega_k) \circ \mathrm{d}B_t^k, \\ J_k = \frac{1}{N} \sum_{j=1}^{N} \omega_j. \end{cases}$$

Stochastic model and its mean-field limit

Orientations ω_k only.

$$\begin{cases} \mathrm{d}\omega_k = (\mathrm{Id} - \omega_k \otimes \omega_k) J_k \, \mathrm{d}t + \sqrt{2\tau}\, (\mathrm{Id} - \omega_k \otimes \omega_k) \circ \mathrm{d}B_t^k, \\ J_k = \frac{1}{N} \sum_{j=1}^N \omega_j. \end{cases}$$

Limit as $N \to \infty$ (Bolley-Cañizo-Carrillo):

$$\partial_t f = -\nabla_\omega \cdot ((\mathrm{Id} - \omega \otimes \omega) J[f] f) + \tau \Delta_\omega f,$$

$$J[f] = \int_{\mathbb{S}} \omega \, f \, \mathrm{d}\omega.$$

Stochastic model and its mean-field limit

Orientations ω_k only.

$$\begin{cases} \mathrm{d}\omega_k = (\mathrm{Id} - \omega_k \otimes \omega_k) J_k \, \mathrm{d}t + \sqrt{2\tau} \, (\mathrm{Id} - \omega_k \otimes \omega_k) \circ \mathrm{d}B_t^k, \\ J_k = \frac{1}{N} \sum_{j=1}^N \omega_j. \end{cases}$$

Limit as $N \to \infty$ (Bolley-Cañizo-Carrillo):

$$\partial_t f = -\nabla_\omega \cdot ((\mathrm{Id} - \omega \otimes \omega) J[f] f) + \tau \Delta_\omega f \; = \nabla \cdot (f \nabla \Psi) + \tau \Delta f,$$

$$J[f] = \int_{\mathbb{S}} \omega \, f \, \mathrm{d}\omega, \quad \Psi(\omega) = -J[f] \cdot \omega = \int_{\mathbb{S}} K(\omega, \bar{\omega}) \, f(\bar{\omega}) \, \mathrm{d}\bar{\omega}.$$

Here $K = -\omega \cdot \bar{\omega}$ (dipolar potential).
Polymers: $|\omega \times \bar{\omega}|$ (Onsager) or $-(\omega \cdot \bar{\omega})^2$ (Maier–Saupe).

Existence, uniqueness, regularity, positivity, bounds

Theorem (AF, J.-G. Liu, 2011)

For an initial probability measure $f_0 \in H^s(\mathbb{S})$, (for an arbitrary s):

- Existence and uniqueness of a weak solution f.
- Global solution, in $C^\infty(\mathbb{R}_+^* \times \mathbb{S})$, and $f > 0$ for $t > 0$.
- Instantaneous regularity estimates and uniform bounds:

$$\|f(t)\|_{H^{s+m}}^2 \leqslant C\left(1 + \frac{1}{t^m}\right)\|f_0\|_{H^s}^2.$$

Tool: spherical harmonics decomposition.
Nonlinearity: finite number of coefficients.

Onsager free energy

Free energy: $\mathcal{F}(f) = \tau \int_{\mathbb{S}} f \ln f - \frac{1}{2}|J[f]|^2$.
Dissipation term: $\mathcal{D}(f) = \int_{\mathbb{S}} f |\nabla_\omega(\tau \ln f - \omega \cdot J[f])|^2 \geqslant 0$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + \mathcal{D} = 0.$$

The free energy $\mathcal{F}(f)$ is decreasing towards $\mathcal{F}_\infty$.

Onsager free energy

Free energy: $\mathcal{F}(f) = \tau \int_{\mathbb{S}} f \ln f - \frac{1}{2}|J[f]|^2$.
Dissipation term: $\mathcal{D}(f) = \int_{\mathbb{S}} f|\nabla_\omega(\tau \ln f - \omega \cdot J[f])|^2 \geqslant 0$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + \mathcal{D} = 0.$$

The free energy $\mathcal{F}(f)$ is decreasing towards $\mathcal{F}_\infty$.

LaSalle's principle

Limit set: $\mathcal{E}_\infty = \{f \in C^\infty(\mathbb{S}) | \mathcal{D}(f) = 0 \text{ and } \mathcal{F}(f) = \mathcal{F}_\infty\}$.

$$\lim_{t\to\infty} \inf_{g\in\mathcal{E}_\infty} \|f(t) - g\|_{H^s} = 0.$$

Equilibria

Equivalent conditions: $\begin{cases} \mathcal{D}(f) = 0 \text{ (no dissipation)} \\ Q(f) = 0 \text{ (stationary solution)} \\ f \text{ critical point of } \mathcal{F} \\ \tau \ln f - J[f] \cdot \omega = Cte \end{cases}$

Amounts to $f = M_{\kappa\Omega}$, with the compatibility condition $\tau\kappa = c(\kappa)$.

Proposition

The function $\kappa \mapsto \frac{c(\kappa)}{\kappa}$ is decreasing, its limit is $\frac{1}{n}$ when $\kappa \to 0$.

- $\tau \geqslant \frac{1}{n}$. Only one solution: $\kappa = 0$.
  Uniform equilibrium (unique minimizer of $\mathcal{F}$).
- $\tau < \frac{1}{n}$: uniform equilibrium for $\kappa = 0$, not minimizing $\mathcal{F}$.
  Only positive solution: $\kappa(\tau)$.
  Manifold of equilibria: $M_{\kappa(\tau)\Omega}$, $\Omega \in \mathbb{S}$.

Additional conservation relation

Conformal Laplacian: $\widetilde{\Delta}_{n-1}\, Y_\ell = \ell(\ell+1)\dots(\ell+n-2)Y_\ell$, for a spherical harmonic Y_ℓ of degree ℓ.

Proposition

For $g \in H^{\frac{n+1}{2}}(\mathbb{S})$ with mean zero, $\int_{\mathbb{S}} \nabla g \widetilde{\Delta}_{n-1} g = 0$.

Additional conservation relation

Conformal Laplacian: $\widetilde{\Delta}_{n-1}\, Y_\ell = \ell(\ell+1)\dots(\ell+n-2)Y_\ell$, for a spherical harmonic Y_ℓ of degree ℓ.

Proposition

For $g \in H^{\frac{n+1}{2}}(\mathbb{S})$ with mean zero, $\int_{\mathbb{S}} \nabla g \widetilde{\Delta}_{n-1} g = 0$.

Norms $\|g\|^2_{\widetilde{H}^{-\frac{n-1}{2}}} = \int_{\mathbb{S}} g \widetilde{\Delta}^{-1}_{n-1} g$, $\|g\|^2_{\widetilde{H}^{-\frac{n-3}{2}}} = \int_{\mathbb{S}} \Delta g \widetilde{\Delta}^{-1}_{n-1} g$:

Conservation relation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f-1\|^2_{\widetilde{H}^{-\frac{n-1}{2}}} = -\tau\|f-1\|^2_{\widetilde{H}^{-\frac{n-3}{2}}} + \frac{1}{(n-2)!}|J[f]|^2.$$

For $\tau > \frac{1}{n}$, it is an "entropy" dissipation!
Global exponential convergence with rate $(n-1)(\tau - \frac{1}{n})$ towards uniform equilibrium (in any H^s).

The "moving" Fisher–von Mises distribution

"ODE" for $J[f(t)]$: $J[f_0] \neq 0 \Rightarrow J[f(t)] \neq 0$ for all t.
We define $\Omega(t) = \dfrac{J[f(t)]}{|J[f(t)]|}$.
Expansion around the "moving equilibrium" $M_{\kappa(\tau)\Omega(t)}$:

$$f = (1+h)M_{\kappa(\tau)\Omega(t)},$$

Notation $\langle\cdot\rangle_{M_{\kappa\Omega}}$ for the mean against this measure.

The "moving" Fisher–von Mises distribution

"ODE" for $J[f(t)]$: $J[f_0] \neq 0 \Rightarrow J[f(t)] \neq 0$ for all t.
We define $\Omega(t) = \dfrac{J[f(t)]}{|J[f(t)]|}$.
Expansion around the "moving equilibrium" $M_{\kappa(\tau)\Omega(t)}$:

$$f = (1 + h) M_{\kappa(\tau)\Omega(t)},$$

Notation $\langle \cdot \rangle_{M_{\kappa\Omega}}$ for the mean against this measure.
Unique decomposition of the form

$$f = (1 + \alpha \left[\omega \cdot \Omega(t) - c(\kappa(\tau))\right] + g) M_{\kappa(\tau)\Omega(t)},$$

with $\langle g \rangle_{M_{\kappa\Omega}} = 0$ and $\langle g\omega \rangle_{M_{\kappa\Omega}} = 0$.
By LaSalle's principle, h, α and g converge to 0.

Exponential convergence to a fixed equilibrium

Poincaré inequality:

$$\langle |\nabla h|^2 \rangle_{M_{\kappa\Omega}} \geqslant \Lambda_\kappa \langle (h - \langle h \rangle_{M_{\kappa\Omega}})^2 \rangle_{M_{\kappa\Omega}}$$

Expansion of $\mathcal{D}(f)$ et $\mathcal{F}(f) - \mathcal{F}_\infty$, in terms of α^2 et $\langle g^2 \rangle_{M_{\kappa\Omega}}$, with $\|h\|_\infty$ sufficiently small: exponential decay of $\|f - M_{\kappa\Omega(t)}\|$ with rate r for all $r < r_\infty(\tau) = (c(\kappa)^2 + n\tau - 1)\Lambda_\kappa$.

Exponential convergence to a fixed equilibrium

Poincaré inequality:

$$\langle|\nabla h|^2\rangle_{M_{\kappa\Omega}} \geqslant \Lambda_\kappa\langle(h-\langle h\rangle_{M_{\kappa\Omega}})^2\rangle_{M_{\kappa\Omega}}$$

Expansion of $\mathcal{D}(f)$ et $\mathcal{F}(f)-\mathcal{F}_\infty$, in terms of α^2 et $\langle g^2\rangle_{M_{\kappa\Omega}}$, with $\|h\|_\infty$ sufficiently small: exponential decay of $\|f-M_{\kappa\Omega(t)}\|$ with rate r for all $r<r_\infty(\tau)=(c(\kappa)^2+n\tau-1)\Lambda_\kappa$.
Dynamics of $\Omega(t)$:

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = -(\mathrm{Id}-\Omega\otimes\Omega)\langle g\,\omega\cdot\Omega\,\omega\rangle_{M_{\kappa\Omega}}$$

So we get $\left|\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right| \leqslant C\sqrt{\langle g^2\rangle_{M_{\kappa\Omega}}} \leqslant \widetilde{C}e^{-rt}$: exponential convergence of Ω to some $\Omega_\infty\in\mathbb{S}$.

Exponential convergence to a fixed equilibrium

Poincaré inequality:

$$\langle|\nabla h|^2\rangle_{M_{\kappa\Omega}} \geqslant \Lambda_\kappa \langle (h - \langle h\rangle_{M_{\kappa\Omega}})^2\rangle_{M_{\kappa\Omega}}$$

Expansion of $\mathcal{D}(f)$ et $\mathcal{F}(f) - \mathcal{F}_\infty$, in terms of α^2 et $\langle g^2\rangle_{M_{\kappa\Omega}}$, with $\|h\|_\infty$ sufficiently small: exponential decay of $\|f - M_{\kappa\Omega(t)}\|$ with rate r for all $r < r_\infty(\tau) = (c(\kappa)^2 + n\tau - 1)\Lambda_\kappa$.

Dynamics of $\Omega(t)$:

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = -(\mathrm{Id} - \Omega\otimes\Omega)\langle g\,\omega\cdot\Omega\,\omega\rangle_{M_{\kappa\Omega}}$$

So we get $\left|\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right| \leqslant C\sqrt{\langle g^2\rangle_{M_{\kappa\Omega}}} \leqslant \widetilde{C}e^{-rt}$: exponential convergence of Ω to some $\Omega_\infty \in \mathbb{S}$.

Interpolation + uniform bounds: exponential convergence to $M_{\kappa\Omega_\infty}$ in any H^s.

Special cases

Critical case $\tau = \frac{1}{n}$. Same type of method, using the decomposition

$$f = 1 + \alpha \cos\theta + \frac{1}{2}\alpha^2(\cos^2\theta - \frac{1}{n}) + \frac{1}{6}\alpha^3(\cos^3\theta - \frac{3}{n+2}\cos\theta) + g,$$

with $\alpha = n\langle f \cos\theta\rangle$ and $\cos\theta = \omega \cdot \Omega$.
Expansion of $\mathcal{D}(f)$ in terms of $\langle g^2\rangle$ and α^6, and of $\mathcal{F}(f)$ in terms of $\langle g^2\rangle$ and α^4: algebraic convergence in $\frac{1}{\sqrt{t}}$ towards the uniform distribution.

Special cases

Critical case $\tau = \frac{1}{n}$. Same type of method, using the decomposition

$$f = 1 + \alpha\cos\theta + \tfrac{1}{2}\alpha^2(\cos^2\theta - \tfrac{1}{n}) + \tfrac{1}{6}\alpha^3(\cos^3\theta - \tfrac{3}{n+2}\cos\theta) + g,$$

with $\alpha = n\langle f\cos\theta\rangle$ and $\cos\theta = \omega\cdot\Omega$.
Expansion of $\mathcal{D}(f)$ in terms of $\langle g^2\rangle$ and α^6, and of $\mathcal{F}(f)$ in terms of $\langle g^2\rangle$ and α^4: algebraic convergence in $\frac{1}{\sqrt{t}}$ towards the uniform distribution.
Case of no noise: $\tau = 0$. The norm of $|J[f]|$ increases and the $H^{-\frac{n-1}{2}}$ norm of f explodes...

Thanks!